NEW ANALYTICAL SOLID GEOMETRY

FOR B.A. & B.Sc. (PASS & HONOURS) STUDENTS

by

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PREFACE TO THE FIFTH EDITION

From the very rapid sale of the first four heavy editions and a roaring demand for the fifth it appears that the book has really been found useful and has supplied a very real need. I take this opportunity of thanking the numerous readers for their unsolicited letters of appreciation, and offer an apology for the unavoidable delay in bringing out the present edition.

In view of the recently raised standard of Examinations the book has been thoroughly revised, recast at some places, and brought up to date. Slight modifications have been effected in the wording of some Articles and Examples set in the University Examinations to make them of more permanent value. Older Examples of lesser importance have been deleted or in a few cases replaced by modern Examples of greater importance.

Copious hints and complete solutions have been given to typical Examples.

It is hoped that the book in its present form will be found more useful to those for whom it is meant.

BANSI LAL

D.A.-V. COLLEGE, JULLUNDUR CITY,
1st January, 1962.

EXTRACT FROM THE PREFACE TO THE FIRST EDITION

Analytical Solid Geometry, with its imaginative character and complicated figures, is generally looked upon with disfavour by the students. The very idea of a *parallelepiped,* which is unfortunately taken as the starting point by almost every writer on the subject, scares them away. This book has, therefore, been written so as to

avoid that bugbear and the like, and make Analytical Solid Geometry easily comprehensible to the students.

The book has the following

Distinguishing Features :

(1) It *just* covers the University Syllabus and is complete in itself, rigidly excluding all matters, however important they might appear from *other* points of view.

(2) Written primarily from the view-point of an *examinee*, it attempts, with its simple figures and neat proofs (*e.g.*, see Arts. 4, 9 and 13, (*a*)), to present the subject in an *easy-to-understand* and *easy-to-reproduce* form.

(3) Wherever possible, the method under consideration has been summarized in a *Rule* stated in consecutive steps.

(4) After an Article or two, only *typical* Examples, whether classical or modern, but mostly taken from the Examination papers, have been given and carefully graded. Their number, it is hoped, is sufficient to meet the requirements of even the most ambitious candidate for the Degree (Pass and Honours) Examination. The most important Examples have been fully solved.

All this has been done to give the students an idea of the type of questions favourite with the Examiners and of the type of answers expected by them to award cent per cent marks.

(5) Each chapter ends with a set of *Miscellaneous Examples*, not to form *necessarily* a part of the class-work, to help the ambitious student to take stock of the knowledge acquired so far.

(6) Articles and Examples which are most important from the *Examination stand-point* have been printed in bold type, and those next to them in importance in italics. This will help the student in his judicious choice of them when the time at his disposal is limited.

(7) The common errors usually made by the students, as noticed by the author in his long experience of about one-third of a century as a teacher and Examiner, have been pointed out as **'Caution'**.

(8) A chapter-wise list of all the important results and formulae to be remembered by the student has been prefixed to the book for ready reference and occasional review.

In short, no pains have been spared to make the book useful for all types of students—ordinary, average and brilliant.

In fine, acknowledgment is due, and is hereby made to all the authors consulted in the preparation of the book particularly to Charles Smith and Robert J. T. Bell.

BANSI LAL

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PANJAB UNIVERSITY SYLLABUS IN ANALYTICAL SOLID GEOMETRY* FOR THE B. A. (OLD TYPE) EXAMINATION, 1962 (*and after*)

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PANJAB UNIVERSITY SYLLABUS IN ANALYTICAL SOLID GEOMETRY† FOR THE B.Sc. (OLD TYPE) EXAMINATION, 1962 (*and after*)

Same as for the B.A. Examination with the following addition :

The interpretation of the following equations :-

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0\ ;$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c},\ \frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}\ ;$$

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}=1 \text{ and } y^2=4ax.$$

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*One-fourth of Mathematics B Course Paper B.

†One-half of Mathematics Paper II.

‡Three-fifths of Mathematics Honours Paper II.

ABBREVIATIONS

A. U.	stands for		*Allahabad University*
Ag. U.	,,	,,	*Agra University*
B. U.	,,	,,	*Bombay University*
Bar. U.	,,	,,	*Baroda University*
B.H.U.	,,	,,	*Benares Hindu University*
C. U.	,,	,,	*Calcutta University*
D. U.	,,	,,	*Delhi University*
Em.	,,	,,	*Emergency Examination*
Eng. 2.	,,	,,	*Engineering Examination Part II*
H.	,,	,,	*Honours Examination*
J. & K.U.	,,	,,	*Jammu & Kashmir University*
L. U.	,,	,,	*London University*
M.	,,	,,	*M.A. Examination*
M(P).	,,	,,	*M.A. (Previous) Examination*
M. T.	,,	,,	*Mathematics Tripos Examination*
N. U.	,,	,,	*Nagpur University*
P. U.	,,	,,	*Panjab (India) University*
Pesh. U.	,,	,,	*Peshawar University*
P(P). U.	,,	,,	*Panjab (Pakistan) University*
S.	,,	,,	*Supplementary Examination*
w. r. t.	,,	,,	*with respect to*

Rene Descartes (1596—1650), a French mathematician, the "father" of Analytical Geometry, who invented the basis of the subject from his dream on the night of 10th November, 1619.

IMPORTANT RESULTS AND FORMULAE

(To Be Remembered)

CHAPTER I

1. **Distance formula.** The distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\mathbf{d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$. [Art. 4]

2. **Section formulae.** The co-ordinates of the point which divides the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m_1 : m_2$ are

$$\mathbf{x=\frac{m_1x_2+m_2x_1}{m_1+m_2},\quad y=\frac{m_1y_2+m_2y_1}{m_1+m_2},\quad z=\frac{m_1z_2+m_2z_1}{m_1+m_2}.} \qquad \text{[Art. 5]}$$

[Remember the **Rule** to write down these from the Fig. See page 6.] $(x_1, y_1, z_1)\ m_1 \ : \ m_2\ (x_2, y_2, z_2)$

Mid-point formulae. The co-ordinates of the mid-point of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) are

$$\mathbf{x=\frac{x_1+x_2}{2},\quad y=\frac{y_1+y_2}{2},\quad z=\frac{z_1+z_2}{2}.} \qquad \text{[Art. 5, Cor.]}$$

3. **Centroid formulae or centre of gravity formulae for the triangle.** The co-ordinates of the centroid or the centre of gravity of the triangle, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), are

$$\mathbf{x=\frac{x_1+x_2+x_3}{3},\quad y=\frac{y_1+y_2+y_3}{3},\quad z=\frac{z_1+z_2+z_3}{3}.} \qquad \text{[Ex. 3, Art. 5]}$$

4. **Centre of gravity formulae for the tetrahedron.** The co-ordinates of the centre of gravity of the tetrahedron, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), are

$$\mathbf{x=\frac{x_1+x_2+x_3+x_4}{4},\quad y=\frac{y_1+y_2+y_3+y_4}{4},\quad z=\frac{z_1+z_2+z_3+z_4}{4}.}$$

[Ex. 4, Art. 5]

CHAPTER II

1. If l, m, n are the direction-cosines of a line OP, and OP$=r$, then the co-ordinates of P are $\mathbf{(lr, mr, nr)}$. [Art. 8]

2. **Relation between the direction-cosines.** If l, m, n are the direction-cosines of a line, then $\mathbf{l^2+m^2+n^2=1}$. [Art. 9]

Cor. Relation between the direction-cosines. (Another form.) If α, β, γ are the angles which a line makes with the axes, then $\mathbf{\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1}$. [Art. 9, Cor.]

3. If the direction-cosines of a line are proportional to $a, b, c,$ the actual direction-cosines are

$$\frac{\mathbf{a}}{\sqrt{\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2}}, \frac{\mathbf{b}}{\sqrt{\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2}}, \frac{\mathbf{c}}{\sqrt{\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2}}.$$

[**Rule to find the actual direction-cosines of a line, which are proportional to a, b, c.**

Divide a, b, c each by $\sqrt{a^2+b^2+c^2}$. The resulting ratios are the actual direction-cosines.] [Art. 10]

****Complete results.** The actual direction-cosines are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}},$$

the ambiguous signs being taken *all* positive or *all* negative.

[Complete results, Art. 10]

4. **Length of the projection.** The projection of a segment AB on a line X'X is $\mathbf{A'B'=AB}\cos\theta,$

where θ is the angle which AB makes with X'X. [Art. 11, (*c*)]

5. **Direction-cosines of the join of two points.** The direction-cosines of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to

$$\mathbf{x}_2-\mathbf{x}_1, \mathbf{y}_2-\mathbf{y}_1, \mathbf{z}_2-\mathbf{z}_1. \quad \text{[Art. 12]}$$

6. **Angle formula for two lines whose actual direction-cosines are given.** *If θ is the angle between the lines whose direction-cosines are l, m, n ; l', m', n', then* $\cos\theta=\mathbf{ll'+mm'+nn'}$. [Art. 13,(*a*)]

7. **Lagrange's identity.**

$$(\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)(\mathbf{l}'^2+\mathbf{m}'^2+\mathbf{n}'^2)-(\mathbf{ll'+mm'+nn'})^2 = (\mathbf{mn'-m'n})^2+(\mathbf{nl'-n'l})^2+(\mathbf{lm'-l'm})^2. \qquad \begin{vmatrix} l, & m, & n \\ l', & m', & n' \end{vmatrix}$$

[Art. 13, (*a*)]

Cor. 1. Angle formula (sine form) for two lines whose actual direction-cosines are given. If θ is the angle between the lines whose direction-cosines are l, m, n ; l', m', n', then

$$\sin\theta=\sqrt{(\mathbf{mn'-m'n})^2+(\mathbf{nl'-n'l})^2+(\mathbf{lm'-l'm})^2}. \qquad \begin{vmatrix} l, & m, & n \\ l', & m', & n' \end{vmatrix}$$

[Art. 13, (*a*), Cor. 1]

****Complete angle formula (sine form) for two lines.** If θ is the angle between the lines whose direction-cosines are l, m, n ; l', m', n', then

$$\sin\theta=\pm\sqrt{(\mathbf{mn'-m'n})^2+(\mathbf{nl'-n'l})^2+(\mathbf{lm'-l'm})^2}.$$

[Complete angle formula, Art. 13, (*a*), Cor. 1]

Cor. 2. Angle formula (tangent form) for two lines whose actual direction-cosines are given. If θ is the angle between the lines whose direction-cosines are l, m, n ; l', m', n', then

$$\tan\theta = \frac{\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}}{ll'+mm'+nn'}. \qquad \begin{matrix} l, & m, & n \\ l', & m', & n' \end{matrix}$$

$$\left[\frac{\sin\theta}{\cos\theta}\right]$$ [Art. 13, (*a*), Cor. 2]

****Complete angle formula (tangent form) for two lines.** If θ is the angle between the lines whose direction-cosines are l, m, n; l', m', n', then

$$\tan\theta = \pm\frac{\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}}{ll'+mm'+nn'}. \qquad \left[\frac{\sin\theta}{\cos\theta}\right]$$

[Complete angle formula, Art. 13, (*a*), Cor. 2]

Cor. 3. Condition of perpendicularity of two lines whose actual direction-cosines are given. The condition that the lines whose direction-cosines are l, m, n; l', m', n', may be perpendicular, is $ll'+mm'+nn'=0$. [Art. 13, (*a*), Cor. 3]

Cor. 4. Conditions of parallelism of two lines whose actual direction-cosines are given. The conditions that the lines whose direction-cosines are l, m, n; l', m', n', may be parallel, are

$$l=l', \; m=m', \; n=n'. \qquad \text{[Art. 13, (a), Cor. 4]}$$

8. Angle formula for two lines whose proportional direction-cosines are given. If θ is the angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c', then

$$\cos\theta = \frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\,\sqrt{a'^2+b'^2+c'^2}}. \qquad \text{[Art. 13, (b)]}$$

****Complete angle formula for two lines whose proportional direction-cosines are given.** If θ is the angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c', then

$$\cos\theta = \pm\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\,\sqrt{a'^2+b'^2+c'^2}}.$$

[Complete angle formula, Art. 13, (*b*)]

Cor. 1. Angle formula (sine form) for two lines whose proportional direction-cosines are given. If θ is the angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c', then

$$\sin\theta = \frac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{\sqrt{a^2+b^2+c^2}\,\sqrt{a'^2+b'^2+c'^2}}. \qquad \left|\begin{matrix} a, & b, & c \\ a', & b', & c' \end{matrix}\right.$$

[Art. 13, (*b*), Cor. 1]

****Complete angle formula (sine form) for two lines.** If θ is the angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c', then

$$\sin\theta = \pm\frac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}.$$

[Complete angle formula, Art. 13, (*b*), Cor. 1]

Cor. 2. Angle formula (tangent form) for two lines whose proportional direction-cosines are given. If θ is the angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c', then

$$\tan\theta = \frac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{aa'+bb'+cc'}. \qquad \left|\begin{array}{l} a, b, c \\ a', b', c' \end{array}\right.$$

$\left[\dfrac{\sin\theta}{\cos\theta}\right]$ [Art. 13, (*b*), Cor. 2]

Note. The **angle formula (tangent form)** for two lines is the same whether the actual direction-cosines are used or their *proportionals.*

****Complete angle formula (tangent form) for two lines.** If θ is the angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c', then

$$\tan\theta = \pm\frac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{aa'+bb'+cc'}. \qquad \left[\frac{\sin\theta}{\cos\theta}\right]$$

[Complete angle formula, Art. 13, (*b*), Cor. 2]

Cor. 3. Condition of perpendicularity of two lines whose proportional direction-cosines are given. The condition that the lines whose direction-cosines are proportional to a, b, c; a', b', c', may be perpendicular, is $aa'+bb'+cc' = 0$. [Art. 13, (*b*), Cor. 3]

Note. The condition of perpendicularity of two lines is the same whether the actual direction-cosines are used or their *proportionals.*

[Note, Art. 13, (*b*), Cor. 3]

Cor. 4. Conditions of parallelism of two lines whose proportional direction-cosines are given. The conditions that the lines whose direction-cosines are proportional to a, b, c; a', b', c', may be parallel, are $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$. [Art. 13, (*b*), Cor. 4]

9. Projection of the join of two points on a line whose direction-cosines are given. The projection of the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on a line whose direction-cosines are l, m, n is

$$(x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)\,n. \qquad [\text{ Art. } 14\ ]$$

CHAPTER III

1. Equation of a plane parallel to one of the co-ordinate planes.

The equation of the plane parallel to the yz-plane and at a distance a from it, is $\mathbf{x = a}$.

The equation of the plane parallel to the zx-plane and at a distance b from it, is $\mathbf{y = b}$.

The equation of the plane parallel to the xy-plane and at a distance c from it, is $\mathbf{z = c}$. [Art. 15]

Cor. Equations of the co-ordinate planes.

The equation of the yz-plane is $\mathbf{x = 0}$.

The equation of the zx-plane is $\mathbf{y = 0}$.

The equation of the xy-plane is $\mathbf{z = 0}$. [Art. 15, Cor.]

2. (*i*) $f(x)=0$ represents a system of planes (parallel to the plane of the *absent* variables, *i.e.*,) parallel to the yz-plane.

(*ii*) $f(x, y)=0$ represents a cylinder generated by a straight line (parallel to the axis of the *absent* variable, *i.e.*,) parallel to the z-axis.

(*iii*) $f(x, y, z)=0$ represents a surface. [Arts. 16, 17, 18]

3. Equations of a curve. $f_1(x, y, z)=0$, $f_2(x, y, z)=0$, together represent the curve of intersection of the two surfaces

$$f_1(x, y, z)=0 \text{ and } f_2(x, y, z)=0. \qquad \text{[Art. 19]}$$

Cor. Equations of the axes.

The equations of the x-axis are $\mathbf{y = 0, z = 0}$.

The equations of the y-axis are $\mathbf{z = 0, x = 0}$.

The equations of the z-axis are $\mathbf{x = 0, y = 0}$. [Art. 19, Cor.1]

CHAPTER IV

1. General form. The general equation of a plane is

$$\mathbf{Ax + By + Cz + D = 0}. \qquad \text{[Art. 21]}$$

Cor. One-point form. The equation of *any* plane through (x_1, y_1, z_1) is $\mathbf{A(x-x_1)+B(y-y_1)+C(z-z_1) = 0}$. [Art. 21, Cor.]

2. Normal form. The equation of a plane in terms of p, the length of the perpendicular from the origin on the plane, and l, m, n, the direction-cosines of this perpendicular, is $\mathbf{lx+my+nz=p}$.

[Art. 22]

3. Intercept form. The equation of a plane in terms of a, b, c, the intercepts of the plane on the axes, is $\dfrac{\mathbf{x}}{\mathbf{a}} + \dfrac{\mathbf{y}}{\mathbf{b}} + \dfrac{\mathbf{z}}{\mathbf{c}} = \mathbf{1}$.

[Art. 23]

4. Three-point form. The equation of the plane through (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) is

$$\begin{vmatrix} \mathbf{x}, & \mathbf{y}, & \mathbf{z}, & \mathbf{1} \\ \mathbf{x_1}, & \mathbf{y_1}, & \mathbf{z_1}, & \mathbf{1} \\ \mathbf{x_2}, & \mathbf{y_2}, & \mathbf{z_2}, & \mathbf{1} \\ \mathbf{x_3}, & \mathbf{y_3}, & \mathbf{z_3}, & \mathbf{1} \end{vmatrix} = \mathbf{0}. \qquad \text{[Art. 26]}$$

5. Direction-cosines of the normal. The direction-cosines of the normal to the plane $Ax+By+Cz+D=0$ are proportional to **A, B, C,** *i.e.*, proportiona to the **coefficients of x, y, z.**

[Art. 24, Cor.]

6. Angle formula for two planes. If θ is the angle between the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$, then

$$\cos\theta = \frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\,\sqrt{A'^2+B'^2+C'^2}}. \qquad \text{[Art. 27]}$$

****Complete angle formula for two planes.** If θ is the angle between the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$, then

$$\cos\theta = \pm\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\,\sqrt{A'^2+B'^2+C'^2}}.$$

[Complete angle formula, Art. 27]

7. Angle formula (tangent form) for two planes. If θ is the angle between the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$, then

$$\tan\theta = \frac{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}}{AA'+BB'+CC'}. \qquad \begin{vmatrix} A, & B, & C \\ A', & B', & C' \end{vmatrix}$$

[Ex. 5, Art. 27]

****Complete angle formula (tangent form) for two planes.** If θ is the angle between the planes

$$Ax+By+Cz+D=0, \quad A'x+B'y+C'z+D'=0,$$

then

$$\tan\theta = \pm\frac{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}}{AA'+BB'+CC'}.$$

[Complete angle formula, Ex. 5, Art. 27]

Cor. 1. Condition of perpendicularity of two planes. The condition that the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$ may be perpendicular, is $\mathbf{AA'+BB'+CC'=0}$.

[In words : *product of coefficients of x + product of coefficients of y + product of coefficients of z=0.*]

[Art. 27, Cor. 1]

Cor. 2. Conditions of parallelism of two planes. The conditions that the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$ may be parallel, are $\dfrac{A}{A'}=\dfrac{B}{B'}=\dfrac{C}{C'}$.

[In words : *ratio of coefficients of x = ratio of coefficients of y = ratio of coefficients of z.*]

[Art. 27, Cor. 2]

Cor. 3. Equation of any plane parallel to a given plane. The equation of *any* plane parallel to the plane $Ax+By+Cz+D=0$ is

$$\mathbf{Ax+By+Cz+k=0,}$$

where k is any constant.

[**Rule to write down the equation of** *any* **plane parallel to a given plane (equation in the general form) :**

In the equation of the given plane, change only the constant term to a new constant k.]

Note. The value of k is found from the second condition satisfied by the plane. [Art. 27, Cor. 3]

8. Position of the origin with respect to the angle between two planes. The quantity $\mathbf{AA'+BB'+CC'}$ **is negative or positive** according as the origin is in the acute angle or obtuse angle between the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$, D, D' being both positive. [Ex. 6, Art. 27]

9. Rule to find whether two points are on the same side or on opposite sides of a plane :

In the L. H. S. of the equation of the plane (R. H. S. being zero) substitute in succession the co-ordinates of the two points ; if the results are of the same sign, the points are on the same side of the plane ; if the results are of opposite signs, the points are on opposite sides. [Art. 28]

10. Perpendicular distance formula for the plane (equation in the normal form). The perpendicular distance of the point (x_1, y_1, z_1) from the plane $lx+my+nz=p$ is

$$\mathbf{d=lx_1+my_1+nz_1-p.} \qquad \text{[Art. 29, (a)]}$$

****Complete perpendicular distance formula for the plane (equation in the normal form).** The perpendicular distance of the point (x_1, y_1, z_1) from the plane $lx+my+nz=p$ is

$$\mathbf{d=\pm(lx_1+my_1+nz_1-p),}$$

that sign being taken on the R.H.S. which gives a *positive* result for d.
[Complete perpendicular distance formula, Art. 29, (a)]

11. Perpendicular distance formula for the plane (equation in the general form). The perpendicular distance of the point (x_1, y_1, z_1) from the plane $Ax+By+Cz+D=0$ is

$$\mathbf{d=\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}.} \qquad \text{[Art. 29, (b)]}$$

****Complete perpendicular distance formula for the plane (equation in the general form).** The perpendicular distance of the point (x_1, y_1, z_1) from the plane $Ax+By+Cz+D=0$ is

$$\mathbf{d} = \pm \frac{\mathbf{Ax_1 + By_1 + Cz_1 + D}}{\sqrt{\mathbf{A^2 + B^2 + C^2}}},$$

that sign being taken on the R.H.S. which gives a *positive* result for *d*.
[Complete perpendicular distance formula, Art. 29, (*b*)]

[Aid to memory. For the complete perpendicular distance formula for the plane (like the complete angle formula for two planes), take the double sign ($\pm$) with the ordinary perpendicular distance formula.]

[**Rule to find the perpendicular distance of a point from a plane (equation in the general form) :**

In the L. H. S. of the equation of the plane (R. H. S. being zero), substitute the co-ordinates of the point, and divide the result by

$$\sqrt{(\text{coeff. of } x)^2 + (\text{coeff. of } y)^2 + (\text{coeff. of } z)^2}.$$

The result gives the perpendicular distance. [Art. 29, (*b*)]

12. **Equations of the planes bisecting the angles between two given planes.** The equations of the planes bisecting the angles between the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$ are

$$\frac{\mathbf{Ax+By+Cz+D}}{\sqrt{\mathbf{A^2+B^2+C^2}}} = \pm \frac{\mathbf{A'x+B'y+C'z+D'}}{\sqrt{\mathbf{A'^2+B'^2+C'^2}}} \quad \dots (1)$$

To distinguish between the two bisecting planes. If the constant terms D, D′ in the equations of the two planes are *both positive*, the positive sign on the R.H.S. of (1) gives the equation of the plane bisecting that angle between the two planes in which the origin lies, and the negative sign gives the equation of the plane bisecting that angle between the two planes in which the origin does not lie. [Art. 30]

13. (*a*) **Length of the projection on a plane.** The projection of a segment AB on a plane π is $\mathbf{A'B' = AB \cos \theta}$,
where θ is the angle which AB makes with the plane π. [Art. 32, (*a*)]

(*b*) **Area of the projection.** The projection of a plane area A on a plane π is $\mathbf{A' = A \cos \theta}$,
where θ is the angle which the plane of the area A makes with the plane π. [Art. 32, (*b*)]

14. If A_x, A_y, A_z are the projections of a plane area A on the co-ordinate planes, then $\mathbf{A = \sqrt{A_x^2 + A_y^2 + A_z^2}}$. [Art. 33]

15. **Volume formula for the tetrahedron.** The volume of the tetrahedron, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), is $\mathbf{V} = \frac{1}{6}\begin{vmatrix} \mathbf{x_1}, & \mathbf{y_1}, & \mathbf{z_1}, & \mathbf{1} \\ \mathbf{x_2}, & \mathbf{y_2}, & \mathbf{z_2}, & \mathbf{1} \\ \mathbf{x_3}, & \mathbf{y_3}, & \mathbf{z_3}, & \mathbf{1} \\ \mathbf{x_4}, & \mathbf{y_4}, & \mathbf{z_4}, & \mathbf{1} \end{vmatrix}$. [Art. 34]

Complete volume formula for the tetrahedron. The volume of the tetrahedron, whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), is $\mathbf{V} = \pm \frac{1}{6} \begin{vmatrix} \mathbf{x_1}, & \mathbf{y_1}, & \mathbf{z_1}, & \mathbf{1} \\ \mathbf{x_2}, & \mathbf{y_2}, & \mathbf{z_2}, & \mathbf{1} \\ \mathbf{x_3}, & \mathbf{y_3}, & \mathbf{z_3}, & \mathbf{1} \\ \mathbf{x_4}, & \mathbf{y_4}, & \mathbf{z_4}, & \mathbf{1} \end{vmatrix}$,

that sign being taken on the R.H.S. which gives a *positive* result for V.
[Complete volume formula, Art. 34]

16. **Condition for the general homogeneous equation of the second degree in x, y, z to represent a pair of planes.** The condition that the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$$

may represent a pair of planes, is $\mathbf{abc+2fgh-af^2-bg^2-ch^2=0.}$
[Art. 35]

Cor. Condition of perpendicularity of a pair of planes. The condition that the pair of planes $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ may be perpendicular, is $\mathbf{a+b+c = 0.}$

[In words :
coefficient of $\mathbf{x^2}$+coefficient of $\mathbf{y^2}$+coefficient of $\mathbf{z^2}$ = 0.]
[Art. 35, Cor.]

CHAPTER V

1. **(Distance form or) Symmetrical form of the equations of a line whose actual direction-cosines are given.** The equations of the straight line passing through (x_1, y_1, z_1) and having direction-cosines l, m, n are $\dfrac{\mathbf{x-x_1}}{\mathbf{l}} = \dfrac{\mathbf{y-y_1}}{\mathbf{m}} = \dfrac{\mathbf{z-z_1}}{\mathbf{n}} (=\mathbf{r}).$ [Art. 37]

Cor. 1. Symmetrical form of the equations of a straight line whose proportional direction-cosines are given. The equations of the straight line passing through (x_1, y_1, z_1) and having direction-cosines proportional to a, b, c, are

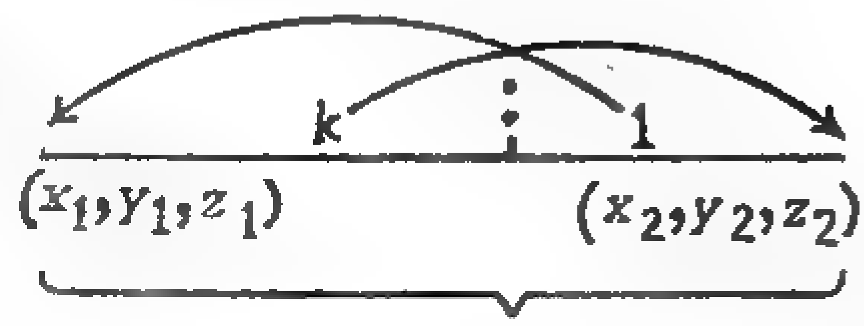
$$\frac{\mathbf{x-x_1}}{\mathbf{a}} = \frac{\mathbf{y-y_1}}{\mathbf{b}} = \frac{\mathbf{z-z_1}}{\mathbf{c}}.$$ [Art. 37, Cor. 1]

Cor. 2. Any point on the line. Any point on the line $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$ is $(\mathbf{x_1+lr, y_1+mr, z_1+nr})$. [Art. 37, Cor.2]

2. **Two-point form.** The equations of the straight line through (x_1, y_1, z_1), (x_2, y_2, z_2) are $\dfrac{\mathbf{x-x_1}}{\mathbf{x_2-x_1}} = \dfrac{\mathbf{y-y_1}}{\mathbf{y_2-y_1}} = \dfrac{\mathbf{z-z_1}}{\mathbf{z_2-z_1}}.$ [Art. 40]

****Cor. Any point on the line through two points.** Any point on the line through (x_1, y_1, z_1), (x_2, y_2, z_2) is

$$\left(\frac{\mathbf{kx_2+x_1}}{\mathbf{k+1}}, \frac{\mathbf{ky_2+y_1}}{\mathbf{k+1}}, \frac{\mathbf{kz_2+z_1}}{\mathbf{k+1}}\right).$$



[Art. 40, Cor.]

****3.** The equations of the bisectors of the angles between the lines $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$, $\frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'}$ (l, m, n ; l', m', n' *actual* direction-cosines) are

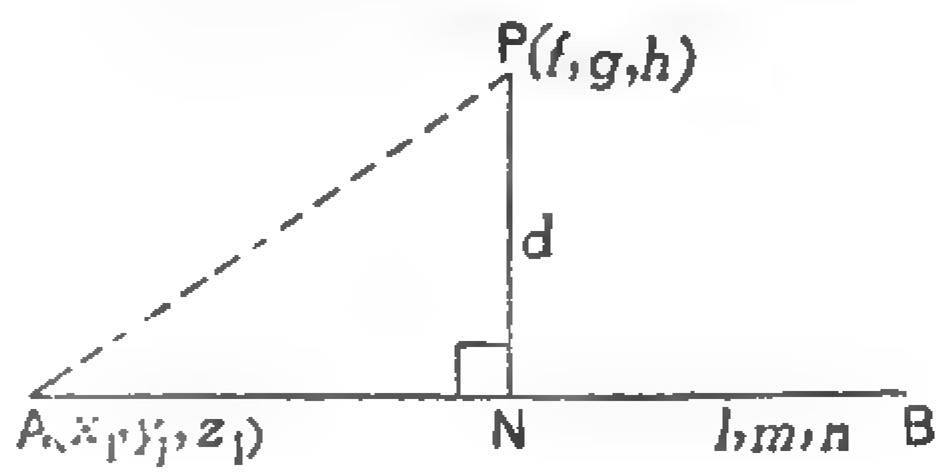
$$\frac{\mathbf{x}}{\mathbf{l+l'}} = \frac{\mathbf{y}}{\mathbf{m+m'}} = \frac{\mathbf{z}}{\mathbf{n+n'}}, \quad \frac{\mathbf{x}}{\mathbf{l-l'}} = \frac{\mathbf{y}}{\mathbf{m-m'}} = \frac{\mathbf{z}}{\mathbf{n-n'}}.$$ [Ex. 3, Art. 40]

4. Perpendicular distance formula for the line [equations in the symmetrical (actual direction-cosines form)]. The perpendicular distance of the point (f, g, h) from the line

$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ (l, m, n actual direction-cosines) is

$$\mathbf{d} = \{(\mathbf{f-x_1})^2 + (\mathbf{g-y_1})^2 + (\mathbf{h-z_1})^2 - [(\mathbf{f-x_1})\mathbf{l} + (\mathbf{g-y_1})\mathbf{m} + (\mathbf{h-z_1})\mathbf{n}]^2\}^{\frac{1}{2}}$$

... **First form** [Art. 41]



$$\begin{vmatrix} f-x_1, & g-y_1, & h-z_1 \\ l, & m, & n \end{vmatrix}$$

or $$\mathbf{d} = \{[(\mathbf{g-y_1})\mathbf{n} - (\mathbf{h-z_1})\mathbf{m}]^2 + [(\mathbf{h-z_1})\mathbf{l} - (\mathbf{f-x_1})\mathbf{n}]^2 + [(\mathbf{f-x_1})\mathbf{m} - (\mathbf{g-y_1})\mathbf{l}]^2\}^{\frac{1}{2}}$$

... **Second form** [Art. 41, Cor.]

5. (*i*) **Conditions of parallelism of a line and a plane.** The conditions that the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ may be parallel to the plane $Ax+By+Cz+D=0$ are [that it should be perpendicular to the normal to the plane, and its point (x_1, y_1, z_1) should not lie on the plane, *i.e.*,]

$\mathbf{Al+Bm+Cn=0}$, and $Ax_1+By_1+Cz_1+D \neq 0$. [Art. 43, (*a*)]

(*ii*) **Conditions of perpendicularity of a line and a plane.** The conditions that the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ may be perpendicular to the plane $Ax+By+Cz+D=0$ are [that it should be parallel to the normal to the plane, *i.e.*,] $\frac{\mathbf{l}}{\mathbf{A}} = \frac{\mathbf{m}}{\mathbf{B}} = \frac{\mathbf{n}}{\mathbf{C}}$. [Art. 43, (*b*)]

(*iii*) **Conditions for a line to lie in a plane.** The conditions that the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ may lie in the plane $Ax+By+Cz+D=0$ are [that it should be perpendicular to the normal to the plane, and its point (x_1, y_1, z_1) should lie on the plane, *i.e.*,]

$\mathbf{Al+Bm+Cn=0}$, **and** $\mathbf{Ax_1+By_1+Cz_1+D=0}$. [Art. 43, (*c*)]

6. (*a*) **Any plane through a given line (equations in the general form).** The equation of *any* plane through the line

$$Ax+By+Cz+D=0,\ A'x+B'y+C'z+D'=0$$

is
$$\mathbf{Ax+By+Cz+D+k\,(A'x+B'y+C'z+D')=0},$$

where k is any constant.

[In words : **one plane+k(other plane)=0,** where 'one plane' stands for the 'L.H.S. of the equation of one plane (R.H.S. being zero)', and so for the 'other plane'.] [Art. 44, (*a*)]

Note. The value of k is found from the second condition satisfied by the plane.

(*b*) **Any plane through a given line (equations in the symmetrical form).** The equation of *any* plane through the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

is
$$\mathbf{A(x-x_1)+B(y-y_1)+C(z-z_1)=0} \quad \ldots (1)$$

where
$$\mathbf{Al+Bm+Cn=0} \quad \ldots (2)$$
[Art. 44, (*b*)]

Cor. Plane through one line and parallel to another line (equations of both lines in the symmetrical form). The equation of the plane through the line $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$,

and parallel to the line $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$

is
$$\begin{vmatrix} \mathbf{x-x_1}, & \mathbf{y-y_1}, & \mathbf{z-z_1} \\ \mathbf{l_1}, & \mathbf{m_1}, & \mathbf{n_1} \\ \mathbf{l_2}, & \mathbf{m_2}, & \mathbf{n_2} \end{vmatrix} = \mathbf{0}.$$
[Art. 44, (*b*), Cor.]

7. **Rule to prove that two given lines intersect (or are coplanar) (equations of both lines in the symmetrical form) :**

(*i*) *Write down the equation of the plane through one line and parallel to the other.*

(*ii*) *Show that this plane passes through one point of the other line.*

Cor. Plane through two intersecting lines. The equation of the plane, in which the intersecting lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1},\ \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

lie, is [the same as the equation of the plane through one line and

parallel to the other, *i.e.,*]

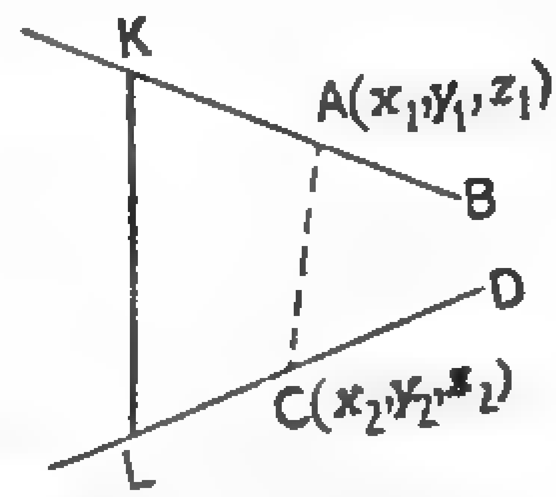
$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} = 0. \qquad \text{[Art. 45, (a), Cor.]}$$

8. *Shortest distance between two lines.*

Rule to find the shortest distance between the lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}, \quad \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}.$$

[Method of projection.]



Find the direction-cosines of KL (shortest distance) which is ⊥ to both the lines.

(*a*) *To find the length of the S. D.*

KL = projection of AC on KL.

(*b*) *To find the equations of the S.D.*

KL is the line of intersection of the planes AKL (thro' AB and KL) and CKL (thro' CD and KL). [Art. 46]

Another Rule to find the length of the shortest distance between two lines :

[Method of parallel plane.]

(*i*) *Find the equation of the plane through the second line and parallel to the first.*

(*ii*) *Find the perpendicular distance of one point on the first line from the plane whose equation has been found above.*

The result gives the required S. D. [Ex. 9, Art. 46]

9. **Equations of two lines in the simplest form.** For problems relating to two given lines, let the equations of the lines be

$$y = mx, \ z = c \ ; \ y = -mx, \ z = -c. \qquad \text{[Note, Art. 47]}$$

10. **Any line intersecting two given lines.** The equations of *any* line intersecting the lines $u_1=0, v_1=0$; $u_2=0, v_2=0$, are $u_1+k_1v_1=0$, $u_2+k_2v_2=0$, where k_1, k_2 are any constants. [Art. 48]

Note. The values of k_1, k_2 are found from the second condition satisfied by the line.

11. **Condition for three planes to intersect at a point.** The condition that the planes

$$a_1x+b_1y+c_1z+d_1=0, \ a_2x+b_2y+c_2z+d_2=0, \ a_3x+b_3y+c_3z+d_3=0$$

may intersect at a point, is $\Delta_4 \neq 0$, where Δ_4 is the determinant obtained by omitting the fourth column from the result of writing down the coefficients in the equations of the planes,

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{vmatrix}. \qquad \text{[Art. 51]}$$

12. Conditions for three planes to form a triangular prism. The conditions that the above planes may form a triangular prism, are

$$\Delta_4 = 0, \ \Delta_3 \neq 0. \qquad [\text{ Art. } 52 \ ]$$

13. Conditions for three planes to have a common line of intersection. The conditions that the above planes may have a common line of intersection, are $\Delta_4=0$, $\Delta_3=0$. [Art. 53]

CHAPTER VI

1. Formulae for change of origin. If (f, g, h) are the co-ordinates of the new origin referred to the old axes, the new axes being parallel to the old, (x, y, z) the co-ordinates of any point referred to the old axes, and (x', y', z') referred to the new, then

$$\mathbf{x=x'+f, \ y=y'+g, \ z=z'+h.} \qquad [\text{ Art. } 55 \ ]$$

2. Formulae for change of directions of axes. If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction-cosines of the new axes referred to the old, the origin remaining the same, (x, y, z) the co-ordinates of any point referred to the old axes, and (x', y', z') referred to the new, then

$$\begin{aligned} \mathbf{x} &= \mathbf{l_1x'+l_2y'+l_3z'}, \\ \mathbf{y} &= \mathbf{m_1x'+m_2y'+m_3z'}, \\ \mathbf{z} &= \mathbf{n_1x'+n_2y'+n_3z'}. \end{aligned} \qquad [\text{ Art. } 56 \ ]$$

3. Relations between the direction-cosines of three mutually perpendicular lines. If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular lines, then

$$\left.\begin{aligned} \mathbf{l_1^2+m_1^2+n_1^2} &= \mathbf{1}, \\ \mathbf{l_2^2+m_2^2+n_2^2} &= \mathbf{1}, \\ \mathbf{l_3^2+m_3^2+n_3^2} &= \mathbf{1}. \end{aligned}\right\} \ldots(\text{A}) \qquad \left.\begin{aligned} \mathbf{l_2l_3+m_2m_3+n_2n_3} &= \mathbf{0}, \\ \mathbf{l_3l_1+m_3m_1+n_3n_1} &= \mathbf{0}, \\ \mathbf{l_1l_2+m_1m_2+n_1n_2} &= \mathbf{0}. \end{aligned}\right\} \ldots(\text{B})$$

[Also l_1, l_2, l_3 ; m_1, m_2, m_3 ; n_1, n_2, n_3 are the direction-cosines of three mutually $\perp$ lines, $\therefore$]

$$\left.\begin{aligned} \mathbf{l_1^2+l_2^2+l_3^2} &= \mathbf{1}, \\ \mathbf{m_1^2+m_2^2+m_3^2} &= \mathbf{1}, \\ \mathbf{n_1^2+n_2^2+n_3^2} &= \mathbf{1}. \end{aligned}\right\} \ldots(\text{C}) \qquad \left.\begin{aligned} \mathbf{m_1n_1+m_2n_2+m_3n_3} &= \mathbf{0}, \\ \mathbf{n_1l_1+n_2l_2+n_3l_3} &= \mathbf{0}, \\ \mathbf{l_1m_1+l_2m_2+l_3m_3} &= \mathbf{0}. \end{aligned}\right\} \ldots(\text{D})$$

[Art. 58]

Cor. If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular lines, then

$$\begin{vmatrix} \mathbf{l_1}, & \mathbf{m_1}, & \mathbf{n_1} \\ \mathbf{l_2}, & \mathbf{m_2}, & \mathbf{n_2} \\ \mathbf{l_3}, & \mathbf{m_3}, & \mathbf{n_3} \end{vmatrix} = \pm\mathbf{1}. \qquad [\text{ Art. 58, Cor. }]$$

**4. If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular lines, then

$$\mathbf{l_1=\pm(m_2n_3-m_3n_2), \ m_1=\pm(n_2l_3-n_3l_2), \ n_1=\pm(l_2m_3-l_3m_2),}$$

and so on.

[In words : In the determinant
$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix},$$
each constituent $=\pm$ **(its co-factor).**] [Art. 59]

CHAPTER VII

1. **Central form.** The equation of the sphere, whose centre is (a, b, c) and radius r, is $\mathbf{(x-a)^2+(y-b)^2+(z-c)^2=r^2}$. [Art. 61]

Cor. Standard form. The equation of the sphere, whose centre is the origin and radius a, is $\mathbf{x^2+y^2+z^2=a^2}$. [Art. 61, Cor. 1]

2. **General form.** The general equation of a sphere is
$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0.}$$
Its centre is $(\mathbf{-u, -v, -w})$,
and radius $=\sqrt{\mathbf{u^2+v^2+w^2-d}}$. [Art. 62]

3. **Diameter form.** The equation of the sphere on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter is
$$\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0.}$$ [Art. 65]

4. **Any sphere through a given circle.** The equation of *any* sphere through the circle of intersection of
the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$,
and the plane $Ax+By+Cz+D=0$,
is $\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d+k(Ax+By+Cz+D)=0}$,
where k is any constant.

[In words : **sphere** $+$ **k (plane)=0**, where 'sphere' stands for the 'L.H.S. of the equation of the sphere (R.H.S. being zero)', and so for the 'plane'.]

Note. The value of k is found from the second condition satisfied by the sphere.

5. (*a*) **Equation of the tangent plane to a sphere (equation in the standard form).** The equation of the tangent plane at the point (x_1, y_1, z_1) of the sphere $x^2+y^2+z^2=a^2$ is $\mathbf{xx_1+yy_1+zz_1=a^2}$. [Art. 70, (*a*)]

(*b*) **Equation of the tangent plane to a sphere (equation in the general form).** The equation of the tangent plane at the point (x_1, y_1, z_1) of the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is $\mathbf{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0}$. [Art. 70, (*b*)]

[**Rule to write down the equation of the tangent plane at the point (x_1, y_1, z_1) of a sphere (or conicoid whose equation contains no* product terms yz, zx, xy).**

In the equation of the surface, change $\mathbf{x^2}$ *to* $\mathbf{xx_1}$, $\mathbf{y^2}$ *to* $\mathbf{yy_1}$, $\mathbf{z^2}$ *to* $\mathbf{zz_1}$, $\mathbf{x}$ *to* $\frac{1}{2}\mathbf{(x+x_1)}$, $\mathbf{y}$ *to* $\frac{1}{2}\mathbf{(y+y_1)}$, $\mathbf{z}$ *to* $\frac{1}{2}\mathbf{(z+z_1)}$.

The resulting equation is the required equation. [Art. 70, (*b*)]

***Note 1.** *If, however, the equation contains the product terms* yz, zx, xy, *change* $\mathbf{yz}$ *to* $\frac{1}{2}(\mathbf{yz_1+y_1z})$, $\mathbf{zx}$ *to* $\frac{1}{2}(\mathbf{zx_1+z_1x})$, $\mathbf{xy}$ *to* $\frac{1}{2}(\mathbf{xy_1+x_1y})$. (See Art. 95.)

Note 2. Important. *If the numerical values of* x_1, y_1, z_1 *are given, substitute them in the above equation.*] [Note, Art. 70, (*b*)]

6. Tangent plane property. If a plane touches a sphere, the perpendicular distance of the centre from the plane = the radius.

Note. For the tangent plane property we use the **complete** perpendicular distance formula for the plane. [Note, Art. 71, (*b*)]

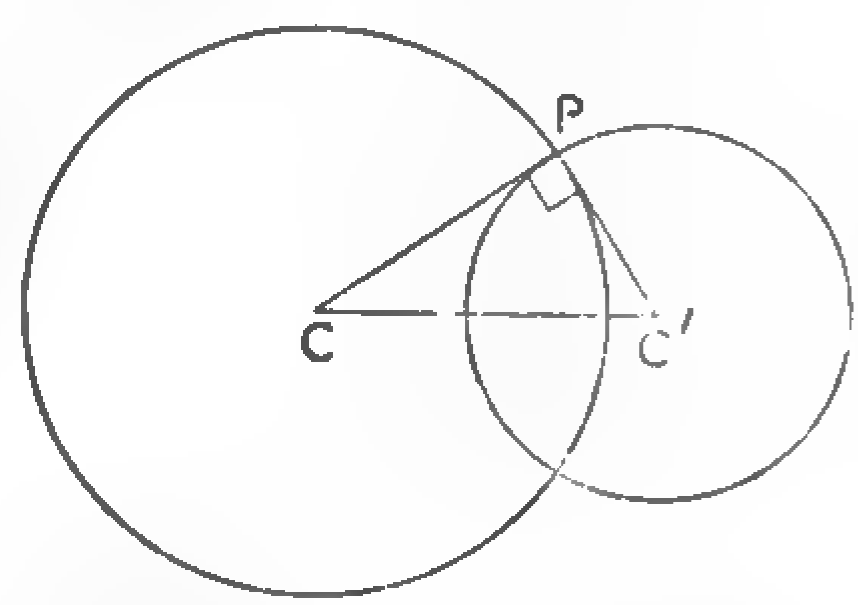
7. (*a*) **Equation of the polar plane w.r.t. a sphere (equation in the standard form).** The equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the sphere $x^2+y^2+z^2=a^2$ is

$$\mathbf{xx_1+yy_1+zz_1=a^2}. \quad \text{[Art. 74]}$$

(*b*) **Equation of the polar plane w.r.t. a sphere (equation in the general form).** The equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is
$\mathbf{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0}$. [Ex. 1, Art. 74]

[Aid to memory. The equation of the polar plane of (x_1, y_1, z_1) w. r. t. a sphere (or conicoid) is of the same *form* as the equation of the tangent plane at (x_1, y_1, z_1).]

8. If two spheres cut orthogonally, the square of the distance between their centres = the sum of the squares of their radii.
[Art. 75, Cor.]



Condition of orthogonality of two spheres. The condition that the spheres
$x^2+y^2+z^2+2ux+2vy+2wz+d=0$, $x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$
may cut orthogonally, is $\mathbf{2uu'+2vv'+2ww'=d+d'}$. [Art. 76]

9. Formula for the power of a point w.r.t. a sphere or the square of the tangent from a point to a sphere. The power of the point (x_1, y_1, z_1) w.r.t. the sphere or the square of the tangent from the point (x_1, y_1, z_1) to the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \text{ is}$$
$$\mathbf{x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d}.$$

[Rule to find the power of a point w. r. t. a sphere or the square of the tangent from a point to a sphere :

(*i*) *Write the equation of the sphere so that the coefficients of*

x^2, y^2, z^2 are each $=1$ on the L.H.S. (by dividing thro' out by the coefficient of x^2, if necessary), R.H.S. being zero.

(ii) In the L.H.S. substitute the co-ordinates of the point. The result is the power of the point or the square of the tangent from the point.]

[Art. 77 and Cor.]

10. Equation of the radical plane of two spheres. The equation of the radical plane of the spheres

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$
$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0, \text{ is}$$
$$\mathbf{2x(u-u')+2y(v-v')+2z(w-w')+d-d'=0.}$$

[Rule to find the equation of the radical plane of two spheres :

(i) Write the equation of each sphere so that the coefficients of x^2, y^2, z^2 are each $=1$ on the L.H.S. (by dividing thro' out by the coefficient of x^2, if necessary), R.H.S. being zero.

(ii) Subtract one equation from the other.

The resulting equation is the required equation.] [Art. 79]

11. Equations of two spheres in the simplest form. For problems relating to two given spheres, let the equations of the spheres be $\mathbf{x^2+y^2+z^2+2u_1x+d=0}$, $\mathbf{x^2+y^2+z^2+2u_2x+d=0}$.

[Note, Art. 82]

12. Equation of a coaxal system. The equation of a coaxal system of spheres is $\mathbf{x^2+y^2+z^2+2ux+d=0}$,
where u is a parameter. [Art. 84]

****Limiting points.** The limiting points of the coaxal system of spheres $x^2+y^2+z^2+2ux+d=0$ are $(\pm\sqrt{\mathbf{d}}, \mathbf{0}, \mathbf{0})$. [Note, Ex. 4, Art. 84]

CHAPTER VIII

1. General equation of a quadric cone whose vertex is the origin. The general equation of a quadric cone, whose vertex is the origin, is $\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$. [Art. 87, Cor.]

2. Equation of a quadric cone through the axes. The equation of a quadric cone through the axes is $\mathbf{fyz+gzx+hxy=0}$.

[Art. 89]

3. Equation of a right circular cone. Standard form. The equation of the right circular cone, whose vertex is the origin, axis the z-axis, and semivertical angle α, is $\mathbf{x^2+y^2=z^2\tan^2\alpha}$.

[Art. 91]

4. Equation of the (tangent cone or) enveloping cone. The equation of the enveloping cone from the point (x_1, y_1, z_1) to a sphere (or conicoid) is $\mathbf{SS_1=T^2}$,
where $S=0$ is the equation of the sphere (or conicoid),

S_1 is the result of substituting the co-ordinates of the point

(x_1, y_1, z_1) in S,

$T=0$ is the equation of the tangent plane at (x_1, y_1, z_1).

[Art. 94 and Ex. 1]

5. Equation of the tangent plane. The equation of the tangent plane at the point (x_1, y_1, z_1) of the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \text{ is}$$

$$\mathbf{x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0.} \quad \text{[Art. 95]}$$

[Rule to write down the equation of the tangent plane at the point (x_1, y_1, z_1) of the cone

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.}$$

See Rule in 5, (b), and Note 1, page xxv.]

6. Notation. (1) $\mathbf{D}=\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=\mathbf{abc+2fgh-af^2-bg^2-ch^2}.$

(2) A,B,C,F,G,H are the co-factors of a, b, c, f, g, h respectively in the determinant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$,

i.e., $\mathbf{A=bc-f^2}$, $B=ca-g^2$, $C=ab-h^2$;

$\mathbf{F=gh-af}$, $G=hf-bg$, $H=fg-ch$.

Cor. 1. $\mathbf{BC-F^2=aD}$, $CA-G^2=bD$, $AB-H^2=cD$.

Cor. 2. $\mathbf{GH-AF=fD}$, $HF-BG=gD$, $FG-CH=hD$,

where $D=\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=abc+2fgh-af^2-bg^2-ch^2.$

(3) $\mathbf{P^2}=\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & 0 \end{vmatrix}$

$$=\mathbf{-(Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv)},$$

where A,B,C,F,G,H are the co-factors of a, b, c, f, g, h respectively in the determinant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$,

i.e., $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$;

$F=gh-af$, $G=hf-bg$, $H=fg-ch$. [Art. 96, and Cors. 1 and 2]

7. Condition of tangency of a plane and a cone. The condition that the plane $lx+my+nz=0$ may touch the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0, \text{ is}$$

$$\mathbf{Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0,}$$

where A,B,C,F,G,H are the co-factors of a, b, c, f, g, h respectively in the determinant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$,

i.e., $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$;

$F=gh-af$, $G=hf-bg$, $H=fg-ch$. [Art. 98]

8. **Equation of the reciprocal cone.** The equation of the cone reciprocal to the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ is

$$\mathbf{A}x^2+\mathbf{B}y^2+\mathbf{C}z^2+2\mathbf{F}yz+2\mathbf{G}zx+2\mathbf{H}xy=0,$$

where A,B,C,F,G,H are the co-factors of a, b, c, f, g, h respectively in the determinant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$,

i.e., $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$;

$F=gh-af$, $G=hf-bg$, $H=fg-ch$. [Art. 100]

9. **Invariants. Practical form.** If, by a change of rectangular axes, origin remaining the same, $ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ is changed into $a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy$,

then $$\mathbf{a}+\mathbf{b}+\mathbf{c}=\mathbf{a}'+\mathbf{b}'+\mathbf{c}',$$

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=\mathbf{A}'+\mathbf{B}'+\mathbf{C}',$$

i.e., $$bc-f^2+ca-g^2+ab-h^2=b'c'-f'^2+c'a'-g'^2+a'b'-h'^2,$$

$$\mathbf{D}=\mathbf{D}',$$

i.e., $$abc+2fgh-af^2-bg^2-ch^2=a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2.$$

[Art. 101, Cor.]

10. **Condition that a cone may have three mutually perpendicular generators.** The condition that the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

may have three mutually perpendicular generators, is

$$\mathbf{a}+\mathbf{b}+\mathbf{c}=0.$$

[In words : *coefficient of x^2 + coefficient of y^2 + coefficient of $z^2=0$.*]

[Art. 102]

11. **Condition that a cone may have three mutually perpendicular tangent planes.** The condition that the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

may have three mutually perpendicular tangent planes, is [that the reciprocal cone should have three mutually perpendicular generators, *i.e.*,]

$$\mathbf{A}+\mathbf{B}+\mathbf{C}=0,$$

i.e., $$bc-f^2+ca-g^2+ab-h^2=0.$$ [Art. 103]

CHAPTER X

Equation of surface	Name	Equation of surface	Name
$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$	Ellipsoid	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$	Hyperboloid of one sheet
$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$	Hyperboloid of two sheets	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$	Cone
$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}.$	Elliptic paraboloid	$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}.$	Hyperbolic paraboloid
$y^2=4ax.$	Parabolic cylinder	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$	Elliptic cylinder
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$	Hyperbolic cylinder	[Arts. 109, 111, 112]	

[☞ **The following important results and formulae of chapters XI—XIII should be omitted by the B.A. & B.Sc. (Pass Course) students of the Panjab University.**]

CHAPTER XI

THE CENTRAL CONICOID

1. Standard form. The equation of the central conicoid, whose centre is the origin and axes are the co-ordinate axes, is

$$\mathbf{ax^2+by^2+cz^2=1.} \qquad \text{[Note 3, Art. 110]}$$

[☞ *The equation of the central conicoid, when not stated, is taken for granted as* $ax^2+by^2+cz^2=1.$]

2. Equation of the tangent plane. The equation of the tangent plane at the point (x_1, y_1, z_1) of the central conicoid is

$$\mathbf{axx_1+byy_1+czz_1=1.} \qquad \text{[Art. 115]}$$

3. Condition of tangency of a plane and a central conicoid. The condition that the plane $lx+my+nz=p$ may touch the central conicoid, is $\dfrac{\mathbf{l^2}}{\mathbf{a}}+\dfrac{\mathbf{m^2}}{\mathbf{b}}+\dfrac{\mathbf{n^2}}{\mathbf{c}}=\mathbf{p^2}.$ [Art. 116]

4. Condition of perpendicularity of a pair of lines. The condition that the two lines whose direction-cosines (l,m,n) are given by

$$ul+vm+wn=0,$$
$$al^2+bm^2+cn^2=0,$$

may be perpendicular, is

$$\mathbf{u^2(b+c)+v^2(c+a)+w^2(a+b)=0.} \qquad \text{[Note, Ex. 5, Art. 116]}$$

5. Equation of the director sphere. The equation of the director sphere of the central conicoid is $\mathbf{x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$.

[Art. 117]

[☞ *The equation of the ellipsoid, when not stated, is taken for granted as* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.]

6. Equations of the normal to an ellipsoid. The equations of the normal at the point (x_1, y_1, z_1) of the ellipsoid are

$$\frac{\mathbf{x-x_1}}{\dfrac{\mathbf{x_1}}{\mathbf{a^2}}}=\frac{\mathbf{y-y_1}}{\dfrac{\mathbf{y_1}}{\mathbf{b^2}}}=\frac{\mathbf{z-z_1}}{\dfrac{\mathbf{z_1}}{\mathbf{c^2}}},$$

i.e.,

$$\frac{\mathbf{x-x_1}}{\dfrac{\partial \mathbf{F}}{\partial \mathbf{x_1}}}=\frac{\mathbf{y-y_1}}{\dfrac{\partial \mathbf{F}}{\partial \mathbf{y_1}}}=\frac{\mathbf{z-z_1}}{\dfrac{\partial \mathbf{F}}{\partial \mathbf{z_1}}},$$

where $F(x, y, z)=0$ is the equation of the ellipsoid (or *any conicoid*).

[Art. 119]

Cor. Equations of the normal in the actual direction-cosines form. The equations of the normal at the point (x_1, y_1, z_1) of the ellipsoid, in the actual direction-cosines form, are

$$\frac{\mathbf{x-x_1}}{\dfrac{\mathbf{px_1}}{\mathbf{a^2}}}=\frac{\mathbf{y-y_1}}{\dfrac{\mathbf{py_1}}{\mathbf{b^2}}}=\frac{\mathbf{z-z_1}}{\dfrac{\mathbf{pz_1}}{\mathbf{c^2}}},$$

where p is the perpendicular from the centre of the ellipsoid on the tangent plane at (x_1, y_1, z_1). [Art. 119, Cor.]

7. Co-ordinates of the foot of a normal from a point to an ellipsoid. The co-ordinates of the foot of a normal from (α, β, γ) to the ellipsoid are

$$\mathbf{x_1=\frac{a^2\alpha}{a^2+\lambda},\ y_1=\frac{b^2\beta}{b^2+\lambda},\ z_1=\frac{c^2\gamma}{c^2+\lambda}}. \qquad \text{[Art. 120, (3)]}$$

8. Quadric cone through the feet of the six normals from a point to an ellipsoid. The feet of the six normals from (α, β, γ) to the ellipsoid lie on a cone of the second degree in whose equation, **coeff. of $\mathbf{x^2=0}$, coeff. of $\mathbf{y^2=0}$, coeff. of $\mathbf{z^2=0}$, constant term$=$0.**

[Ex. 2, Cor., Art. 120]

9. Equation of the polar plane w.r.t. a central conicoid. The equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the central conicoid is $\mathbf{axx_1+byy_1+czz_1=1}$. [Art. 121]

10. Equations of the polar of a line. The equations of the polar of the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ w.r.t. the central conicoid are

$$\mathbf{axx_1+byy_1+czz_1=1,\ axl+bym+czn=0}. \qquad \text{[Art. 123]}$$

11. Equations of the polar of a line through two points. The equations of the polar of the line through (x_1, y_1, z_1) and (x_2, y_2, z_2) w.r.t. the central conicoid are

$$axx_1+byy_1+czz_1=1,\ axx_2+byy_2+czz_2=1. \quad \text{[Ex. 6, (a), Art. 123]}$$

12. Equation of the enveloping cone. The equation of the enveloping cone from the point (x_1, y_1, z_1) to the central conicoid is

$$SS_1 = T^2,$$

where $S=0$ is the equation of the central conicoid,

S_1 is the result of substituting the co-ordinates of the point (x_1, y_1, z_1) in S,

$T=0$ is the equation of the tangent plane at (x_1, y_1, z_1). [Art. 124]

13. Equation of the cone whose vertex is the origin and generators parallel to those of a given cone. If the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d = 0$$

represents a cone, the equation of the cone whose vertex is the origin and generators parallel to those of the above cone, is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

i.e., **second degree terms = 0.** [Ex. 5, Cor., Art. 124]

14. Equation of the enveloping cylinder. The equation of the enveloping cylinder of the central conicoid, whose generators are parallel to the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$, is

$$(ax^2+by^2+cz^2-1)\,(al^2+bm^2+cn^2)=(axl+bym+czn)^2,$$

i.e., $Ss_1=t^2$,

where $S=0$ is the equation of the central conicoid,

s_1 is the result of substituting the co-ordinates of the point (l, m, n) in S, *the constant term* (-1) *being omitted,*

$t=0$ is the equation of the tangent plane at (l, m, n), *the constant term* (-1) *being omitted.* [Art. 125]

15. Equation of the plane of the section with a given centre. The equation of the plane of the section of the central conicoid, whose centre is (x_1, y_1, z_1), is $T=S_1$,

where $T=0$ is the equation of the tangent plane at (x_1, y_1, z_1),

$S=0$ is the equation of the central conicoid,

S_1 is the result of substituting the co-ordinates of the point (x_1, y_1, z_1) in S. [Art. 126]

16. Equation of the diametral plane. The equation of the diametral plane of the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ w.r.t. the central conicoid is

$$lax+mby+ncz=0,$$

i.e., $\quad \mathbf{l}\dfrac{\partial \mathbf{F}}{\partial \mathbf{x}}+\mathbf{m}\dfrac{\partial \mathbf{F}}{\partial \mathbf{y}}+\mathbf{n}\dfrac{\partial \mathbf{F}}{\partial \mathbf{z}}=\mathbf{0},$

where $F(x, y, z)=0$ is the equation of the central conicoid. [Art. 129]

17. Relations between the co-ordinates of the extremities of three conjugate semi-diameters of an ellipsoid. If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are the extremities of three conjugate semi-diameters of the ellipsoid, then

$$\frac{\mathbf{x_1}^2}{\mathbf{a}^2}+\frac{\mathbf{y_1}^2}{\mathbf{b}^2}+\frac{\mathbf{z_1}^2}{\mathbf{c}^2}=\mathbf{1}, \text{ and so on; } \frac{\mathbf{x_1x_2}}{\mathbf{a}^2}+\frac{\mathbf{y_1y_2}}{\mathbf{b}^2}+\frac{\mathbf{z_1z_2}}{\mathbf{c}^2}=\mathbf{0}, \text{ and so on...;}$$

[Art. 131, (A), (B)]

$$\mathbf{x_1}^2+\mathbf{x_2}^2+\mathbf{x_3}^2=\mathbf{a}^2, \text{ and so on ; } \mathbf{y_1z_1}+\mathbf{y_2z_2}+\mathbf{y_3z_3}=\mathbf{0}, \text{ and so on.}$$

[Art. 131, (C), (D)]

****Cor. A useful relation.**

$$\left(\frac{\mathbf{x_1}+\mathbf{x_2}+\mathbf{x_3}}{\mathbf{a}}\right)^2+\left(\frac{\mathbf{y_1}+\mathbf{y_2}+\mathbf{y_3}}{\mathbf{b}}\right)^2+\left(\frac{\mathbf{z_1}+\mathbf{z_2}+\mathbf{z_3}}{\mathbf{c}}\right)^2=\mathbf{3}.$$

[Note 2, Mis. Ex. 24, Chap. XI]

****18. Equation of the plane through the extremities of three conjugate semi-diameters.** If $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ are the extremities of three conjugate semi-diameters of the ellipsoid, then the equation of the plane PQR is

$$\mathbf{x}\left(\frac{\mathbf{x_1}+\mathbf{x_2}+\mathbf{x_3}}{\mathbf{a}^2}\right)+\mathbf{y}\left(\frac{\mathbf{y_1}+\mathbf{y_2}+\mathbf{y_3}}{\mathbf{b}^2}\right)+\mathbf{z}\left(\frac{\mathbf{z_1}+\mathbf{z_2}+\mathbf{z_3}}{\mathbf{c}^2}\right)=\mathbf{1}.$$

[Ex. 1, (*b*), Art. 131]

****19. Equation of a plane in terms of the foot of the perpendicular from the origin on it.** The equation of the plane, on which the foot of the perpendicular from the origin is $P(x_1, y_1, z_1)$ is

$$\mathbf{xx_1}+\mathbf{yy_1}+\mathbf{zz_1}=\mathbf{x_1}^2+\mathbf{y_1}^2+\mathbf{z_1}^2.$$

The plane thro' $P(x_1, y_1, z_1)$ and $\perp$ to OP (direction-cosines proportional to x_1, y_1, z_1)] [Note, Ex. 7, Art. 131]

20. Three important properties of three conjugate semi-diameters. If OP, OQ, OR are three conjugate semi-diameters of the ellipsoid, then

(1) $\mathbf{OP^2+OQ^2+OR^2=a^2+b^2+c^2}$,

(2) the **volume of the parallelepiped whose coterminous edges are OP, OQ, OR**

$$=\mathbf{abc},$$

**and (3) if A_1, A_2, A_3 are the areas of the triangles QOR, ROP, POQ, then

$$\mathbf{A_1}^2+\mathbf{A_2}^2+\mathbf{A_3}^2=\tfrac{1}{4}(\mathbf{b^2c^2+c^2a^2+a^2b^2}).$$ [Arts. 132, (*a*), (*b*), (*c*)]

****21. Conjugate semi-diameters of the ellipsoid.** If P is any point on an ellipsoid, and OQ, OR are any two conjugate semi-

diameters of the ellipse in which the diametral plane of OP meets the ellipsoid, then OP, OQ, OR are conjugate semi-diameters of the ellipsoid. [Art. 133]

THE CONE

1. Standard form. The equation of the cone, whose vertex is the origin, and axes are the co-ordinate axes, is

$$\mathbf{ax^2+by^2+cz^2=0}. \qquad [\text{ Art. } 134]$$

[☞ *The equation of the cone, when not stated, is taken for granted as* $ax^2+by^2+cz^2=0$.]

2. Equation of the tangent plane. The equation of the tangent plane at the point (x_1, y_1, z_1) of the cone is $\mathbf{axx_1+byy_1+czz_1=0}$. [Art. 135, (1)]

3. Condition of tangency of a plane and a cone. The condition that the plane $lx+my+nz=0$ may touch the cone, is

$$\frac{\mathbf{l^2}}{\mathbf{a}}+\frac{\mathbf{m^2}}{\mathbf{b}}+\frac{\mathbf{n^2}}{\mathbf{c}}=\mathbf{0}. \qquad [\text{ Art. } 135, (2)]$$

****Cor.** The condition that the plane $l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$ may touch the cone $a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$, is

$$\frac{\mathbf{l^2}}{\mathbf{a}}+\frac{\mathbf{m^2}}{\mathbf{b}}+\frac{\mathbf{n^2}}{\mathbf{c}}=\mathbf{0}. \qquad [\text{ Art. } 135, (2), \text{ Cor. }]$$

4. Equation of the polar plane. The equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the cone is $\mathbf{axx_1+byy_1+czz_1=0}$. [Art. 135, (3)]

5. Equation of the plane of the section with a given centre. The equation of the plane of the section of the cone, whose centre is (x_1, y_1, z_1), is $\mathbf{T=S_1}$,

where T=0 is the equation of the tangent plane at (x_1, y_1, z_1),

S=0 is the equation of the cone,

S_1 is the result of substituting the co-ordinates of the point (x_1, y_1, z_1) in S. [Art. 135, (5)]

6. Equation of the diametral plane. The equation of the diametral plane of the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ w.r.t. the cone is

$$\mathbf{lax+mby+ncz=0},$$

i.e., $\mathbf{l}\dfrac{\partial \mathbf{F}}{\partial \mathbf{x}}+\mathbf{m}\dfrac{\partial \mathbf{F}}{\partial \mathbf{y}}+\mathbf{n}\dfrac{\partial \mathbf{F}}{\partial \mathbf{z}}=\mathbf{0}$,

where $F(x, y, z)=0$ is the equation of the cone. [Art. 135, (6)]

7. Equation of the asymptotic cone of the central conicoid. The equation of the asymptotic cone of the central conicoid $ax^2+by^2+cz^2=1$ is $\mathbf{ax^2+by^2+cz^2=0}$. [Ex. 10, Art. 135]

THE PARABOLOID

1. Standard form. The equation of the paraboloid, whose vertex is the origin and axis the z-axis, is $\mathbf{ax^2+by^2=2z}$.

[Note 3, Art. 111, (*b*)]

[☞ *The equation of the paraboloid, when not stated, is taken for granted as* $ax^2+by^2=2z$.]

2. Equation of the tangent plane. The equation of the tangent plane at the point (x_1, y_1, z_1) of the paraboloid is $\mathbf{axx_1+byy_1=z+z_1}$.

[Art. 137, (1)]

3. Condition of tangency of a plane and a paraboloid. The condition that the plane $lx+my+nz=p$ may touch the paraboloid, is

$$\frac{\mathbf{l}^2}{\mathbf{a}}+\frac{\mathbf{m}^2}{\mathbf{b}}=-\mathbf{2np}. \qquad [\text{ Art. 137, (2) }]$$

4. Equation of the polar plane. The equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the paraboloid is

$$\mathbf{axx_1+byy_1=z+z_1}. \qquad [\text{ Art. 137, (3) }]$$

5. Equation of the enveloping cone. The equation of the enveloping cone from the point (x_1,y_1,z_1) to the paraboloid is $\mathbf{SS_1=T^2}$,
where $S=0$ is the equation of the paraboloid,

S_1 is the result of substituting the co-ordinates of the point (x_1, y_1, z_1) in S,

$T=0$ is the equation of the tangent plane at (x_1, y_1, z_1).

[Art. 137, (4)]

6. Equation of the plane of the section with a given centre. The equation of the plane of the section of the paraboloid, whose centre is (x_1, y_1, z_1), is $\mathbf{T=S_1}$,
where $T=0$ is the equation of the tangent plane at (x_1, y_1, z_1),

$S=0$ is the equation of the paraboloid,

S_1 is the result of substituting the co-ordinates of the point (x_1, y_1, z_1) in S. [Art. 137, (5)]

7. Equations of the normal to an elliptic paraboloid. The equations of the normal at the point (x_1, y_1, z_1) of the elliptic paraboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z$ are

$$\frac{\mathbf{x-x_1}}{\dfrac{\mathbf{x_1}}{\mathbf{a^2}}}=\frac{\mathbf{y-y_1}}{\dfrac{\mathbf{y_1}}{\mathbf{b^2}}}=\frac{\mathbf{z-z_1}}{-\mathbf{1}},$$

i.e.,

$$\frac{\mathbf{x-x_1}}{\dfrac{\partial \mathbf{F}}{\partial \mathbf{x_1}}}=\frac{\mathbf{y-y_1}}{\dfrac{\partial \mathbf{F}}{\partial \mathbf{y_1}}}=\frac{\mathbf{z-z_1}}{\dfrac{\partial \mathbf{F}}{\partial \mathbf{z_1}}},$$

where $F(x, y, z)=0$ is the equation of the elliptic paraboloid.

[Art. 139]

8. Equation of the diametral plane w.r.t. a paraboloid. The equation of the diametral plane of the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ w.r.t. the paraboloid $ax^2+by^2=2z$ is $\mathbf{lax+mby-n=0}$,

i.e., $$\mathbf{l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0},$$

where $F(x, y, z)=0$ is the equation of the paraboloid. [Art. 143]

Cor. Equation of any diametral plane. The equation of *any* diametral plane of the paraboloid is $\mathbf{lax+mby-n=0}$.

[Art. 143, Cor. 2]

CHAPTER XII

1. All parallel plane sections of a conicoid are similar and similarly situated conics. [Art. 145]

2. Axes of a central section of a central conicoid. The equation giving the lengths (r) of the semi-axes of the section of the central conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$, is

$$\mathbf{\frac{l^2}{a-\frac{1}{r^2}}+\frac{m^2}{b-\frac{1}{r^2}}+\frac{n^2}{c-\frac{1}{r^2}}=0} \qquad \text{[Art. 146, (5)]}$$

or $$\mathbf{\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2}[\,l^2(b+c)+m^2(c+a)+n^2(a+b)\,]}$$
$$\mathbf{+(l^2.bc+m^2.ca+n^2.ab)=0.} \qquad \text{[Art. 146, (6)]}$$

and the equations giving the direction-ratios (λ, μ, ν) of the axis of semi-length r, are

$$\mathbf{\frac{\lambda\left(a-\frac{1}{r^2}\right)}{l}=\frac{\mu\left(b-\frac{1}{r^2}\right)}{m}=\frac{\nu\left(c-\frac{1}{r^2}\right)}{n}}. \qquad \text{[Art. 146,(9)]}$$

Cor. 1. Area of a central section of an ellipsoid. The area of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $lx+my+nz=0$, is $\mathbf{\pi abc\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}}$. [Art. 146, Cor. 1]

Cor. 2. Condition for a central section of a central conicoid to be a rectangular hyperbola. The condition that the section of the central conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ may be a rectangular hyperbola, is $\mathbf{l^2(b+c)+m^2(c+a)+n^2(a+b)=0}$.

[Art. 146, Cor. 2]

3. Axes of any section of a central conicoid. The equation giving the lengths (r) of the semi-axes of the section of the central conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$, is

$$\frac{\mathbf{l}^2}{\frac{\mathbf{a}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}}+\frac{\mathbf{m}^2}{\frac{\mathbf{b}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}}+\frac{\mathbf{n}^2}{\frac{\mathbf{c}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}}=0, \quad \text{[Art. 147, (11)]}$$

or $$\frac{1}{\mathbf{r}^4}(\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)-\frac{1}{\mathbf{r}^2\mathbf{k}^2}[\,\mathbf{l}^2(\mathbf{b}+\mathbf{c})+\mathbf{m}^2(\mathbf{c}+\mathbf{a})+\mathbf{n}^2(\mathbf{a}+\mathbf{b})\,]$$
$$+\frac{1}{\mathbf{k}^4}(\mathbf{l}^2.\mathbf{bc}+\mathbf{m}^2.\mathbf{ca}+\mathbf{n}^2.\mathbf{ab})=0, \quad \text{[Art. 147, (12)]}$$

and the equations giving the direction-ratios (λ, μ, ν) of the axis of semi-length r, are

$$\frac{\lambda\left(\frac{\mathbf{a}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{l}}=\frac{\mu\left(\frac{\mathbf{b}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{m}}=\frac{\nu\left(\frac{\mathbf{c}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{n}},$$

where $\mathbf{k}^2=1-\dfrac{\mathbf{p}^2}{\mathbf{p}_0^{\,2}}$, and $\mathbf{p}_0^{\,2}=\dfrac{\mathbf{l}^2}{\mathbf{a}}+\dfrac{\mathbf{m}^2}{\mathbf{b}}+\dfrac{\mathbf{n}^2}{\mathbf{c}}$. [Art. 147, (13)]

Cor. 1. Relation between the semi-axes of a central section and a parallel section. If r_1, r_2 are the lengths of the semi-axes of the section of the central conicoid $ax^2+by^2+cz^2=1$ by a central plane $lx+my+nz=0$, and r_1', r_2' the lengths of the corresponding semi-axes of the section of the conicoid by a parallel plane $lx+my+nz=p$, then [$\mathbf{r}'=\mathbf{kr}$, *i.e.*,]

$$\mathbf{r}_1'=\mathbf{kr}_1,\ \mathbf{r}_2'=\mathbf{kr}_2,$$

where $\mathbf{k}^2=1-\dfrac{\mathbf{p}^2}{\mathbf{p}_0^{\,2}}$, and $\mathbf{p}_0^{\,2}=\dfrac{\mathbf{l}^2}{\mathbf{a}}+\dfrac{\mathbf{m}^2}{\mathbf{b}}+\dfrac{\mathbf{n}^2}{\mathbf{c}}$,

and the corresponding axes of the two sections are parallel.

[Art. 147, Cor. 1]

Cor. 2. Relation between the areas of a central section and a parallel section. If A is the area of the section of the central conicoid $ax^2+by^2+cz^2=1$ by a central plane $lx+my+nz=0$, and A′ the area of the section of the conicoid by a parallel plane $lx+my+nz=p$, then

$$\mathbf{A}'=\mathbf{k}^2\mathbf{A},$$

where $\mathbf{k}^2=1-\dfrac{\mathbf{p}^2}{\mathbf{p}_0^{\,2}}$, and $\mathbf{p}_0^{\,2}=\dfrac{\mathbf{l}^2}{\mathbf{a}}+\dfrac{\mathbf{m}^2}{\mathbf{b}}+\dfrac{\mathbf{n}^2}{\mathbf{c}}$. [Art. 147, Cor. 2]

Cor. 3. Area of any section of an ellipsoid. The area of the section of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ by the plane $lx+my+nz=p$, is

$$\pi\,\mathbf{abc}\,\frac{\sqrt{\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2}}{\sqrt{\mathbf{a}^2\mathbf{l}^2+\mathbf{b}^2\mathbf{m}^2+\mathbf{c}^2\mathbf{n}^2}}\left[1-\frac{\mathbf{p}^2}{\mathbf{a}^2\mathbf{l}^2+\mathbf{b}^2\mathbf{m}^2+\mathbf{c}^2\mathbf{n}^2}\right].$$

[Art. 147, Cor. 3]

4. Axes of any section of a paraboloid. The equation giving the lengths (r) of the semi-axes of the section of the paraboloid

$ax^2+by^2=2z$ by the plane $lx+my+nz=p$, is

$$\frac{\mathbf{l}^2}{\frac{\mathbf{a}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}}+\frac{\mathbf{m}^2}{\frac{\mathbf{b}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}}+\frac{\mathbf{n}^2}{-\frac{1}{\mathbf{r}^2}}=0,$$

or $\frac{1}{\mathbf{r}^4}(\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)-\frac{1}{\mathbf{r}^2\mathbf{k}^2}[\mathbf{l}^2(\mathbf{b})+\mathbf{m}^2(\mathbf{a})+\mathbf{n}^2(\mathbf{a}+\mathbf{b})]+\frac{1}{\mathbf{k}^4}(\mathbf{n}^2\,.\,\mathbf{ab})=0,$

and the equations giving the direction-ratios (λ, μ, ν) of the axis of semi-length r, are

$$\frac{\lambda\left(\frac{\mathbf{a}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{l}}=\frac{\mu\left(\frac{\mathbf{b}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{m}}=\frac{\nu\left(-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{n}},$$

where $\mathbf{k}^2=\frac{\mathbf{p}_0^2}{\mathbf{n}^2}$, and $\mathbf{p}_0^2=\frac{\mathbf{l}^2}{\mathbf{a}}+\frac{\mathbf{m}^2}{\mathbf{b}}+2\mathbf{np}$. [Art. 148]

5. **Area of any section of a paraboloid.** The area of the section of the paraboloid $ax^2+by^2=2z$ by the plane $lx+my+nz=p$, is

$$\frac{\pi}{\mathbf{n}^3}\frac{\sqrt{\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2}}{\sqrt{\mathbf{ab}}}\left[\frac{\mathbf{l}^2}{\mathbf{a}}+\frac{\mathbf{m}^2}{\mathbf{b}}+2\mathbf{np}\right].\qquad \text{[Art. 148, Cor. 1]}$$

6. **Condition for any section of a paraboloid to be a rectangular hyperbola.** The condition that the section of the paraboloid $ax^2+by^2=2z$ by the plane $lx+my+nz=p$ may be a rectangular hyperbola, is $\mathbf{l}^2(\mathbf{b})+\mathbf{m}^2(\mathbf{a})+\mathbf{n}^2(\mathbf{a}+\mathbf{b})=0$. [Art. 148, Cor. 2]

7. **Central circular sections of an ellipsoid.** The equations of the planes of the central circular sections of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ are}$$

$$\frac{\mathbf{x}}{\mathbf{a}}\sqrt{\mathbf{a}^2-\mathbf{b}^2}\pm\frac{\mathbf{z}}{\mathbf{c}}\sqrt{\mathbf{b}^2-\mathbf{c}^2}=0. \qquad \text{[Art. 149]}$$

Circular sections of an ellipsoid. The equations of the planes of the circular sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ are

$$\frac{\mathbf{x}}{\mathbf{a}}\sqrt{\mathbf{a}^2-\mathbf{b}^2}+\frac{\mathbf{z}}{\mathbf{c}}\sqrt{\mathbf{b}^2-\mathbf{c}^2}=\lambda,\ \frac{\mathbf{x}}{\mathbf{a}}\sqrt{\mathbf{a}^2-\mathbf{b}^2}-\frac{\mathbf{z}}{\mathbf{c}}\sqrt{\mathbf{b}^2-\mathbf{c}^2}=\mu,$$

for all values of λ and μ. [Art. 149]

8. **Co-ordinates of the umbilics of an ellipsoid.** The co-ordinates of the umbilics of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ are given by

$$\frac{\mathbf{x}}{\mathbf{a}}=\pm\frac{\sqrt{\mathbf{a}^2-\mathbf{b}^2}}{\sqrt{\mathbf{a}^2-\mathbf{c}^2}},\ \mathbf{y}=0,\ \frac{\mathbf{z}}{\mathbf{c}}=\pm\frac{\sqrt{\mathbf{b}^2-\mathbf{c}^2}}{\sqrt{\mathbf{a}^2-\mathbf{c}^2}}.\ (\mathbf{a}^2>\mathbf{b}^2>\mathbf{c}^2) \qquad \text{[Art. 154]}$$

CHAPTER XIII

[☞ *The equation of the hyperboloid of one sheet, when not stated, is taken for granted as* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.]

1. Equations of any generators of the λ-, and μ-systems. The equations of any generator of the λ-system, of the hyperboloid of one sheet, are

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\ \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right);$$

and the equations of any generator of the μ-system are

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right).$$ [Art. 156]

2. Equations of any generator. The equations of *any* generator of one system, of the hyperboloid of one sheet, [through ($a\cos\alpha$, $b\sin\alpha$, 0)], are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c},$$

and the equations of *any* generator of the other system are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{-c}.$$ [Ex. 4, Cor., Art. 158]

The equations of *any* generator of the λ-system [through ($a\cos\alpha$, $b\sin\alpha$, 0)] are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{-c},$$

and the equations of *any* generator of the μ-system [through ($a\cos\alpha$, $b\sin\alpha$, 0)] are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}.$$

3. (*a*) No two generators of the same system, of a hyperboloid of one sheet, intersect. [Art. 159, (*a*)]

(*b*) Any generator of the λ-system, of a hyperboloid of one sheet, intersects any generator of the μ-system. [Art. 159, (*b*)]

4. (*a*) **Section of a hyperboloid of one sheet by the tangent plane at a point.** The section of a hyperboloid of one sheet by the tangent plane at a point is the two generating lines through the point. [Art. 160, (*a*)]

(*b*) **Any plane through a generating line.** Any plane through a generating line of a hyperboloid of one sheet is a tangent plane. [Art. 160, (*b*)]

5. Locus of the points of intersection of perpendicular generators. The locus of the points of intersection of perpendicular generators of the hyperboloid of one sheet is the curve of intersection of the hyperboloid and its director sphere $x^2+y^2+z^2=a^2+b^2-c^2$. [Art. 161]

6. Co-ordinates of any point "θ, ϕ" on a hyperboloid of one sheet. The co-ordinates of any point on the hyperboloid of one sheet are $(\mathbf{a} \cos \theta \sec \phi, \mathbf{b} \sin \theta \sec \phi, \mathbf{c} \tan \phi)$. [Art. 162]

Cor. For all points "θ, ϕ" on a generator of the λ-system, of the hyperboloid of one sheet, $\theta-\phi$ **is constant,** and for all points on a generator of the μ-system, $\theta+\phi$ **is constant.** [Art. 162, Cor.]

7. Co-ordinates of the point of intersection of two generators of opposite systems through two points on the principal elliptic section. If α, β are the eccentric angles of two points P, Q on the principal elliptic section of the hyperboloid of one sheet, then the co-ordinates of R, the point of intersection of two generators of opposite systems, one drawn through P and the other through Q, are

$$\mathbf{x}=\mathbf{a} \cos \frac{\beta+\alpha}{2} \sec \frac{\beta-\alpha}{2}, \quad \mathbf{y}=\mathbf{b} \sin \frac{\beta+\alpha}{2} \sec \frac{\beta-\alpha}{2},$$

$$\mathbf{z}=\pm \mathbf{c} \tan \frac{\beta-\alpha}{2}. \; (\beta > \alpha)$$ [Note, Art. 163, (*b*)]

[☞ *The equation of the hyperbolic paraboloid, when not stated, is taken for granted as* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$.]

8. Equations of any generators of the λ- and μ-systems. The equations of any generator of the λ-system, of the hyperbolic paraboloid, are

$$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=\frac{\mathbf{z}}{\lambda}, \quad \frac{\mathbf{x}}{\mathbf{a}}-\frac{\mathbf{y}}{\mathbf{b}}=2\lambda;$$

and the equations of any generator of the μ-system are

$$\frac{\mathbf{x}}{\mathbf{a}}-\frac{\mathbf{y}}{\mathbf{b}}=\frac{\mathbf{z}}{\mu}, \quad \frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}=2\mu.$$ [Note, Art. 165]

9. (*a*) No two generators of the same system, of a hyperbolic paraboloid, intersect. [Art. 167, (1), (*a*)]

(*b*) Any generator of the λ-system, of a hyperbolic paraboloid, intersects any generator of the μ-system. [Art. 167, (1), (*b*)]

10. (*a*) **Section of a hyperbolic paraboloid by the tangent plane at a point.** The section of a hyperbolic paraboloid by the tangent plane at a point is the two generating lines through the point. [Art. 167, (2), (*a*)]

(*b*) **Any plane through a generating line.** Any plane through a generating line of a hyperbolic paraboloid is a tangent plane. [Art. 167, (2), (*b*)]

11. Angle between a pair of planes. If θ is the angle between the pair of planes $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, then

$$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-bc-ca-ab}}{a+b+c}. \qquad \text{[Art. 35, (b)]}$$

[Aid to memory. From Analytical Plane Geometry, if θ is the angle between the pair of lines $ax^2+2hxy+by^2=0$, then

$$\tan\theta = \frac{2\sqrt{h^2-ab}}{a+b}.$$

Here instead of h^2, we have $f^2+g^2+h^2$, and instead of ab, we have $ab+bc+ca$, and instead of $a+b$, we have $a+b+c$.]

****12. Co-ordinates of any point "r, θ" on a hyperbolic paraboloid.** The co-ordinates of any point on the hyperbolic paraboloid are **($ar\cos\theta$, $br\sin\theta$, $\frac{1}{2}r^2\cos 2\theta$).** [Art. 167, (4)]

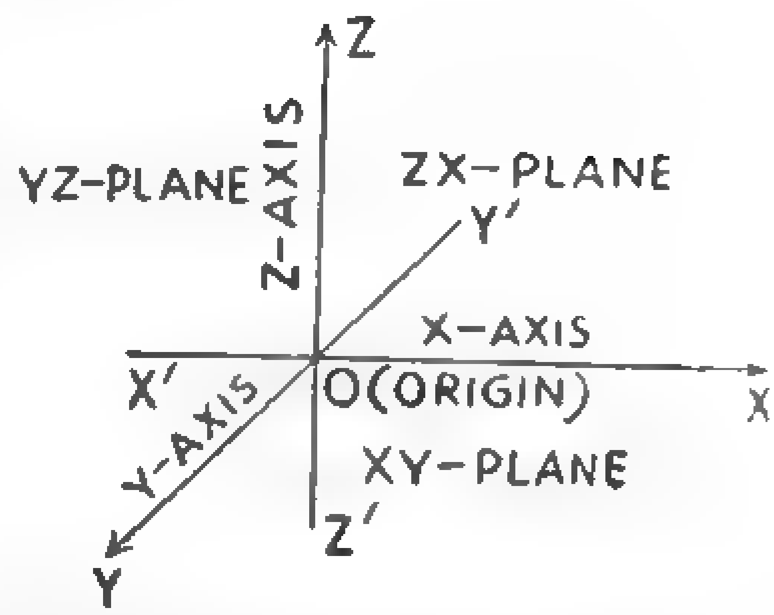
CHAPTER I

THE POINT

SECTION I

CO-ORDINATES

1. (*a*) **Origin. Def.** *Let X′OX, Y′OY, Z′OZ be three fixed mutually perpendicular straight lines whose positive directions are X′OX, Y′OY, Z′OZ as indicated by arrow-heads in the Fig., and which intersect in a point O.



Then O is called the **origin.**

(*b*) **Co-ordinate axes. Def.**

X′OX is called the *x-axis* [or, more fully, the *axis of x*], Y′OY the *y-axis,* Z′OZ the *z-axis,* and the three together, taken in this particular order, are called the **co-ordinate axes** or, more shortly, the **axes.**

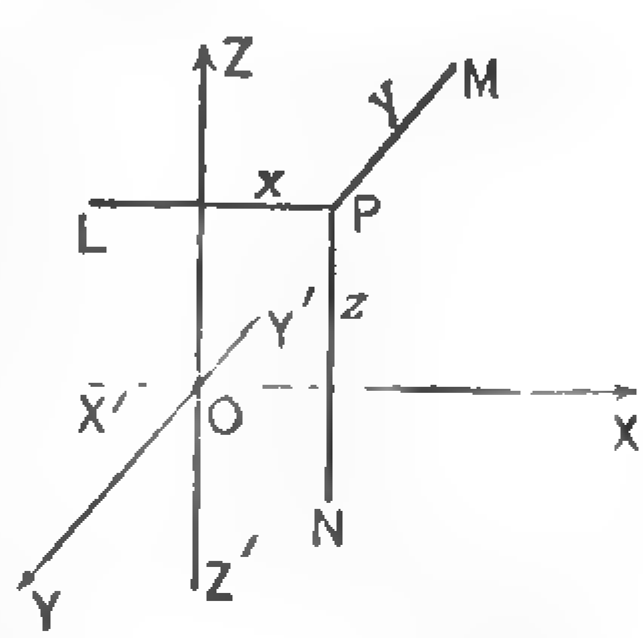
Note. (*i*) **Rectangular axes.** Since the axes are mutually *at right angles,* they are also called **rectangular axes.**

(*ii*) **Oblique axes.** If the axes are not mutually at right angles, they are called **oblique axes.**

In this book we shall deal with rectangular axes only.

(*c*) **Co-ordinate planes. Def.** The plane YOZ is called the *yz-plane,* the plane ZOX the *zx-plane,* the plane XOY the *xy-plane,* and the three together, taken in this particular order, are called the **co-ordinate planes.**

2. (*a*) **Co-ordinates. Def.** Let P be any point in space. From P draw PL, PM, PN perpendiculars respectively on the *yz-, zx-, xy-*planes. Then LP is called the *x-coordinate* of P, MP the *y-coordinate,* NP the *z-coordinate,* and the three together, taken in this particular order, are called the **co-ordinates** of P [or, more fully, the **Cartesian co-ordinates** of P].



Thus **the co-ordinates of a point are**

---

* **How to draw the Fig.** Draw two perpendicular straight lines X′OX, Z′OZ, and through their point of intersection, O, draw a third straight line Y′OY perpendicular to the plane in which X′OX and Z′OZ lie.

How to imagine the Fig. Take O, a corner of the floor of a room, looked at by the reader, X′OX the edge of the floor running towards his right, **Y′OY** the **edge** running towards him and Z′OZ the edge running towards the ceiling.

its perpendicular distances *from* **the three co-ordinate planes.**

(*b*) **Convention for signs of co-ordinates.**

(*i*) *LP, or the x-coordinate, is considered positive if it is measured in the direction OX* [*as in the Fig. of Art.* 2, (*a*)], *and negative if measured in the opposite direction OX′.* [Elementary Geometry]

(*ii*) *MP, or the y-coordinate, is considered positive if it is measured in the direction OY* [*as in the Fig. of Art.* 2, (*a*)], *and negative if measured in the opposite direction OY′.*

(*iii*) *NP, or the z-coordinate, is considered positive if it is measured in the direction OZ* [*as in the Fig. of Art.* 2, (*a*)], *and negative if measured in the opposite direction OZ′.*

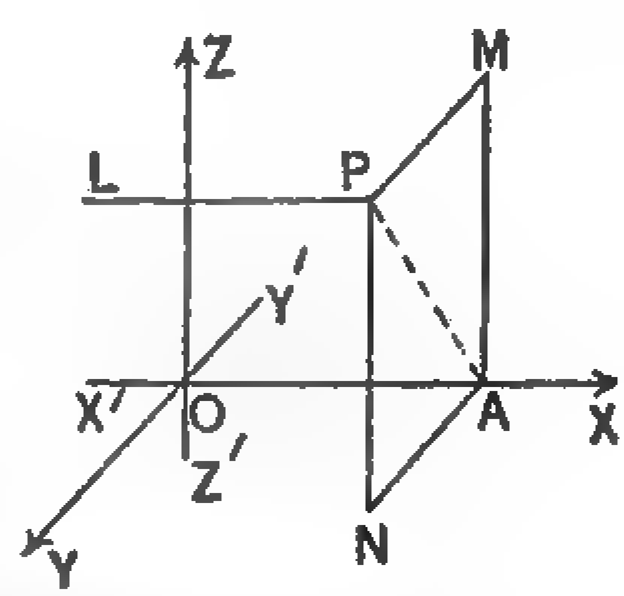
Point (x, y, z). Def. The point whose co-ordinates are (x, y, z) is called the **point (x, y, z)** or, more shortly, **(x, y, z).**

Thus the **origin is (0, 0, 0).**

(*c*) **Octants.** The three co-ordinate planes divide space into $2\times2\times2=8$ parts called **octants.** The octant OXYZ in which the three co-ordinates are all positive is called the **first octant.** There is no established order in numbering the other octants.

Two important facts about co-ordinates.

3. (*a*) *If A is the foot of the perpendicular from P(x, y, z) on the x-axis, then* **OA=x.**



Proof. From P draw PL, PM, PN $\perp$s respectively on the *yz*-, *zx*-, *xy*- planes.

Let the plane MPN which is $\parallel$ to the plane YOZ [$\because$ MP is $\parallel$ to OY, NP $\parallel$ to OZ] meet the *x*-axis in A.

Then OA is $\perp$ to the plane MPN

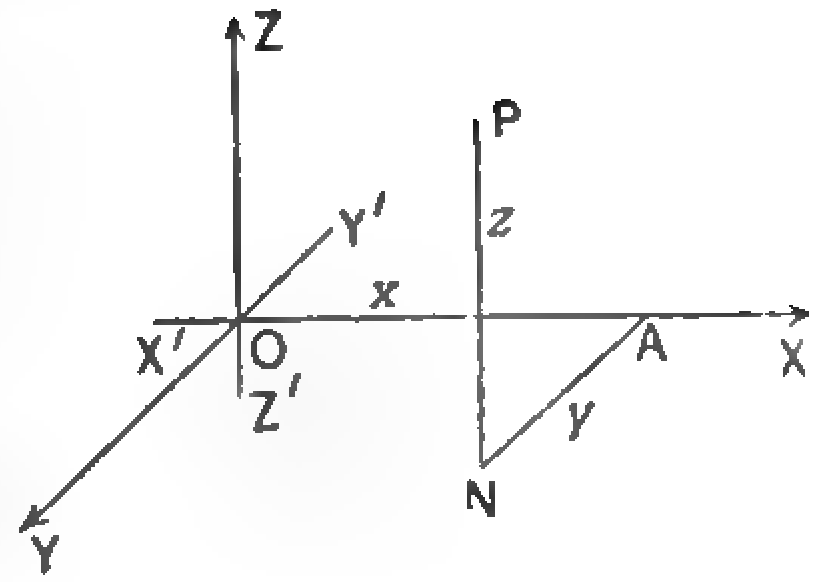
[$\because$ OA is $\perp$ to plane YOZ and plane YOZ is $\parallel$ to plane MPN]

$\therefore$ OA is $\perp$ to AP which meets it in that plane, or PA is $\perp$ to OA, *i.e.*, A is the foot of the $\perp$ from P on the *x*-axis.

Also OA=LP [Distance between two $\parallel$ planes YOZ and MPN]
$=x.$

Note. *Similarly if B is the foot of the perpendicular from P(x, y, z) on the y-axis, then* **OB=y,**
and if C is the foot of the perpendicular from P(x, y, z) on the z-axis, then **OC = z.**

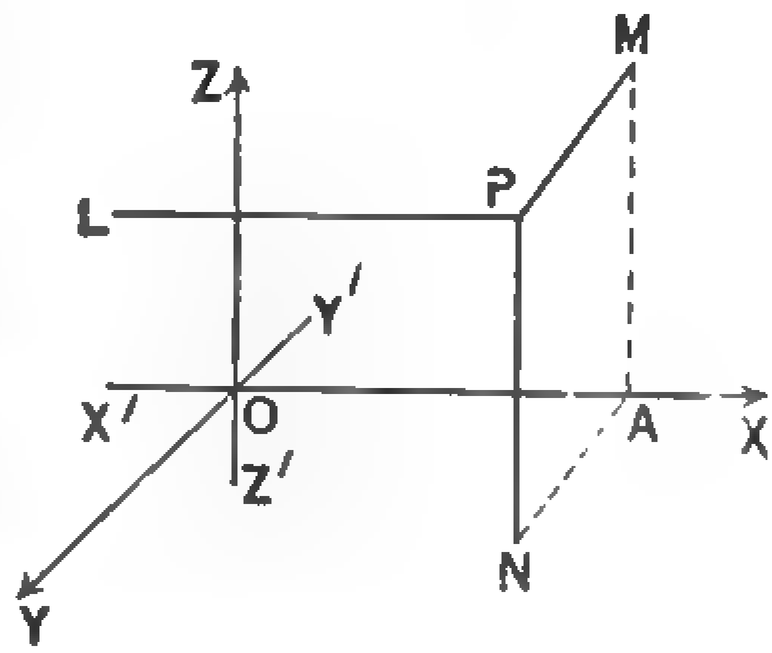
Cor. 1. Rule to find the co-ordinates of a point P :



From P draw PN ⊥ on the xy-plane ; from N, the foot of this ⊥, draw NA ⊥ on the x-axis (i.e., ∥ to the y-axis).

Then if OA=x, AN=y, NP=z [observing the convention for signs of co-ordinates (Art. 2, (b))], (x, y, z) are the required co-ordinates of P.

[Proof.** From P draw PL, PM, PN ⊥s respectively on the *yz-, zx-, xy-*planes.



Let the plane MPN meet the *x*-axis in A.

Then OA=*x* (Proved in Art. 3, (*a*)).

Also OA is ⊥ to the plane MPN (Proved in Art. 3, (*a*)).

∴ OA is ⊥ to AN which meets it in that plane, or NA is ⊥ to OA.

Again the two ∥ planes MPL, XOY are cut by the plane MPNA in the lines of section MP, AN,

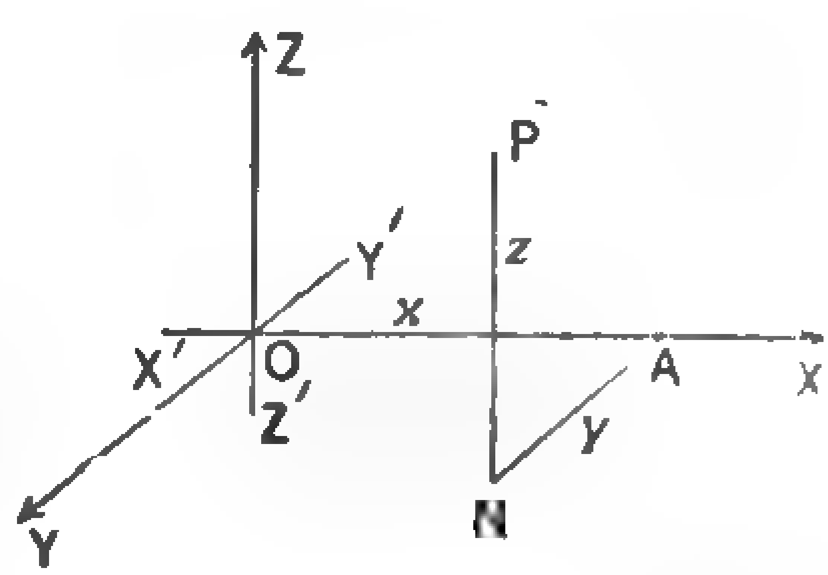
∴ MP is ∥ to AN. (Elementary Geometry)

Similarly NP is ∥ to AM.

∴ MPNA is a ∥gm., ∴ AN=MP=*y*.

Also NP=*z*.]

Cor. 2. Rule to plot the point (x, y, z) :



Measure OA along the x-axis and =x ; in the xy-plane, measure AN ⊥ to the x-axis (i.e., ∥ to the y-axis) and =y, measure NP ⊥ to the xy-plane and =z [observing the convention for signs of co-ordinates (Art. 2, (b))]. Then P is the required pt.

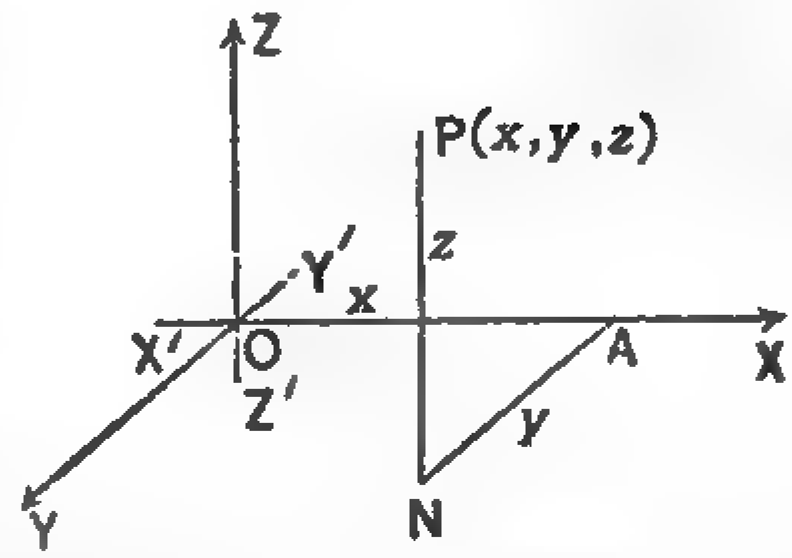
EXAMPLES

1. In the Fig. of Cor. 1, Art. 3, (*a*), write down the co-ordinates of the points N and A.

2. Sketch in a figure the positions of the points :

(1, 0, 2), (−1, −2, 3), (−2, −3, 0), (0, 0, −1).

3. (*b*) *If N is the foot of the perpendicular from P(x, y, z) on the xy-plane, then the co-ordinates of N,* **in the xy-plane,** *are* **(x, y).**



Proof. From N draw NA ⊥ on the x-axis (*i.e.*, ∥ to the y-axis). Then $OA=x$, $AN=y$. [Art. 3, (*a*), Cor. 1]

∴ by Analytical Plane Geometry, the co-ordinates of N, *in the xy-plane,* (referred to X'OX, Y'OY as axes) are (x, y).

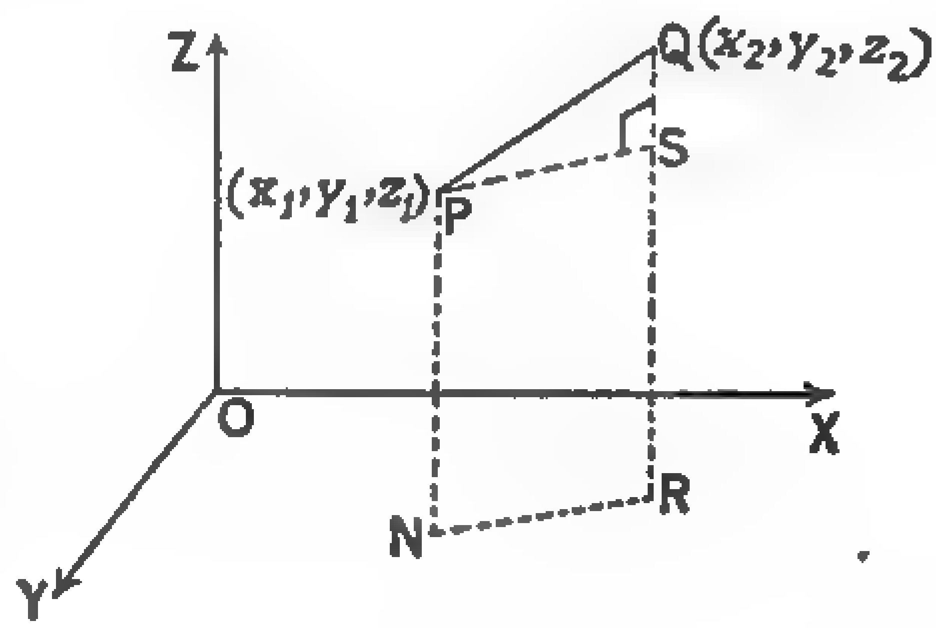
Note. *Similarly if L is the foot of the perpendicular from P(x,y,z) on the yz-plane, then the co-ordinates of L,* **in the yz-plane,** *are* **(y, z),** *and if M is the foot of the perpendicular from P(x, y, z) on the zx-plane, then the co-ordinates of M,* **in the zx-plane,** *are* **(z, x).**

SECTION II

DISTANCE BETWEEN TWO POINTS

4. **Distance formula.** *To find the distance between two points whose co-ordinates are given.*

Let P, Q be the pts. and (x_1, y_1, z_1), (x_2, y_2, z_2) their co-ordinates.



Let d be the required distance PQ.

Let N, R be the feet of the ⊥s from P, Q on the xy-plane, so that the co-ordinates of N, R, *in the xy-plane,* are (x_1, y_1), (x_2, y_2). [Art. 3, (*b*)]

∴ by Analytical Plane Geometry, $NR=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$. [Distance formula]

Thro' P draw PS ∥ to NR to meet QR in S.

[∵ PN, QR, being ⊥s on the xy-plane, are ∥ and ∴ lie in the same plane, ∴ a line thro' P ∥ to NR also lies in that plane and ∴ meets QR]

Then from the rt. ∠d △ PSQ,

$$PQ^2 = PS^2+SQ^2 \quad \dots (1)$$

But $PS=NR=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$, [Proved above]

$SQ=RQ-RS=RQ-NP=z_2-z_1$,

∴ from (1), $d^2 = (x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$

or $$\mathbf{d} = \sqrt{(\mathbf{x}_2-\mathbf{x}_1)^2+(\mathbf{y}_2-\mathbf{y}_1)^2+(\mathbf{z}_2-\mathbf{z}_1)^2}.$$

****Note 1. Complete distance formula.**

$\because\ d^2=(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$

$\therefore\ d=\pm\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$,

that sign being taken on the R.H.S.† which gives a positive* result for d.

Note 2. The student should notice the close analogy between the distance formula $[d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}\,]$ in Analytical Plane Geometry and that $[d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\,]$ in Analytical Solid Geometry. He will constantly come across examples of this kind. **He should make use of this analogy as an aid to memory for standard results and formulae in Analytical Solid Geometry.**

Cor. Distance of a point from the origin. *The distance of the point* (x_1, y_1, z_1) *from the origin is* $\sqrt{x_1^2+y_1^2+z_1^2}$.

Proof. Let d be the required distance. (1) $(0, 0, 0)$ (2) (x_1, y_1, z_1)

Then $d=\sqrt{(x_1-0)^2+(y_1-0)^2+(z_1-0)^2}$

$[\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ (Art. 4)]

$=\sqrt{x_1^2+y_1^2+z_1^2}$.

EXAMPLES

1. (*a*) *Find the distance between* $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$, *the axes being rectangular.*

(*b*) *Find the distance between the points* $(1, 2, -3)$ *and* $(3, -2, 1)$.

2. A, B are the points $(1, 2, 3)$ and $(-1, 4, -3)$ and P a variable point whose co-ordinates are (x, y, z). Find the equations satisfied by x, y, z, if (*i*) PA=PB, (*ii*) $PA^2+PB^2=2k^2$, (*iii*) $PA^2-PB^2=2k^2$.

3. *Find the locus of points which are equidistant from the points* $(-1, 2, 3)$ *and* $(3, 2, 1)$.

SECTION III

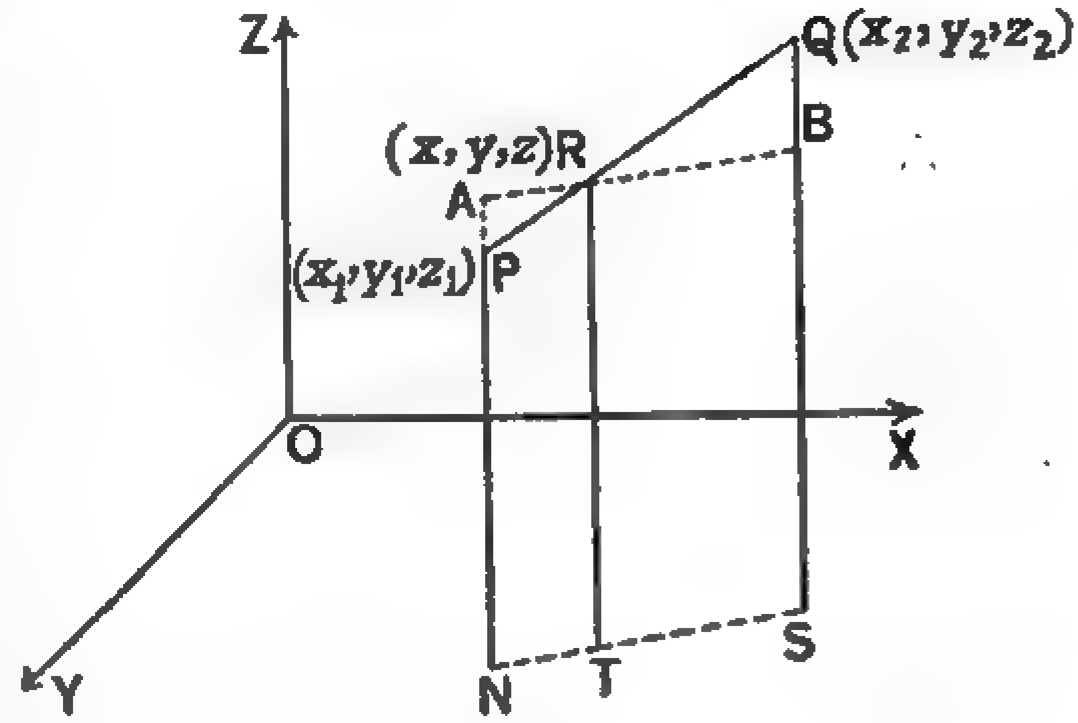
CO-ORDINATES OF THE POINT DIVIDING THE JOIN OF TWO GIVEN POINTS IN A GIVEN RATIO

5. Section formulae. *To find the co-ordinates of the point which divides the straight line joining two given points in a given ratio.*

---

*Except when PQ is parallel to one of the co-ordinate axes, there is no convention for signs with regard to the direction which is to be considered positive. We have, however, considered the direction PQ to be positive.

†The letters 'R.H.S.' stand, as usual, for 'Right Hand Side'.



Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ be the given pts.

Let (x, y, z) be the required co-ordinates of R, the pt. which divides PQ in the given ratio $m_1 : m_2$.

From P, Q, R draw PN QS, RT $\perp$s on the xy-plane.

Thro' R draw a st. line ARB $\parallel$ to NTS to meet NP produced and SQ in A and B.

[$\because$ PN, QS, RT, being $\perp$s on the xy-plane, are $\parallel$, and, being cut by the same st. line PRQ, lie in the same plane,

(Elementary Geometry)

$\therefore$ a line thro' R $\parallel$ to NTS also lies in that plane and $\therefore$ meets NP produced and SQ]

Then from the similar $\triangle$s APR, BQR,

$$\frac{PA}{BQ} = \frac{PR}{RQ} = \frac{m_1}{m_2} \quad ...(1)$$

But $\quad PA = NA-NP = TR-NP = z-z_1,$

$\quad\quad BQ = SQ-SB = SQ-TR = z_2-z,$

$\therefore$ from (1),

$$\frac{z-z_1}{z_2-z} = \frac{m_1}{m_2}, \text{ or } m_2z-m_2z_1 = m_1z_2-m_1z,$$

or $\quad z(m_1+m_2) = m_1z_2+m_2z_1$

$$\therefore \quad \mathbf{z} = \frac{\mathbf{m_1z_2+m_2z_1}}{\mathbf{m_1+m_2}}.$$

Similarly $\mathbf{x} = \dfrac{\mathbf{m_1x_2+m_2x_1}}{\mathbf{m_1+m_2}}$, $\mathbf{y} = \dfrac{\mathbf{m_1y_2+m_2y_1}}{\mathbf{m_1+m_2}}$.

Note. The section formulae give the co-ordinates of the point of division, for all real values of $m_1 : m_2$.

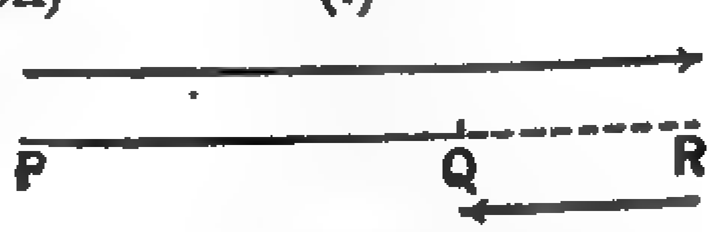
If $m_1 : m_2$ is +ve, $\dfrac{PR}{RQ}\left(=\dfrac{m_1}{m_2}\right)$ is +ve.

(*i*)

$\therefore$ PR, RQ have the same sign (direction)

$\therefore$ R divides PQ internally as in Fig. (*i*).

Similarly if $m_1 : m_2$ is −ve, R divides PQ *externally* as in Fig. (*ii*).



(*ii*)

[**Rule to find the co-ordinates of the point which divides the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $m_1 : m_2$.**

Multiply m_1 by the x-coordinate of the point remote from m_1, and m_2 by the x-coordinate of the point remote from m_2

$$\underbrace{(x_1, y_1, z_1)\ m_1 \quad : \quad m_2\ (x_2, y_2, z_2)}$$

as shown by the arrows, add these products, and divide the result by m_1+m_2.

This gives the x-coordinate of the point of division.

Similarly for the y- and z-coordinates.

Cor. Mid-point formulae. *The co-ordinates of the mid-point of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) are*

$$\mathbf{x}=\frac{\mathbf{x}_1+\mathbf{x}_2}{2}, \quad \mathbf{y}=\frac{\mathbf{y}_1+\mathbf{y}_2}{2}, \quad \mathbf{z}=\frac{\mathbf{z}_1+\mathbf{z}_2}{2}.$$

Proof. Let P, Q be the pts. (x_1, y_1, z_1), (x_2, y_2, z_2), and R the mid-pt. of PQ.

$$\underbrace{\underset{P}{(x_1, y_1, z_1)} \quad 1 \underset{R}{:} 1 \quad \underset{Q}{(x_2, y_2, z_2)}}$$

Now PR=RQ, $\therefore \ \dfrac{PR}{RQ}=\dfrac{1}{1}$

$\therefore$ the co-ordinates of R are

$$x=\frac{1(x_2)+1(x_1)}{1+1} \qquad \text{[Rule (Art. 5)]}$$

$$=\frac{x_1+x_2}{2}.$$

Similarly $y=\dfrac{y_1+y_2}{2}$, $z=\dfrac{z_1+z_2}{2}$.

EXAMPLES

1. (*a*) *Find the co-ordinates of the point which divides the join of $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ in a given ratio $\lambda : 1$.*

(*b*) Find the co-ordinates of the points which divide the join of (1, −2, 3), (3, 4, −5) in the ratios 1 : 3, −1 : 3, 2 : −3.

2. Converse problem. *Given that P(3, 2, −4), Q(5, 4, −6), R(9, 8, −10) are collinear, find the ratio in which Q divides PR.*

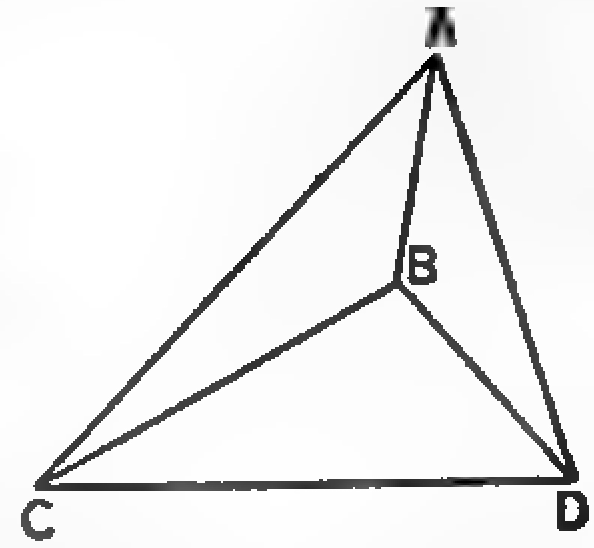
[*P(P). U. B. Sc. 1956 S*]

3. *Centroid formulae (or centre of gravity) formulae for the triangle.* **Prove that the co-ordinates of the centroid (or the centre of gravity) of the triangle, whose vertices are**

$$(\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1), \quad (\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2), \quad (\mathbf{x}_3, \mathbf{y}_3, \mathbf{z}_3),$$

are

$$\mathbf{x}=\frac{\mathbf{x}_1+\mathbf{x}_2+\mathbf{x}_3}{3}, \quad \mathbf{y}=\frac{\mathbf{y}_1+\mathbf{y}_2+\mathbf{y}_3}{3}, \quad \mathbf{z}=\frac{\mathbf{z}_1+\mathbf{z}_2+\mathbf{z}_3}{3}.$$



[**Note on tetrahedron.** (*i*) **Def.** A **tetrahedron*** is a solid bounded by four triangular faces.

Thus ABCD† is a tetrahedron bounded by four triangular faces, the $\triangle$s ABC, ACD, ADB, BCD.

(*ii*) **Vertices.** It has four vertices, the points A, B, C, D.

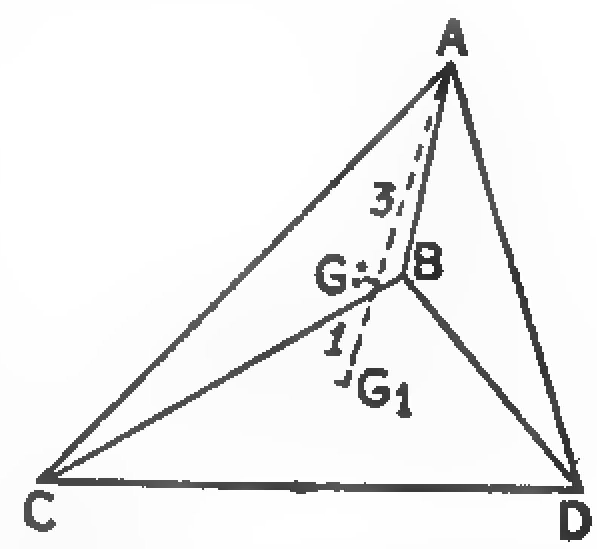
(*iii*) **Edges.** It has six edges, the lines AB, AC, AD ; BC, CD, DB.

Opposite edges. It has *three pairs of opposite edges, i.e.,* edges which do not meet, AB and CD, BC and AD, CA and BD.]

4. *Centre of gravity formulae for the tetrahedron.*

Prove that the co-ordinates of the centre of gravity of the tetrahedron whose vertices are (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4), are

$$x=\frac{x_1+x_2+x_3+x_4}{4},\quad y=\frac{y_1+y_2+y_3+y_4}{4},\quad z=\frac{z_1+z_2+z_3+z_4}{4}.$$



[**Centre of gravity of a tetrahedron.** *If ABCD is a tetrahedron, G_1 the centre of gravity of the face $\triangle$ BCD, then G, the centre of gravity of the tetrahedron, divides AG_1 in the ratio 3 : 1.* (*From Statics*)]

5. ABCD is a tetrahedron, and A′, B′, C′, D′ the centroids of the triangles BCD, CDA, DAB, ABC. Prove that—

(*i*) AA′, BB′, CC′, DD′ divide one another in the ratio 3 : 1.

(*ii*) The centres of gravity of the tetrahedra‡ ABCD, A′B′C′D′ coincide.

6. **Show that the three lines joining the middle points of opposite edges of a tetrahedron meet in a point.** [*L. U.*]

Show also that this point is on the line joining any vertex to the centre of gravity of the opposite face, and divides that line in the ratio 3 : 1.

---

* Greek, *tetra*, four ; *hedra* a base.

† How to draw the Fig Take a $\triangle$ BCD, and a point A outside its plane. Join AB, AC, AD

‡ *Plural* of tetrahedron.

MISCELLANEOUS EXAMPLES ON CHAPTER I

1. Prove by distances that the three points $(1, 2, 3)$, $(-2, 3, 4)$, $(7, 0, 1)$ are collinear.

2. Show that the points $(0, 1, 2)$, $(2, -1, 3)$ and $(1, -3, 1)$ are the vertices of an isosceles right-angled triangle.

3. Show that the four points $(1, 1, 1)$, $(-2, 4, 1)$, $(-1, 5, 5)$, $(2, 2, 5)$ are the vertices of a square.

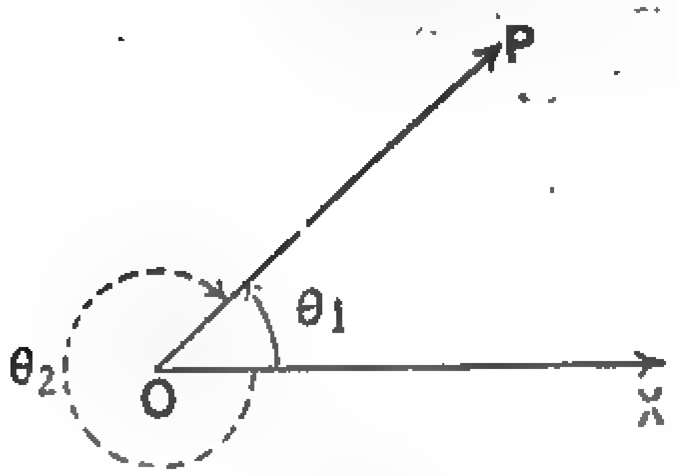
CHAPTER II

DIRECTION-COSINES. PROJECTION ON A LINE

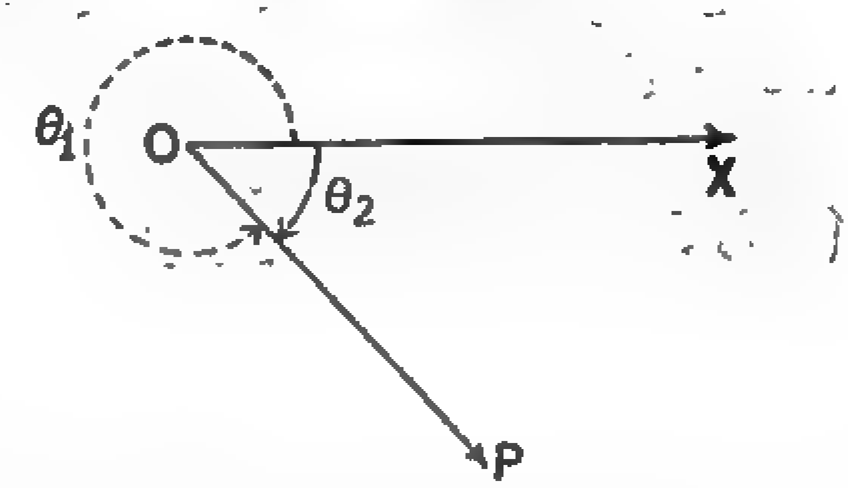
SECTION I

DIRECTION-COSINES

6. (*a*) **Angle between two intersecting lines. Def.**



(*i*)



(*ii*)

The angle which a directed line OP* makes with another directed line OX is the *smallest* angle traced out by a revolving line revolving about O in the plane XOP from the position OX to the position OP.

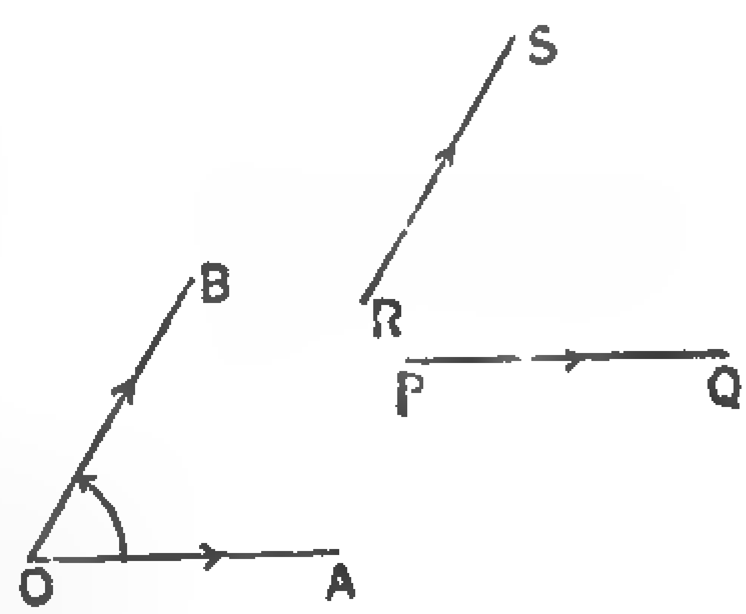
The *sign of the angle* is determined by the usual convention adopted in Trigonometry.

Thus in Fig. (*i*), θ_1 is the angle traced out in the positive (*i.e.*, counter-clockwise) direction and θ_2 the angle traced out in the negative (*i.e.*, clock-wise direction), and θ_1 is numerically $<\theta_2$,

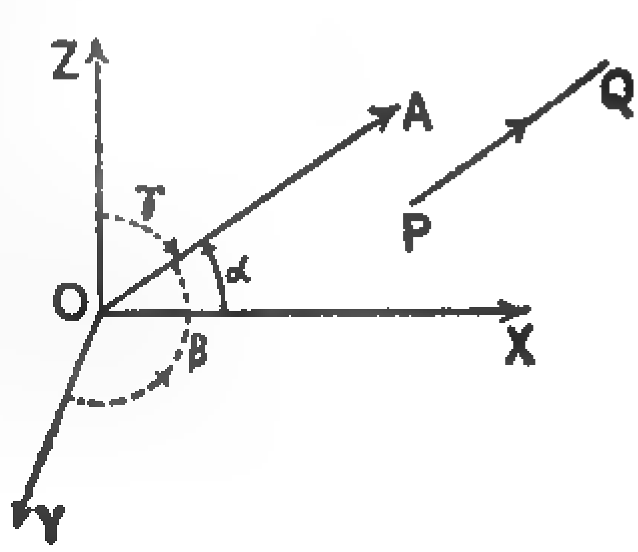
$\therefore$ θ_1 is the angle which OP makes with OX, and it is positive.

Similarly in Fig. (*ii*), θ_2 is the angle which OP makes with OX, and it is negative.

(*b*) **Angle between two non-intersecting** (*i.e.*, **non-coplanar**) **lines. Def.** *If PQ, RS are two directed non-intersecting lines, then the angle which RS makes with PQ is the angle which the directed line OB (‖ in the same sense to RS) makes with the directed line OA (‖ in the same sense to PQ), i.e., $\angle AOB$.*



Thus the angles which a directed line PQ makes with the positive directions of the axes are the angles α, β, γ which the directed line OA (‖ in the same sense to PQ) makes with the positive directions of the axes.



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*I.e., described in the sense from O to P.

7. Direction-cosines. Def. *If α, β, γ are the angles which a directed line makes with the positive directions of the axes, then $\cos\alpha$, $\cos\beta$, $\cos\gamma$ are called the* **direction-cosines** *of the line.*

Note. The expression "the angles which a line makes with the positive directions of the axes" is generally shortened into "the angles which a line makes with the axes".

Notation. The direction-cosines of a line are usually denoted by the letters l, m, n.

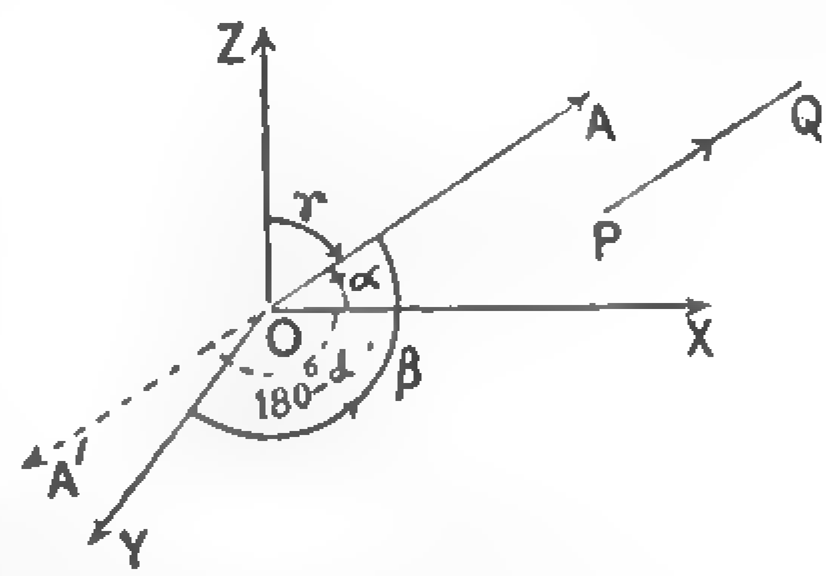
Cor. 1. Direction-cosines of the axes. *The direction-cosines of the x-axis are* **1, 0, 0.**

Proof. The x-axis makes with the axes, angles $0°, 90°, 90°$.

$\therefore$ its direction-cosines are $\cos 0°$, $\cos 90°$, $\cos 90°$, *i.e.*, **1, 0, 0.**

Note. *Similarly the direction-cosines of the y-axis are* **0, 1, 0,**
and the direction-cosines of the z-axis are **0, 0, 1.**

Cor. 2. *If the direction-cosines of a line PQ are l, m, n, then the direction-cosines of QP are $-l, -m, -n$.*



Proof. Let PQ make $\angle$s α, β, γ with the axes, so that $\cos\alpha = l$, $\cos\beta = m$, $\cos\gamma = n$. [Def. (Art. 7)]

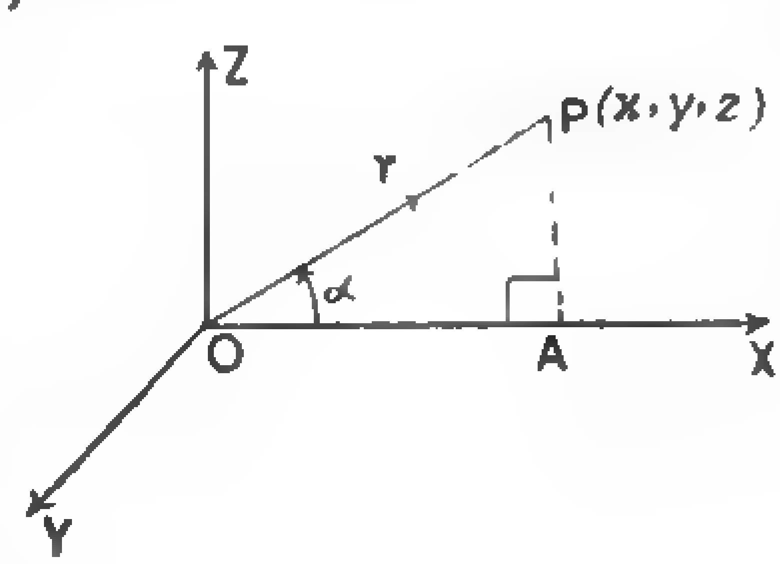
Thro' O draw OA $\parallel$ in the same sense to PQ, so that OA makes $\angle$s α, β, γ with the axes [Art. 6, (*b*) end].

Now the angles which QP makes with the axes are the angles which AO produced, *i.e.*, OA′ makes with the axes [Art. 6, (*b*) end].

$\therefore$ the direction-cosines of QP are $\cos(180°-\alpha)$, $\cos(180°-\beta)$, $\cos(180°-\gamma)$, *i.e.*, $-\cos\alpha$, $-\cos\beta$, $-\cos\gamma$, *i.e.*, $-l, -m, -n$.

Note. Important. *If, however, we ignore the two senses of a line (PQ and QP), we may take l, m, n (or $-l, -m, -n$) the direction-cosines of one and the same line.*

8. *If l, m, n are the direction-cosines of a line OP, and $OP = r$, then the co-ordinates of P are* **(lr, mr, nr).**



Proof. Let (x, y, z) be the co-ordinates of P, and A the foot of the $\perp$ from P on the x-axis.

Then $x = OA$ [Art. 3, (*a*)]

[But $\dfrac{OA}{OP} = \cos\alpha$, where α is the angle which OP makes with the x-axis]

$= OP\cos\alpha = rl = lr.$

Similarly $y=mr$, $z=nr$.

$\therefore$ the co-ordinates of P are (lr, mr, nr).

9. Relation between the direction-cosines. *If l, m, n are the direction-cosines of a line, then* $\mathbf{l^2+m^2+n^2=1}$.

[In words : *The sum of the squares of the direction-cosines of a line is equal to one.*]

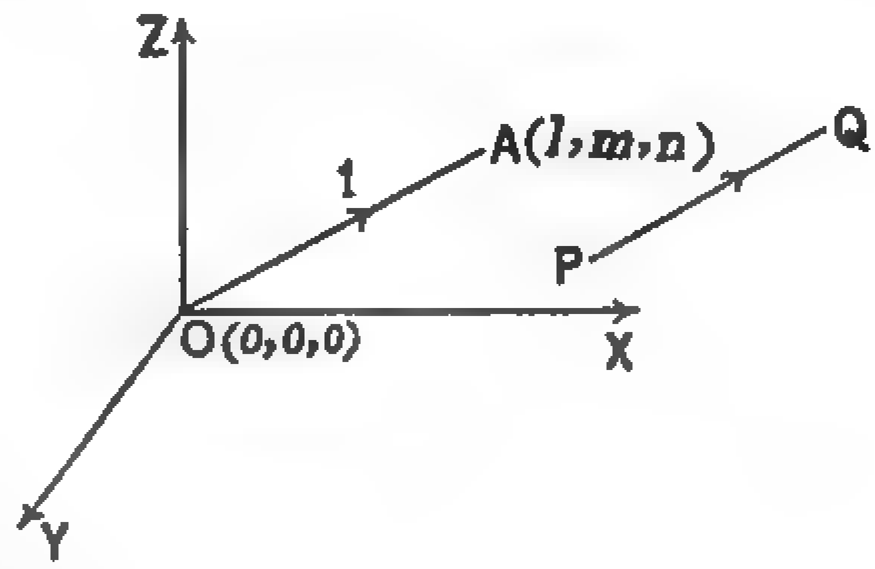
Let PQ be the line.

Thro' O draw OA $\parallel$ in the same sense to PQ, and $=1$.

Then the direction-cosines of OA are l, m, n, and OA$=1$,

$\therefore$ the co-ordinates of A are (l, m, n).

[(lr, mr, nr) (Art. 8), here $r=1$]



and the co-ordinates of O are $(0, 0, 0)$

$\therefore$ $OA^2=(l-0)^2+(m-0)^2+(n-0)^2$

[$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$ (Art. 4)]

or $(1)^2=l^2+m^2+n^2$ [$\because$ OA $=1$]

i. e., $l^2+m^2+n^2=1$.

Cor. Relation between the direction-cosines. (Another form.)

If α, β, γ are the angles which a line makes with the axes, then

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1.$$

Proof. Let l, m, n be the direction-cosines of the line.

Then $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$. [Def. (Art. 7)]

Now $l^2+m^2+n^2=1$ [Art. 9]

$\therefore$ $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$.

EXAMPLES

1. Obtain the relation between the squares of the direction-cosines. [*P(P). U. 1955*]

If α, β, γ are the angles which a line makes with the axes, prove that $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2$.

2. A line makes angles of 45° and 60° with the positive axes of x and y respectively ; what angle does it make with the positive axis of z ?

10. *The direction-cosines of a line are proportional to a, b, c ; to find the actual direction-cosines.*

The direction-cosines of the line are proportional to a, b, c.

Let the actual direction-cosines be l, m, n.

Then $\dfrac{l}{a}=\dfrac{m}{b}=\dfrac{n}{c}=\dfrac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}^{*}=\dfrac{1}{\sqrt{a^2+b^2+c^2}}$ ... (1)

$\therefore\quad l=\dfrac{a}{\sqrt{a^2+b^2+c^2}},\ m=\dfrac{b}{\sqrt{a^2+b^2+c^2}},\ n=\dfrac{c}{\sqrt{a^2+b^2+c^2}},$

$\therefore$ the actual direction-cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}},\ \frac{b}{\sqrt{a^2+b^2+c^2}},\ \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

[**Rule to find the actual direction-cosines of a line, which are proportional to a, b, c :**

Divide a, b, c each by $\sqrt{a^2+b^2+c^2}$. *The resulting ratios are the actual direction-cosines.*]

****Complete results.** $\because\ l^2+m^2+n^2=1,\ \therefore\ \sqrt{l^2+m^2+n^2}=\pm 1.$

In the above work we have taken $\sqrt{l^2+m^2+n^2}=+1$ in (1).

If, however, we take $\sqrt{l^2+m^2+n^2}=-1$ in (1),

then $$\frac{l}{a}=\frac{m}{b}=\frac{n}{c}=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2+b^2+c^2}}=\frac{-1}{\sqrt{a^2+b^2+c^2}}$$

$\therefore\ l=-\dfrac{a}{\sqrt{a^2+b^2+c^2}},\ m=-\dfrac{b}{\sqrt{a^2+b^2+c^2}},\ n=-\dfrac{c}{\sqrt{a^2+b^2+c^2}}.$

$\therefore$ the actual direction-cosines are

$$-\frac{a}{\sqrt{a^2+b^2+c^2}},\ -\frac{b}{\sqrt{a^2+b^2+c^2}},\ -\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

Combining these with the results of Art. 10, the actual direction-cosines are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{b}{\sqrt{a^2+b^2+c^2}},\ \pm\frac{c}{\sqrt{a^2+b^2+c^2}},$$

the ambiguous signs being taken *all* +ve or *all* −ve.

Note 1. Since there are two senses of the line (PQ and QP), therefore, we get two sets of the actual direction-cosines. If, however, we ignore the two senses of the line (Note, Cor. 2, Art. 7), the actual direction-cosines of the line are

$$\frac{a}{\sqrt{a^2+b^2+c^2}},\ \frac{b}{\sqrt{a^2+b^2+c^2}},\ \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

Note 2. Direction-ratios. Def. A set of three numbers a, b, c to which the direction-cosines of a line are proportional, are called the **direction-ratios** of the line.

---

*If $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}$, then each member $=\dfrac{\sqrt{a^2+c^2+e^2}}{\sqrt{b^2+d^2+f^2}}$. [Elementary Algebra]

Caution. *It is only when l, m, n are the* **actual** *direction-cosines (and not merely proportional to the direction-cosines, i.e., not the direction-ratios) of a line that the relation* $l^2+m^2+n^2=1$ *holds.*

EXAMPLES

1. **The direction-cosines of a line are proportional to 2, −3, 6, find the actual values.**

The direction-cosines of the line are proportional to 2, −3, 6.

Dividing by $\sqrt{(2)^2+(-3)^2+(6)^2}$ [Rule (Art. 10)]

$$=\sqrt{4+9+36}=\sqrt{49}=7,$$

the actual direction-cosines are $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$.

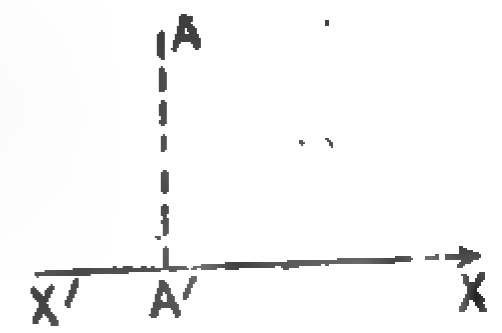
2. (*a*) Define the direction-cosines and direction-ratios of a straight line. What is the difference between them ? [*P(P). U. 1956*]

(*b*) **Find the direction cosines of a line which makes equal angles with the axes.**

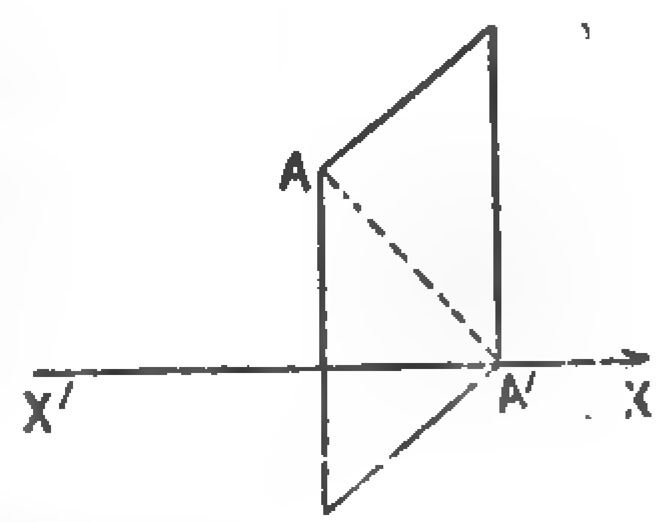
SECTION II

PROJECTION ON A LINE

11. (*a*) **Projection of a point. Def.** *The* **projection of a point A on a line** *X'X is A', the foot of the perpendicular from A on X'X.*



Cor. *The projection of a point A on a line X'X is A', the point in which the plane through A perpendicular to X'X meets X'X.*

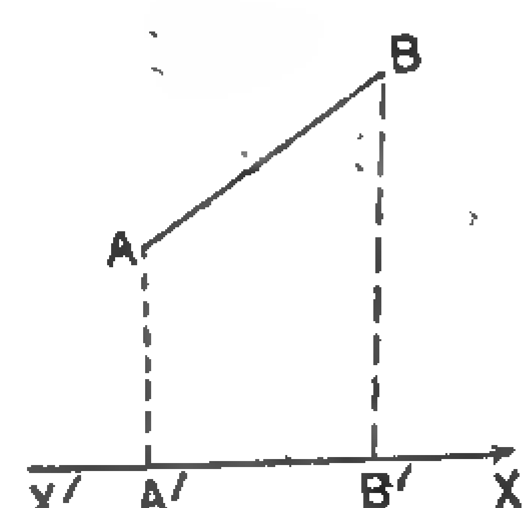


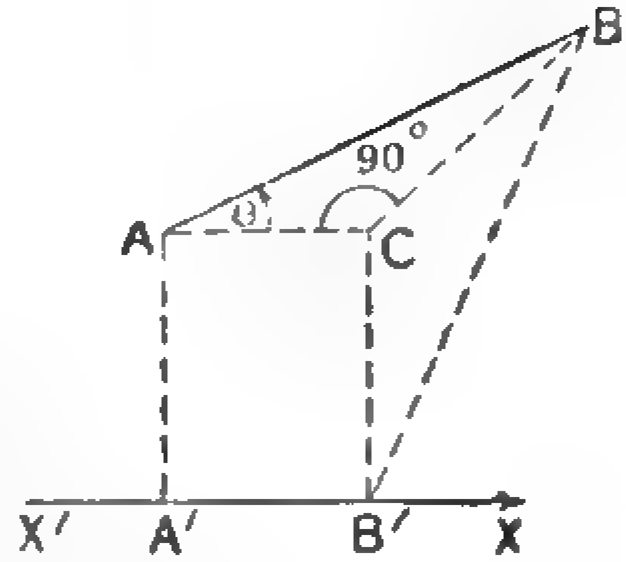
Proof. Let the plane thro' A ⊥ to X'X meet X'X in A'.

Then X'X, being ⊥ to the plane, is ⊥ to A'A which meets it in that plane, or AA' is ⊥ to X'X, *i.e.*, A' is the foot of the ⊥ from A on X'X.

∴ the projection of A is A'.

(*b*) **Projection of a segment. Def.** *The* **projection of a segment AB on a line** *X'X is the segment A'B', where A', B' are the feet of the perpendiculars from A, B on X'X.*





(c) **Length of the projection.** *The projection of a segment AB on a line X′X is*

$$A'B' = AB \cos \theta,$$

where θ is the angle which AB makes with X′X.

Proof. *Let A′ be the foot of the ⊥ from A on X′X, and B′ the pt. in which the plane BCB′, drawn thro′ B ⊥ to X′X, meets X′X, so that B′ is the foot of the ⊥ from B on X′X [Art. 11, (a), Cor.].

Thro′ A draw AC ∥ to X′X to meet the plane BCB′ in C, so that $\angle CAB = \theta$ [Def. (Art. 6, (b))].

Then AC is ⊥ to the plane BCB′

[∵ AC is ∥ to X′X, and X′X is ⊥ to the plane BCB′]

∴ AC is ⊥ to CB which meets it in that plane.

∴ $AC = AB \cos \theta$. $\left[\because \frac{AC}{AB} = \cos \theta \right]$

∴ $A'B' = AC$ [Opposite sides of rect. AA′B′C]

$= AB \cos \theta$.

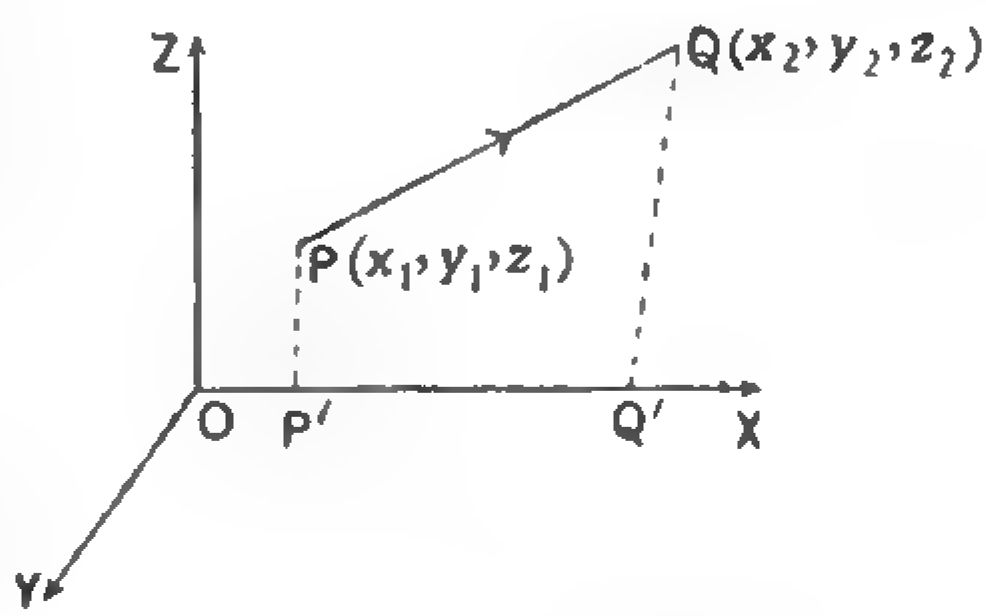
EXAMPLE

The projections of a line on the axes are 3, 4, 12. What is the length of the line?

12. Direction-cosines of the join of two points. *The direction-cosines of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) are proportional to*

$$x_2 - x_1, \; y_2 - y_1, \; z_2 - z_1.$$

Proof. Let P, Q be the pts. (x_1, y_1, z_1), (x_2, y_2, z_2), and l, m, n the direction-cosines of PQ.



Let P′, Q′ be the feet of the ⊥s from P, Q on the x-axis, so that $OP' = x_1$, $OQ' = x_2$ [Art. 3, (a)]

∴ $P'Q' = OQ' - OP'$

$= x_2 - x_1$.

Also $P'Q'$ = projection of PQ on X′X

$= PQ \cos \alpha$,

where α is the angle which PQ makes with the x-axis [Art. 11, (c)]

$= PQ.l$.

---

*How to draw the Fig. Draw lines AB, X′X, and take A′, the foot of the ⊥ from A on X′X. Take a segment A′B′ on X′X, and thro′ A draw AC ∥ and = A′B′. Join BC, CB′, BB′.

$\therefore$ PQ.$l = x_2 - x_1$, or $l = \dfrac{x_2 - x_1}{\text{PQ}}$.

Similarly $m = \dfrac{y_2 - y_1}{\text{PQ}}$, $n = \dfrac{z_2 - z_1}{\text{PQ}}$.

$\therefore$ l, m, n are proportional to $x_2 - x_1, y_2 - y_1, z_2 - z_1$.

EXAMPLES

1. *If P is the point (a, b, c), find the direction-cosines of the directed line OP, also the direction-cosines of PO.*

The direction-cosines of OP are proportional to $a-0, b-0, c-0$, i.e., proportional to a, b, c.

(1) O(0, 0, 0) (2) P(a, b, c)

[$x_2 - x_1, y_2 - y_1, z_2 - z_1$ (Art. 12)]

Dividing by $\sqrt{a^2+b^2+c^2}$, [Rule (Art. 10)]
the actual direction-cosines of OP are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}.$$

$\therefore$ the direction-cosines of PO are

$$-\frac{a}{\sqrt{a^2+b^2+c^2}}, -\frac{b}{\sqrt{a^2+b^2+c^2}}, -\frac{c}{\sqrt{a^2+b^2+c^2}}.$$

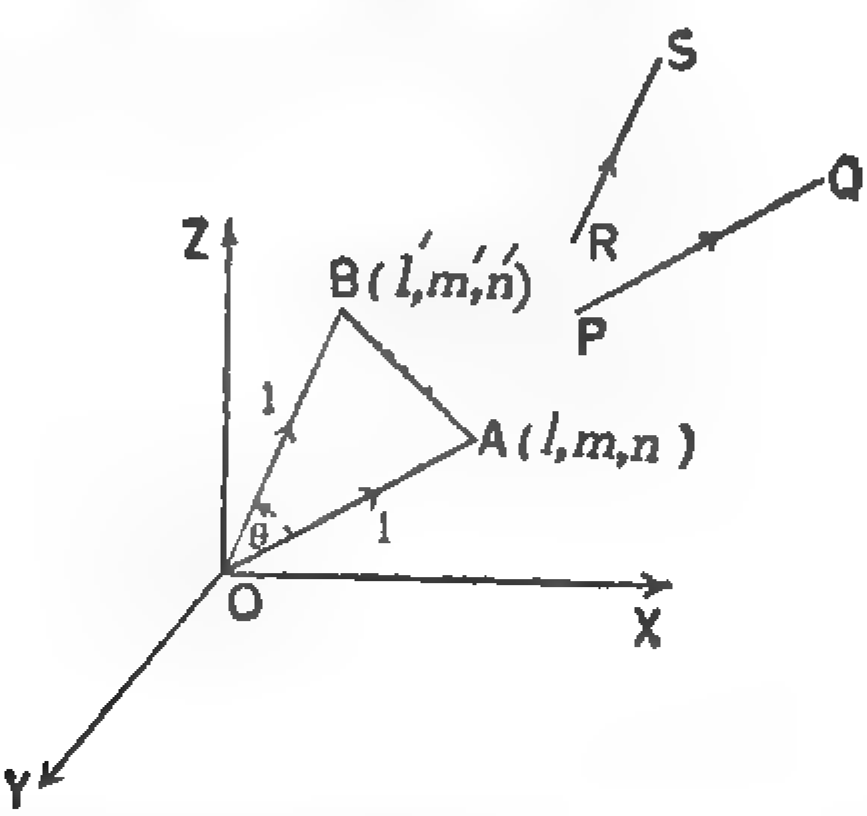
[$-l, -m, -n$ (Art. 7, Cor. 2)]

2. A, B are $(1, 2, -2)$, $(3, -4, 5)$. Find the direction-cosines of OA, OB, AO.

3. *If P, Q are $(1, 2, 3)$, $(-1, 2, 1)$, find the direction-cosines of PQ.*

Angle between two lines.

13. (*a*) *Angle formula.* **To find the angle between two straight lines whose direction-cosines are given.**



Let PQ, RS be the lines, and l, m, n ; l', m', n' their direction-cosines.

Let θ be the required angle between them.

Thro' O draw OA, OB $\parallel$ in the same sense respectively to PQ, RS, and each $=1$, so that $\angle AOB=\theta$. [Def. (Art. 6, (b))]

Now the co-ordinates of A, B are (l, m, n), (l', m', n').

[(lr, mr, nr) (Art. 8), here $r=1$]

$\therefore$ $AB=\sqrt{(l'-l)^2+(m'-m)^2+(n'-n)^2}$

$=\sqrt{l'^2-2ll'+l^2+m'^2-2mm'+m^2+n'^2-2nn'+n^2}$

[But $l^2+m^2+n^2=1$, $l'^2+m'^2+n'^2=1$]

$=\sqrt{2-2(ll'+mm'+nn')}$

$\therefore$ from the $\triangle OAB$, by the cosine formula,

$$\cos\theta=\frac{OA^2+OB^2-AB^2}{2OA.OB} \qquad \text{[Elementary Trigonometry]}$$

$$=\frac{(1)^2+(1)^2-[2-2(ll'+mm'+nn')]}{2(1)(1)} \qquad [\because OA=1, OB=1]$$

$$=\frac{2(ll'+mm'+nn')}{2}$$

$\therefore$ **cos θ=ll′+mm′+nn′**, or $\theta=\cos^{-1}(ll'+mm'+nn')$.

[*Lagrange's identity.* **Important.**

$(\mathbf{l^2+m^2+n^2})(\mathbf{l'^2+m'^2+n'^2})-(\mathbf{ll'+mm'+nn'})^2$

$=(\mathbf{mn'-m'n})^2+(\mathbf{nl'-n'l})^2+(\mathbf{lm'-l'm})^2$*.

{ **Proof. $(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2$

$=l^2l'^2+l^2m'^2+l^2n'^2+m^2l'^2+m^2m'^2+m^2n'^2+n^2l'^2+n^2m'^2+n^2n'^2$

$-(l^2l'^2+m^2m'^2+n^2n'^2+2ll'mm'+2mm'nn'+2nn'll')$

[Cancel $l^2l'^2+m^2m'^2+n^2n'^2$]

$=(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2$.}

Note. This identity is called **Lagrange's identity** *and is very useful.*]

Cor. 1. *Angle formula (sine form) for two lines.* **If θ is the angle between the lines whose direction-cosines are l, m, n ; l′, m′, n′, to prove that**

$$\sin\theta=\sqrt{(\mathbf{mn'-m'n})^2+(\mathbf{nl'-n'l})^2+(\mathbf{lm'-l'm})^2}.\dagger$$

---

*How to write down the R.H.S. Write down the direction-cosines of the lines in order in two rows :

$l, \; m, \; n,$

$l', \; m', \; n'.$

Draw diagonals mentally as in cross-multiplication,

$$\begin{array}{cccc} m & n & l & m \\ & \times \quad \times \quad \times & & \\ m' & n' & l' & m' \end{array}$$

thus getting, $mn'-m'n$, $nl'-n'l$, $lm'-l'm$, square these, and add, thus getting

$(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2$.

†How to write down the R.H.S. Proceed as in the above footnote, and take the square-root of the result.

$=\sqrt{\Sigma(mn'-m'n)^2}$.

$\sin^2\theta=1-\cos^2\theta=1-(ll'+mm'+nn')^2$

$=(l^2+m^2+n^2)(l'^2+m'^2+n'^2)-(ll'+mm'+nn')^2$

$[\because l^2+m^2+n^2=1,\ l'^2+m'^2+n'^2=1]$

which, by Lagrange's identity,

$=(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2$ ...(1)

$\therefore \sin\theta=\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}$

$=\sqrt{\Sigma(mn'-m'n)^2}$.

****Complete angle formula (sine form) for two lines.** *If θ is the angle between the lines whose direction-cosines are l, m, n ; l', m', n', then*

$$\sin\theta=\pm\sqrt{(\mathbf{mn'-m'n})^2+(\mathbf{nl'-n'l})^2+(\mathbf{lm'-l'm})^2} \qquad \text{[From (1)]}$$

$$=\pm\sqrt{\Sigma(mn'-m'n)^2}.$$

Cor. 2. Angle formula (tangent form) for two lines. *If θ is the angle between the lines whose direction-cosines are l, m, n ; l', m', n', then*

$$\tan\theta=\frac{\sqrt{(\mathbf{mn'-m'n})^2+(\mathbf{nl'-n'l})^2+(\mathbf{lm'-l'm})^2}}{\mathbf{ll'+mm'+nn'}}$$

$$=\frac{\sqrt{\Sigma(mn'-m'n)^2}}{\Sigma ll'}.$$

Proof. $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}}{ll'+mm'+nn'}$

[Art. 13, (a) and Cor. 1]

$$=\frac{\sqrt{\Sigma(mn'-m'n)^2}}{\Sigma ll'}.$$

****Complete angle formula (tangent form) for two lines.** If θ is the angle between the lines whose direction-cosines are l, m, n ; l', m', n', then

$$\tan\theta=\pm\frac{\sqrt{(\mathbf{mn'-m'n})^2+(\mathbf{nl'-n'l})^2+(\mathbf{lm'-l'm})^2}}{\mathbf{ll'+mm'+nn'}}$$

$$=\pm\frac{\sqrt{\Sigma(mn'-m'n)^2}}{\Sigma ll'}.$$

Proof. $\tan\theta=\dfrac{\sin\theta}{\cos\theta}=\dfrac{\pm\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}}{ll'+mm'+nn'}$.

[Art. 13, (a) and complete angle formula (sine form) of Cor. 1]

Cor. 3. Condition of perpendicularity of two lines. *The condition that the lines whose direction-cosines are l, m, n ; l', m', n', may be perpendicular, is*

$$\mathbf{ll'+mm'+nn'}=0.$$

Proof. If the lines are $\perp$, θ, the angle between them $=90°$

$\therefore\ \cos\theta=\cos 90°=0$, *i.e.*, $ll'+mm'+nn'=0$, [Art. 13, (a)]

which is the required condition.

Note. The converse is also true, *i.e.*, if $ll'+mm'+nn'=0$, the lines are perpendicular.

For the order of the steps in the above proof can be reversed.

Cor. 4. Conditions of parallelism of two lines. *The conditions that the lines whose direction-cosines are l, m, n; l', m', n', may be parallel, are*

$$\mathbf{l = l',\ m = m',\ n = n'.}$$

Proof. If the lines are $\parallel$, θ, the angle between them, $= 0°$

$\therefore\ \sin\theta = \sin 0° = 0$

i.e., $\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2} = 0$ [Art. 13, (*a*), Cor. 1]

or, squaring,

$$(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2 = 0$$

$\therefore\ mn'-m'n = 0,\ nl'-n'l = 0,\ lm'-l'm = 0$*

or $\dfrac{m}{m'} = \dfrac{n}{n'},\ \dfrac{n}{n'} = \dfrac{l}{l'},\ \dfrac{l}{l'} = \dfrac{m}{m'}$

$$\therefore\ \frac{l}{l'} = \frac{m}{m'} = \frac{n}{n'} = \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{l'^2+m'^2+n'^2}} = \frac{1}{1}$$

or $l = l',\ m=m',\ n=n'$, which are the required conditions.

Independent method. If the lines are $\parallel$, they make equal angles with the x-axis, *i.e.*, $\alpha=\alpha'$, $\therefore\ \cos\alpha=\cos\alpha'$, or $l=l'$.

Similarly $m=m', n=n'$.

$\therefore\ l=l',\ m=m',\ n=n'$, which are the required conditions.

Note 1. The converse is also true, *i.e.*, if $l=l',\ m=m',\ n=n'$, the lines are parallel.

For the order of the steps in the above proofs can be reversed.

Note 2. It is assumed that the lines are parallel in the same sense.

Note 3. *The condition of perpendicularity of two lines and the conditions of their parallelism are deduced respectively from the values of $\cos\theta$ and $\sin\theta$, where θ is the angle between the lines.*

13. (*b*) **Angle formula for two lines.** *If θ is the angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c', to prove that*

$$\cos\theta = \frac{\mathbf{aa'+bb'+cc'}}{\sqrt{\mathbf{a^2+b^2+c^2}}\ \sqrt{\mathbf{a'^2+b'^2+c'^2}}}.$$

The direction-cosines of the lines are proportional to a, b, c;

*Explanation. If the sum of the squares of any number of real quantities is zero, then each quantity is separately zero.

Thus, if $A^2+B^2+C^2+\dots = 0$, then $A=0,\ B=0,\ C=0, \dots$.

a', b', c', $\therefore$ the actual direction-cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}};$$
$$\frac{a'}{\sqrt{a'^2+b'^2+c'^2}}, \frac{b'}{\sqrt{a'^2+b'^2+c'^2}}, \frac{c'}{\sqrt{a'^2+b'^2+c'^2}}$$

[Rule (Art. 10)]

$$\therefore \quad \cos\theta = \frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}.$$

[$ll'+mm'+nn'$ (Art. 13, (*a*))]

****Complete angle formula for two lines.** *If θ is the angle between the lines whose direction-cosines are proportional to* **a, b, c ; a′, b′, c′,** *to prove that*

$$\cos\theta = \pm\frac{\mathbf{aa'+bb'+cc'}}{\sqrt{\mathbf{a^2+b^2+c^2}}\sqrt{\mathbf{a'^2+b'^2+c'^2}}}.$$

The direction-cosines of the lines are proportional to a, b, c ; a', b', c', $\therefore$ the actual direction-cosines are

$$\pm\frac{a}{\sqrt{a^2+b^2+c^2}}, \pm\frac{b}{\sqrt{a^2+b^2+c^2}}, \pm\frac{c}{\sqrt{a^2+b^2+c^2}};$$
$$\pm\frac{a'}{\sqrt{a'^2+b'^2+c'^2}}, \pm\frac{b'}{\sqrt{a'^2+b'^2+c'^2}}, \pm\frac{c'}{\sqrt{a'^2+b'^2+c'^2}},$$

the ambiguous signs being taken all +ve or all −ve.

[Complete results (Art. 10)]

$$\therefore \quad \cos\theta = \pm\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\sqrt{a'^2+b'^2+c'^2}}.$$

[$ll'+mm'+nn'$ (Art. 13, (*a*))]

Cor. 1. Angle formula (sine form) for two lines. *If θ is the angle between the lines whose direction-cosines are proportional to* **a, b, c ;** *a′, b′, c′, to prove that*

$$\sin\theta = \frac{\sqrt{\mathbf{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}}{\sqrt{\mathbf{a^2+b^2+c^2}}\sqrt{\mathbf{a'^2+b'^2+c'^2}}}$$

$$= \frac{\sqrt{\Sigma(bc'-b'c)^2}}{\sqrt{\Sigma a^2}\sqrt{\Sigma a'^2}}.$$

The direction-cosines of the lines are proportional to a, b, c ; a', b', c', $\therefore$ the actual direction-cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}};$$
$$\frac{a'}{\sqrt{a'^2+b'^2+c'^2}}, \frac{b'}{\sqrt{a'^2+b'^2+c'^2}}, \frac{c'}{\sqrt{a'^2+b'^2+c'^2}}$$

[Rule (Art. 10)]

$$\therefore\ \sin\theta=\frac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}$$

[$\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}$ (Art. 13, (a), Cor. 1)]

$$=\frac{\sqrt{\Sigma(bc'-b'c)^2}}{\sqrt{\Sigma a^2}\ \sqrt{\Sigma a'^2}}.$$

****Complete angle formula (sine form) for two lines.** *If θ is the angle between the lines whose direction-cosines are proportional to a, b, c ; a', b', c', then*

$$\mathbf{\sin}\ \theta=\pm\frac{\sqrt{(\mathbf{bc'-b'c})^2+(\mathbf{ca'-c'a})^2+(\mathbf{ab'-a'b})^2}}{\sqrt{\mathbf{a^2+b^2+c^2}}\ \ \sqrt{\mathbf{a'^2+b'^2+c'^2}}}$$

$$=\pm\frac{\sqrt{\Sigma\,(bc'-b'c)^2}}{\sqrt{\Sigma a^2}\ \sqrt{\Sigma a'^2}}.$$

[$\pm\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}$
(Complete angle formula (sine form), Art. 13, (a), Cor. 1)]

Cor. 2. Angle formula (tangent form) for two lines. *If θ is the angle between the lines whose direction-cosines are proportional to a, b, c ; a', b', c', to prove that*

$$\mathbf{\tan}\ \theta=\frac{\sqrt{(\mathbf{bc'-b'c})^2+(\mathbf{ca'-c'a})^2+(\mathbf{ab'-a'b})^2}}{\mathbf{aa'+bb'+cc'}}$$

$$=\frac{\sqrt{\Sigma\,(bc'-b'c)^2}}{\Sigma aa'}.$$

The direction-cosines of the lines are proportional to a, b, c ; a', b', c', $\therefore$ the actual direction-cosines are

$$\frac{a}{\sqrt{a^2+b^2+c^2}},\ \frac{b}{\sqrt{a^2+b^2+c^2}},\ \frac{c}{\sqrt{a^2+b^2+c^2}};$$

$$\frac{a'}{\sqrt{a'^2+b'^2+c'^2}},\ \frac{b'}{\sqrt{a'^2+b'^2+c'^2}},\ \frac{c'}{\sqrt{a'^2+b'^2+c'^2}}\quad\ldots(1)$$

[Rule (Art. 10)]

$$\therefore\quad \tan\theta=\frac{\dfrac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}}{\dfrac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}}$$

[$\tan\theta\left(=\dfrac{\sin\theta}{\cos\theta}\right)=\dfrac{\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}}{ll'+mm'+nn'}$
(Art. 13, (a), Cor. 2)]

$$=\frac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{aa'+bb'+cc'}$$

$$=\frac{\sqrt{\Sigma\,(bc'-b'c)^2}}{\Sigma aa'}.$$

Note. The angle formula (tangent form) for two lines is the same whether the actual direction-cosines are used or their *proportionals.*

****Complete angle formula (tangent form) for two lines.** *If θ is the angle between the lines whose direction-cosines are proportional to a, b, c; a', b', c', then*

$$\tan\theta=\pm\frac{\sqrt{(\mathbf{bc'-b'c})^2+(\mathbf{ca'-c'a})^2+(\mathbf{ab'-a'b})^2}}{\mathbf{aa'+bb'+cc'}}$$

$$=\pm\frac{\sqrt{\Sigma(bc'-b'c)^2}}{\Sigma aa'}.$$

$$\tan\theta=\frac{\dfrac{\pm\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{\sqrt{a^2+b^2+c^2}\;\sqrt{a'^2+b'^2+c'^2}}}{\dfrac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\;\sqrt{a'^2+b'^2+c'^2}}}$$

$$\left[\tan\theta\left(=\frac{\sin\theta}{\cos\theta}\right)=\frac{\pm\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}}{ll'+mm'+nn'}\right.$$

(Complete angle formula (tangent form), Art. 13, (*a*), Cor. 2)]

$$=\pm\frac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{aa'+bb'+cc'}$$

$$=\pm\frac{\sqrt{\Sigma(bc'-b'c)^2}}{\Sigma aa'}.$$

Note. Important. *The cosine of the angle between two lines whose* **proportional** *direction-cosines are given, the sine of the angle between them, and the tangent of the angle between them, are all deduced from the corresponding results for two lines whose* **actual** *direction-cosines are given.*

Cor. 3. Condition of perpendicularity of two lines. *The condition that the lines whose direction-cosines are proportional to a, b, c; a', b', c', may be perpendicular, is* $\mathbf{aa'+bb'+cc'=0}$.

Proof. If the lines are $\perp$, θ, the angle between them $=90°$

$\therefore\;\cos\theta=\cos 90°=0,$

i.e., $\dfrac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\;\sqrt{a'^2+b'^2+c'^2}}=0$ [Art. 13, (*b*)]

or $aa'+bb'+cc'=0$, which is the required condition.

Note. The condition of perpendicularity of two lines is the same whether the actual direction-cosines are used or their *proportionals.*

Cor. 4. Conditions of parallelism of two lines. *The conditions that the lines whose direction-cosines are proportional to a, b, c ; a', b', c', may be parallel, are*

$$\frac{\mathbf{a}}{\mathbf{a'}} = \frac{\mathbf{b}}{\mathbf{b'}} = \frac{\mathbf{c}}{\mathbf{c'}}.$$

Proof. If the lines are ∥, θ, the angle between them, $= 0°$

$\therefore\ \sin\theta = \sin 0° = 0$

i.e., $\dfrac{\sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2}}{\sqrt{a^2+b^2+c^2}\,\sqrt{a'^2+b'^2+c'^2}} = 0$ [Art. 13, (b), Cor. 1]

$\therefore\ \sqrt{(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2} = 0$

or, squaring,

$$(bc'-b'c)^2+(ca'-c'a)^2+(ab'-a'b)^2 = 0$$

$\therefore\ bc'-b'c = 0,\ ca'-c'a = 0,\ ab'-a'b = 0$

or $\dfrac{b}{b'} = \dfrac{c}{c'},\ \dfrac{c}{c'} = \dfrac{a}{a'},\ \dfrac{a}{a'} = \dfrac{b}{b'}$

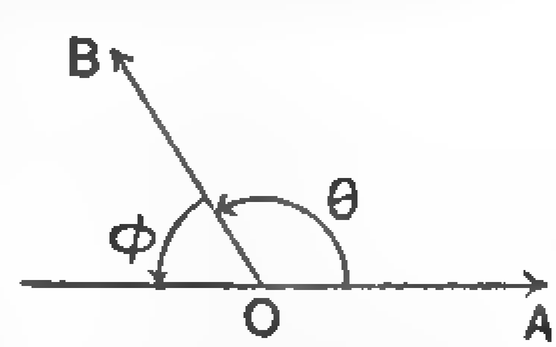
or $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{c}{c'}$, which are the required conditions.

EXAMPLES

1. (*a*) If OA and OB have direction-cosines $\cos\alpha, \cos\beta, \cos\gamma$; $\cos\alpha', \cos\beta', \cos\gamma'$, and θ is the angle which OA makes with OB, prove that $\cos\theta = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$.

(*b*) Find the acute angle between the lines whose direction-cosines are $\dfrac{\sqrt{3}}{2}, \dfrac{1}{4}, \dfrac{\sqrt{3}}{4}$; $-\dfrac{\sqrt{3}}{2}, \dfrac{1}{4}, \dfrac{\sqrt{3}}{4}$.

[(*b*) **Note. Acute and obtuse angles between two lines.** In a numerical example (*i*) if the angle formula, $\cos\theta = ll' + mm' + nn'$ gives a *positive* result, it is the cosine of the *acute* angle between the lines; but (*ii*) if the angle formula gives a *negative* result, it is the cosine of the *obtuse* angle between the lines. In this case, if ϕ is the *acute* angle between them, then $\phi = 180° - \theta$,



$\therefore\ \cos\phi = \cos(180°-\theta) = -\cos\theta = -(ll' + mm' + nn')$, *i.e.*, we must *change the sign in the angle formula*, $\cos\theta = ll' + mm' + nn'$, *to get* $\cos\phi$, *the cosine of the acute angle between the lines.*]

2. Find the angle between two straight lines whose direction-cosines are given. [*P(P). U. 1956*]

Find the angles between the lines whose direction-cosines are proportional to (*i*) 1, 2, 3 ; 3, 4, 5 ; (*ii*) 1, 2, 1 ; 2, −3, 4.

3. Show that the lines whose direction-cosines are proportional to 1, 1, 2 ; $\sqrt{3}-1, -\sqrt{3}-1, 4$; $-\sqrt{3}-1, \sqrt{3}-1, 4$ are inclined to one another at an angle $\pi/3$.

4. *If A, B are (2, 1, −2), (3, −4, 5), find the angle that OA makes with OB.* [*P. U. B. Sc. H.*]

5. *A, B, C are the points (1, 4, 2), (−2, 1, 2), (2, −3, 4). Find the angles of the triangle ABC.*

6. *If a variable line in two adjacent positions has direction-cosines l, m, n ; $l+\delta l, m+\delta m, n+\delta n$, show that the small angle, $\delta\theta$, between the two positions is given by* $(\delta\theta)^2=(\delta l)^2+(\delta m)^2+(\delta n)^2$. [*P. U. 1954*]

The direction-cosines of the line in the two positions are

$$l, m, n\,;\ l+\delta l,\ m+\delta m,\ n+\delta n.$$

$$\therefore\quad l^2+m^2+n^2=1 \quad \ldots \quad \ldots \quad \ldots (1)$$

$$(l+\delta l)^2+(m+\delta m)^2+(n+\delta n)^2=1 \quad \ldots (2)$$

Subtracting (1) from (2),

$$2(l\delta l+m\delta m+n\delta n)+(\delta l)^2+(\delta m)^2+(\delta n)^2=0 \quad \ldots (3)$$

Now $\cos\delta\theta = l(l+\delta l)+m(m+\delta m)+n(n+\delta n)$

[$ll'+mm'+nn'$ (Art. 13, (a))]

$$=l^2+m^2+n^2+l\delta l+m\delta m+n\delta n$$

$$=1+l\delta l+m\delta m+n\delta n \qquad [\because\ l^2+m^2+n^2=1]$$

$$\therefore\quad 1-\cos\delta\theta=-(l\delta l+m\delta m+n\delta n) \ \ldots (4)$$

But $1-\cos\delta\theta=2\sin^2\dfrac{\delta\theta}{2}=2\left(\dfrac{\delta\theta}{2}\right)^2$ nearly

[$\because$ when θ is small, $\sin\theta=\theta$ nearly
(Elementary Trigonometry)]

$$=\frac{(\delta\theta)^2}{2}.$$

$\therefore$ from (4), $\dfrac{(\delta\theta)^2}{2}=-(l\delta l+m\delta m+n\delta n)$ [Substitute from (3)]

$$=\tfrac{1}{2}\left[(\delta l)^2+(\delta m)^2+(\delta n)^2\right]$$

$$\therefore\quad (\delta\theta)^2=(\delta l)^2+(\delta m)^2+(\delta n)^2.$$

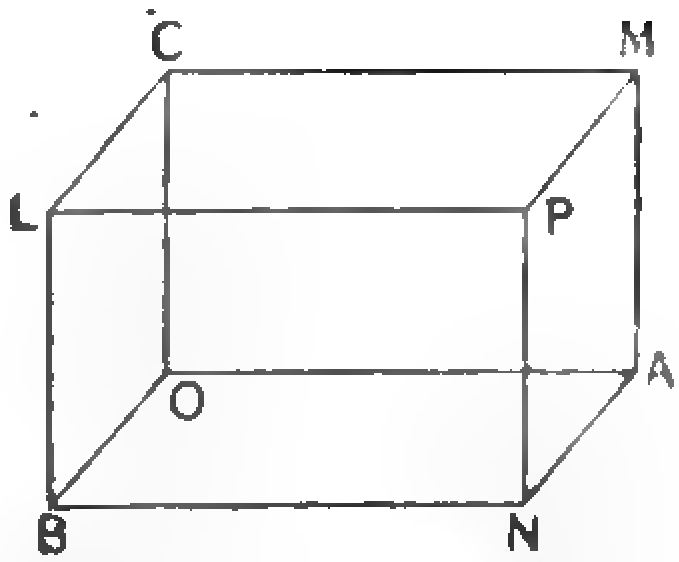
7. *Lines OP, OQ are drawn from O with direction-cosines proportional to (1, −1, −1) and (2, −1, 1). Find the direction-cosines of the normal to the plane POQ.*

[**Note on parallelepiped. Defs.**

(i) A **parallelepiped** is a solid bounded by three pairs of parallel plane faces.

(ii) A **rectangular parallelepiped*** is a parallelepiped whose faces are all rectangles.

(iii) A **cube** † is a parallelepiped whose faces are all squares.

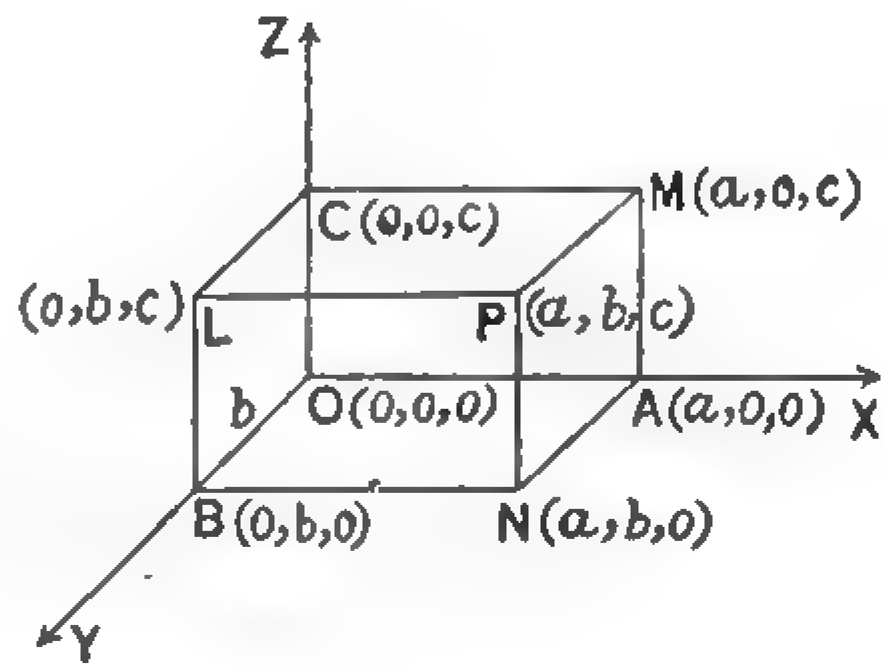


8. *If the edges of a rectangular parallelepiped are a, b, c, show that the angles between the four diagonals are given by*

$$\cos^{-1}\frac{a^2\pm b^2\pm c^2}{a^2+b^2+c^2}.$$ [*P(P). U. 1956*]

[Let OA$=a$, OB$=b$, OC$=c$.

Take O, a corner of the parallelepiped, as origin and OA, OB, OC, the three edges thro' it, as axes.



Then the co-ordinates of O, A, B, C are respectively (0,0,0), $(a,0,0)$, $(0, b, 0)$,$(0, 0, c)$; those of P,L,M,N are respectively (a, b, c), $(0, b, c)$, $(a, 0, c)$, $(a, b, 0)$.

The four diagonals are OP, AL, BM, CN. Find the angles between the diagonals OP and AL ; OP and BM ; OP and CN. It will be found that the angle between the diagonals OP and AL= the angle between the remaining two diagonals BM and CN, and so on.

Note 1. Important. For problems relating to the diagonals of a rectangular parallelepiped or a cube, take a corner as origin and the three edges through it as axes.

****Note 2.** In Ex. 8, the ambiguous signs cannot all be positive. For, if they are all +ve, angle$=\cos^{-1}\dfrac{a^2+b^2+c^2}{a^2+b^2+c^2}=\cos^{-1}1=0$,

$\therefore$ the two diagonals are $\parallel$, which is impossible as is evident from the Fig.]

9. *Find the angle between two diagonals of a cube.*

---

* **How to draw the Fig.** Take three mutually $\perp$ lines OA, OB, OC. Complete the rectangles OBLC, OCMA, OANB, BLPN. Join MP.

† **How to draw the Fig.** Take three mutually $\perp$ lines OA, OB, OC such that OA = OB = OC, and proceed as above.

10. *A line makes angles α, β, γ, δ with the four diagonals of a cube ; prove that* $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=\frac{4}{3}$. [*P(P). U. 1955*]

Angle between two lines whose direction-cosines are given by two numerical homogeneous equations in l, m, n, one of the first degree and the other of the second degree.

11. Find the angle between the two lines whose direction-cosines are given by the equations $l+m+n=0$, $l^2+m^2-n^2=0$. [*P. U. 1943 S*]

The equations giving the direction-cosines of the two lines are

$$l+m+n=0 \quad ...(1)$$

$$l^2+m^2-n^2=0 \quad ...(2)$$

Eliminating n from (1) and (2) [by substituting the value of $n[=-(l+m)]$ from (1) in (2)],

$$l^2+m^2-(l+m)^2=0 \quad \text{or} \quad -2lm=0$$

$\therefore$ *either* $l=0$	*or* $m=0$
i.e., $1.l+0.m+0.n=0,$	*i.e.,* $0.l+1.m+0.n=0,$
also $l+m+n=0.$	also $l+m+n=0.$
Solving by cross-multiplication,	Solving by cross-multiplication,
$\frac{l}{0}=\frac{m}{-1}=\frac{n}{1}$	$\frac{l}{1}=\frac{m}{0}=\frac{n}{-1}$

$\therefore$ the direction-cosines of the lines are proportional to $0, -1, 1$; $1, 0, -1$.

$\therefore$ if θ is the angle between the lines, then

$$\cos\theta=\frac{0\,(1)+(-1)\,(0)+1\,(-1)}{\sqrt{(0)^2+(-1)^2+(1)^2}\,\sqrt{(1)^2+(0)^2+(-1)^2}}$$

$$\left[\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\,\sqrt{a'^2+b'^2+c'^2}}\ \text{(Art. 13, (b))}\right]$$

$$=\frac{-1}{2} \quad ...(1)$$

If ϕ is the *acute* angle between the lines, then changing the sign in (1) [Ex. 1, (*b*)], $\cos\phi=\frac{1}{2}=\cos 60°$, $\therefore\ \phi=60°$.

12. Show that the straight lines whose direction-cosines are given by the equations $2\,l-m+2\,n=0$, and $mn+nl+lm=0$ are perpendicular.

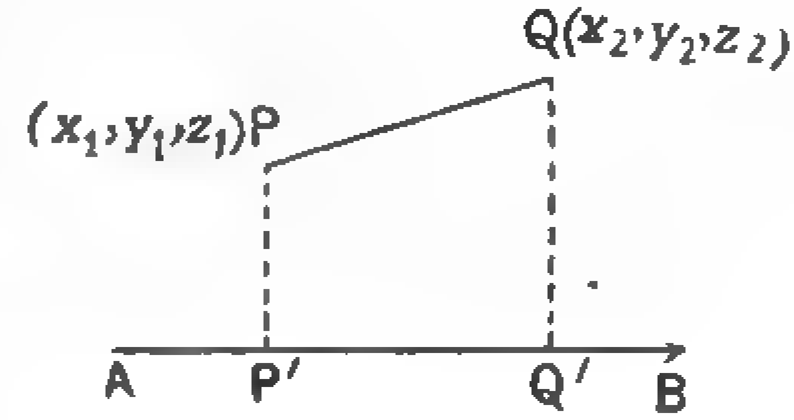
13. Find the expression for the sine of the angle between two straight lines whose direction-cosines are l_1, m_1, n_1 and l_2, m_2, n_2 respectively, the axes being rectangular. [*B. U. 1953*]

Prove by direction-cosines that the points (1, 2, 3), (4, 0, 4), (−2, 4, 2), (7, −2, 5) are collinear.

14. Projection of the join of two points on a line whose direction-cosines are given. *To prove that the projection of the join of the points (x_1, y_1, z_1) and (x_2, y_2, z_2) on a line whose direction-cosines are l, m, n is*

$$(\mathbf{x_2-x_1})\,\mathbf{l}+(\mathbf{y_2-y_1})\,\mathbf{m}+(\mathbf{z_2-z_1})\,\mathbf{n}.$$

Let P, Q be the pts. (x_1, y_1, z_1), (x_2, y_2, z_2), and AB the line whose direction-cosines are l, m, n.



Let P′Q′ be the projection of PQ on AB, so that

P′Q′=PQ cos θ...(1), [Art. 11, (*c*)]

where θ is the angle which PQ makes with AB.

[To find cos θ.]

Now the direction-cosines of PQ are proportional to x_2-x_1, y_2-y_1, z_2-z_1. [Art. 12]

Dividing by $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}=PQ$,

the actual direction-cosines of PQ are $\dfrac{x_2-x_1}{PQ}$, $\dfrac{y_2-y_1}{PQ}$, $\dfrac{z_2-z_1}{PQ}$, and the direction-cosines of AB are l, m, n,

$$\therefore\ \cos\theta=\left(\frac{x_2-x_1}{PQ}\right)l+\left(\frac{y_2-y_1}{PQ}\right)m+\left(\frac{z_2-z_1}{PQ}\right)n$$

[$ll'+mm'+nn'$ (Art. 13, (*a*))]

$$=\frac{1}{PQ}\Big[(x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)n\Big]\ \ldots(2)$$

$\therefore$ from (1),

$$P'Q'=PQ\,\frac{1}{PQ}\Big[(x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)\,n\Big]$$

$$=(x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)\,n.$$

EXAMPLES

1. *If P is the point (x_1, y_1, z_1), prove that the projection of OP on a line whose direction-cosines are l, m, n, is $lx_1+my_1+nz_1$.*

2. If A, B, C, D are the points (1, 2, 3), (2, 4, 1), (−1, 2, 3), (1, 0, 4), find the projection of CD on AB.

3. If P, Q, R, S are the points (−2, 3, 4), (−4, 4, 6), (4, 3, 5), (0, 1, 2), prove by projections that PQ is at right angles to RS.

MISCELLANEOUS EXAMPLES ON CHAPTER II

1. The co-ordinates of the angular points A, B, C, D of a tetrahedron are (−2, 1, 3), (3, −1, 2), (2, 4, −1) and (1, 2, 3) respectively. Find the angle between the edges AC and BD. [*P. U. 1951 S*]

2. **Direction-cosines of a line perpendicular to two lines.** *Show that the line perpendicular to both of the lines whose direction-cosines are proportional to l, m, n ; l', m', n', has direction-cosines proportional to $mn'-m'n, nl'-n'l, lm'-l'm$.* [*L. U.*]

3. *Prove that if two pairs of opposite edges of a tetrahedron are perpendicular, then the third pair are also perpendicular.* [*P.U. 1949 S*]

[**Note. Important. For problems relating to perpendicular opposite edges of a tetrahedron ABCD, take D, one vertex, as origin, and let A, B, C, the remaining vertices, be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).**]

4. Two edges AB, CD of a tetrahedron ABCD are perpendicular. Prove that the distance between the mid-points of AC and BD is equal to the distance between the mid-points of AD and BC. [*L. U.*]

5. **Show that the straight lines whose direction-cosines are given by the equations $ul+vm+wn=0$, $al^2+bm^2+cn^2=0$, are perpendicular if**

$$u^2(b+c)+v^2(c+a)+w^2(a+b)=0.$$

The equations giving the direction-cosines of the two lines are

$$ul+vm+wn=0 \quad ...(1)$$

$$al^2+bm^2+cn^2=0 \quad ...(2)$$

Eliminating n from (1) and (2) [by substituting the value of $n\left(=-\dfrac{ul+vm}{w}\right)$ from (1) in (2)],

$$al^2+bm^2+c\left(\frac{ul+vm}{w}\right)^2=0$$

or $\quad aw^2l^2+bw^2m^2+c(u^2l^2+2\,uvlm+v^2m^2)=0$

or $\quad l^2(aw^2+cu^2)+2\,cuvlm+m^2(bw^2+cv^2)=0.$

Dividing thro' out by m^2,

$$\frac{l^2}{m^2}(aw^2+cu^2)+2\,cuv\,\frac{l}{m}+(bw^2+cv^2)=0 \quad ...(3)$$

which is a quadratic in $\dfrac{l}{m}$.

If l_1, m_1, n_1 ; l_2, m_2, n_2 are the direction-cosines of the lines, then $\dfrac{l_1}{m_1}, \dfrac{l_2}{m_2}$ are the roots of the quadratic (3).

$\therefore$ (product of the roots, *i.e.*,) $\dfrac{l_1}{m_1}\cdot\dfrac{l_2}{m_2}=\dfrac{bw^2+cv^2}{aw^2+cu^2}$

or $\dfrac{l_1 l_2}{bw^2+cv^2}=\dfrac{m_1m_2}{cu^2+aw^2}$, which from symmetry,

$$=\frac{n_1n_2{}^*}{av^2+bu^2}=k \text{ (say)}$$

$\therefore\ l_1l_2+m_1m_2+n_1n_2=k\,[bw^2+cv^2+cu^2+aw^2+av^2+bu^2]$ ...(4)

The lines are $\perp$ if $l_1l_2+m_1m_2+n_1n_2=0$,

or, if, from (4), $k[\,bw^2+cv^2+cu^2+aw^2+av^2+bu^2\,]=0$,

or, if $\qquad bw^2+cv^2+cu^2+aw^2+av^2+bu^2=0$,

or, if $\qquad u^2(b+c)+v^2(c+a)+w^2(a+b)=0$.

6. *Show that the straight lines whose direction-cosines are given by the equations $ul+vm+wn=0$, $al^2+bm^2+cn^2=0$, are parallel, if*

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=0. \qquad \text{[P. U. H. 1952]}$$

[Proceed as in Ex. 5.

The lines are $\parallel$ if $l_1=l_2$, $m_1=m_2$, $n_1=n_2$

or, if $\dfrac{l_1}{m_1}=\dfrac{l_2}{m_2}$, or if the quadratic (3) obtained in Ex. 5, has equal roots, etc.]

7. *Show that he straight lines whose direction-cosines are given by the equations $ul+vm+wn=0$, $fmn+gnl+hlm=0$, are*

(*i*) *perpendicular, if* $\dfrac{f}{u}+\dfrac{g}{v}+\dfrac{h}{w}=0$,

and (*ii*) *parallel, if* $\sqrt{uf}+\sqrt{vg}+\sqrt{wh}=0$.

8. Find the projections of the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) on the axes of co-ordinates. [*P. U. 1935 S*]

***How to write this step.** Since from the first member $\dfrac{l_1\, l_2}{bw^2+cv^2}$, changing l, m, n ; a, b, c ; u, v, w *cyclically, i.e.*, changing l_1 to m_1, l_2 to m_2 ; b to c, w to u ; c to a, v to w, we get the second member $\dfrac{m_1\, m_2}{cu^2+aw^2}$.

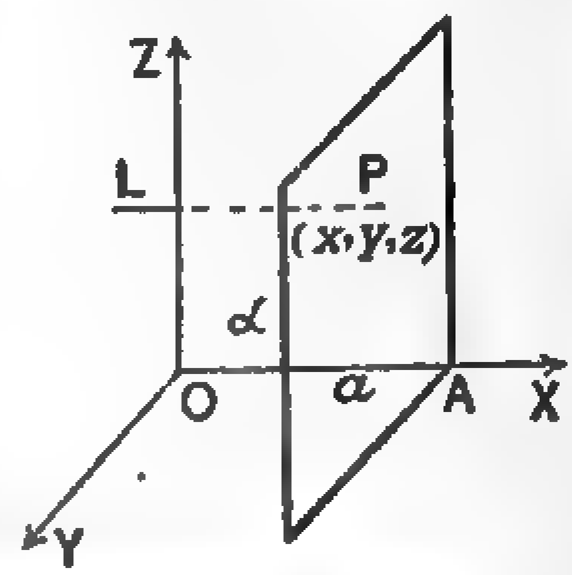
$\therefore$ from the second member $\dfrac{m_1\, m_2}{cu^2+aw^2}$, changing m_1 to n_1, m_2 to n_2 ; c to a, u to v ; a to b, w to u, we get $\dfrac{n_1\, n_2}{av^2+bu^2}$.

CHAPTER III

THE LOCUS OF AN EQUATION

Equation of a plane parallel to one of the co-ordinate planes.

15. Equation of a plane parallel to the yz-plane. *To find the equation of the plane parallel to the yz-plane and at a distance a from it.*



Let α be the plane meeting the x-axis in A, so that $OA = a$.

Let $P(x, y, z)$ be *any* pt. on the plane.

From P draw PL $\perp$ on the yz-plane.

Then $LP = OA$ [Distance between two ∥ planes YOZ and α]

$\therefore$ $\mathbf{x = a}$, which is the required equation.

Note 1. Similarly *the equation of the plane parallel to the zx-plane and at a distance b from it, is* $\mathbf{y = b}$, *and the equation of the plane parallel to the xy-plane and at a distance c from it, is* $\mathbf{z = c}$.

Equations of the co-ordinate planes.

Cor. *The equation of the yz-plane is* $\mathbf{x = 0}$.

For, if the plane coincides with the yz-plane, then $a = 0$.

Note 2. Similarly *the equation of the zx-plane is* $\mathbf{y = 0}$, *and the equation of the xy-plane is* $\mathbf{z = 0}$.

[Aid to memory. The equation of a plane ∥ to the yz-plane contains only the co-ordinate of the absent variable, thus the equation is $x = a$.

Similarly for the equation of a plane ∥ to the zx-plane or ∥ to the xy-plane.

Again the equation of the yz-plane contains only the co-ordinate of the absent variable, thus the equation is $x = 0$.

Similarly for the equation of the zx-plane or the xy-plane.

Equation involving one or more of the three variables x, y, z.

16. Case I. Equation involving one variable. *To find the locus of the equation* $f(x) = 0$.*

The equation is $f(x) = 0$ ... (1)

Let $a, b, c, \dots, k$ be the roots.

Then (1) is equivalent to

$$(x-a)(x-b)(x-c)\dots(x-k) = 0$$

or $\quad x = a, \quad x = b, \quad x = c, \dots, \quad x = k$

which represent a system of planes ∥ to the yz-plane. [Art. 15]

$\therefore$ (1) *represents a system of planes ∥ to the yz-plane.*

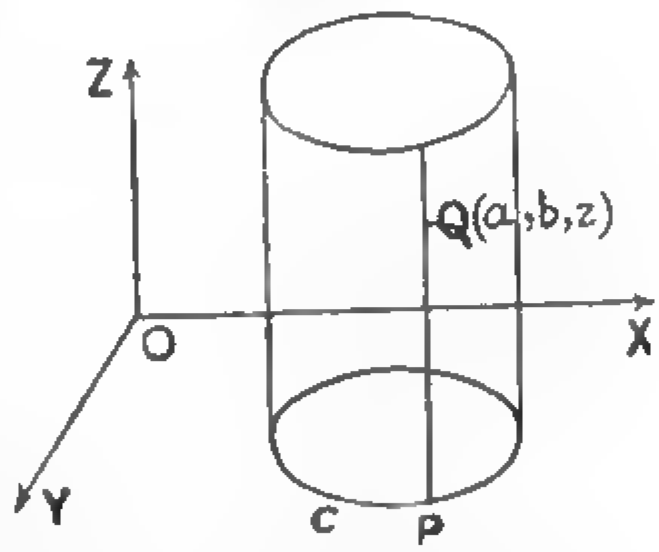
Note 1. These planes may be real or imaginary.

Note 2. Similarly *$f(y)=0$ represents a system of planes parallel to the zx-plane,*
and $f(z)=0$ represents a system of planes parallel to the xy-plane.

***Note 3.** It is assumed that $f(x)$ is a *polynomial* in x, *i.e.*, an expression containing only positive, integral powers of x.

Similarly for $f(x, y)$ and $f(x, y, z)$ in the following Arts.

17. Case II Equation involving two variables. *To find the locus of the equation $f(x, y) = 0$.*



The equation is $f(x, y) = 0$ ... (1)

Let c be the curve whose equation *in the xy-plane* is $f(x, y) = 0$ ... (2)

Let P be *any* pt. on this curve and (a, b) its co-ordinates *in the xy-plane*, so that from (2), $f(a, b) = 0$ ... (3)

Thro' P draw a st. line PQ ∥ to the z-axis, and let Q (a, b, z) be any pt. on this line. [Art. 3, (*b*)]

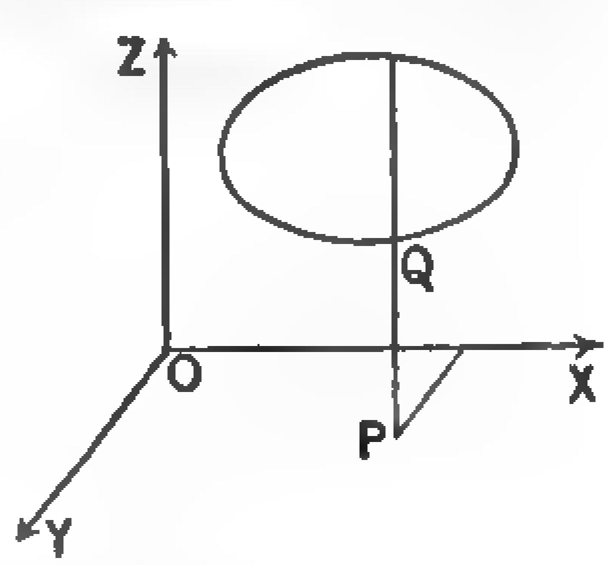
Then from (3), Q lies on the locus of (1), *i.e.*, *any* pt. on the line thro' P ∥ to the z-axis and ∴ the line itself lies on the locus of (1).

∴ (1) represents a cylinder* generated by a st. line which is ∥ to the z-axis and intersects the curve whose equation in the xy-plane is $f(x, y) = 0$.

Similarly *$f(y, z)=0$ represents a cylinder generated by a st. line which is ∥ to the x-axis and intersects the curve whose equation in the yz-plane is $f(y, z) = 0$,*
and $f(z, x) = 0$ represents a cylinder generated by a st. line which is ∥ to the y-axis and intersects the curve whose equation in the zx-plane is $f(z, x) = 0$.

[**Aid to memory.** The equation $f(x, y) = 0$, from which z is absent, represents a cylinder generated by a st. line which is ∥ to the z-axis, *i.e.*, ∥ to the axis of the absent variable.]

18. Case III. Equation involving three variables. *To find the locus of the equation $f(x, y, z) = 0$.*



The equation is $f(x, y, z) = 0$ ... (1)

Let P be *any* pt. in the xy-plane, and (a, b) its co-ordinates *in that plane.*

Thro' P draw a st. line ∥ to the z-axis, and let Q(a, b, z) be *any* pt. on this line. [Art. 3, (*b*)]

If Q lies on the locus of (1), then

***Def.** A cylinder is a surface generated by a straight line which is parallel to a fixed line and satisfies some other condition, *e.g.*, it may intersect a fixed curve.

$$f(a, b, z) = 0,$$

which is an equation in z, having a finite number of roots.

$\therefore$ the line thro' *any* pt. P in the xy-plane $\parallel$ to the z-axis meets the locus of (1) in a finite number of pts.

$\therefore$ (1) *represents a surface* * and not a solid figure.

[**Summary.** (*i*) $f(x)=0$ represents a system of planes ($\parallel$ to the plane of the *absent variables, i.e.,*) $\parallel$ to the yz-plane.

(*ii*) $f(x, y)=0$ represents a cylinder generated by a st. line which is ($\parallel$ to the axis of the *absent variable, i.e.,*) $\parallel$ to the z-axis.

(*iii*) $f(x, y, z)=0$ represents a surface.]

EXAMPLE

What surfaces are represented by (*i*) $\mathbf{x^2+y^2=a^2}$, (*ii*) $\mathbf{y^2=4ax}$, **the axes being rectangular ?** [*B. H. U. 1940*]

19. Locus represented by two equations. *To find the locus represented by* $f_1(x, y, z) = 0$, $f_2(x, y, z) = 0$.

The equations are $f_1(x, y, z)=0$, $f_2(x, y, z) = 0$ ... (1)

The pts. whose co-ordinates satisfy both the equations lie on both the surfaces represented by these equations separately,

$\therefore$ these pts. lie on the curve of intersection of the two surfaces,

$\therefore$ the equations (1), taken together, represent the curve of intersection of the two surfaces $f_1(x, y, z)=0$ and $f_2(x, y, z)=0$.

Cor. 1. Equations of the axes. *The equations of the x-axis are*

$$\mathbf{y=0,\ z=0.}$$

Proof. The x-axis is the line of intersection of the zx-plane and the xy-plane,

$\therefore$ its equations are $y=0$, $z=0$. [Art. 19]

Similarly *the equations of the y-axis are* $\mathbf{z=0,\ x=0,}$

and *the equations of the z-axis are* $\mathbf{x=0,\ y=0.}$

[**Aid to memory.** The equations of the x-axis contain only the co-ordinates of the absent variables, thus the equations are $y=0$, $z=0$.

Similarly for the equations of the y-axis or the z-axis.]

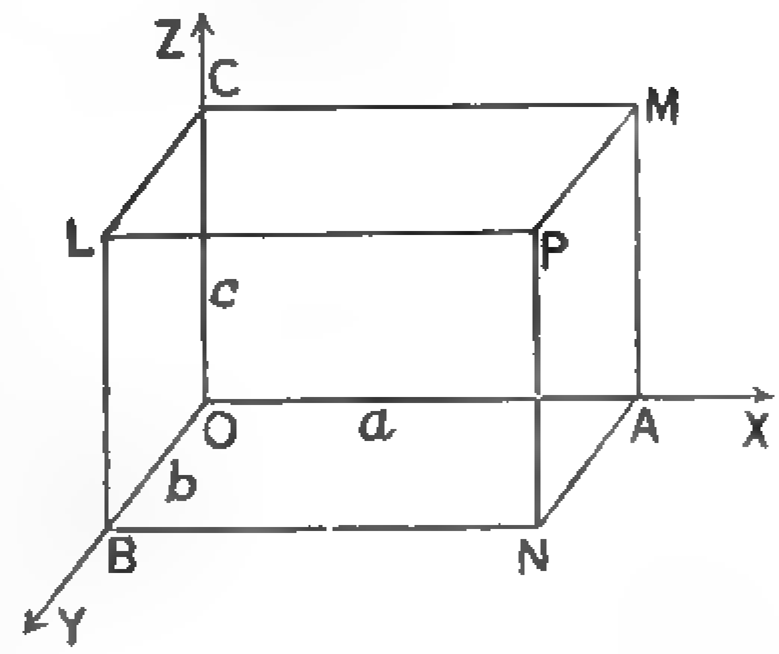
Cor. 2. *To find the equations of a curve in the xy-plane.*

A curve in the xy-plane is the curve of intersection of the cylinder $f(x, y) = 0$ and the xy-plane : $z = 0$.

$\therefore$ the equations of the curve are $f(x, y) = 0$, $z=0$. [Art. 19]

*Compare. A straight wire (*line*) meets a hollow rubber ball (*surface*) in two (*i.e.*, finite number of) points but it meets a solid clay ball (*solid figure*) in an infinite number of points.

EXAMPLES



1. *In the adjoining Fig. of a parallelepiped, if* $OA=a$, $OB=b$, $OC=c$,

(*i*) *find the equations of the planes PNBL, PLCM, PNAM ;*

(*ii*) *find the equations of PN.*

2. What is the locus of the point,

(*i*) whose x-co-ordinate is 2,

(*ii*) whose x-co-ordinate is 2 and y-co-ordinate is -3 ?

3. *What is the locus of a point whose co-ordinates satisfy*

(*i*) $x=a$ *and* $y=b$; (*ii*) $y=b$ *and* $z=c$; (*iii*) $x=a$ *and* $y=0$;

(*iv*) $x=0$ *and* $y=0$?

4. What curves are represented by

(*i*) $x^2+y^2=a^2,\ z=0$; (*ii*) $x^2+y^2=a^2,\ z=b$; (*iii*) $z^2=4ax,\ y=c$?

[*P. U. 1937 S*]

5. If the axes are rectangular, what loci are represented by

(*i*) $x^2+y^2=a^2,\ z^2=b^2$, (*ii*) $x^2+y^2=a^2,\ x^2=b^2\ (a^2>b^2)$?

6. *Find the equations of the cylinders with generators parallel to OX, OY, OZ, which pass through the curve of intersection of the surfaces*

$$x^2+y^2+2z^2=3,\quad x-y+z=1.$$

Note. *The three equations* $x=a, y=b, z=c$, *taken together,* represent a point, the point of intersection of the three planes $x=a$, $y=b$, $z=c$, *i.e.,* the point (a, b, c).

****20. Locus represented by three equations.** *To find the locus represented by* $f_1(x, y, z)=0$, $f_2(x, y, z)=0$, $f_3(x, y, z)=0$.

The equations are

$$f_1(x, y, z)=0,\ f_2(x, y, z)=0,\ f_3(x, y, z)=0 \ \ldots (1)$$

Solving the three equations, we get one or more sets of values of x, y, z, which represent one or more pts.

$\therefore$ the equations (1), taken together, represent one or more pts.

MISCELLANEOUS EXAMPLES ON CHAPTER III

1. (*i*) What locus is represented by $f(z)=0$?

(*ii*) Describe completely the nature of the surface $f(y, z)=0$.

[*P. U. H.*]

2. Explain what is meant in three dimensions by $z^2=4ax$.

[*P. U. 1937 S*]

3. Explain what is represented by the following equations in two and three dimensions :—

(*a*) $f(x, y)=0$, (*b*) $x^2-y^2=a^2$. [*P. U. 1937*]

CHAPTER IV

THE PLANE

SECTION I

EQUATION OF A PLANE

21. **General form.** *To prove that the general equation of the first degree in x, y, z represents a plane.*

The general equation of the first degree in x, y, z is

$$Ax+By+Cz+D=0^* \quad ...(1)$$

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be *any* two pts. on the locus of (1).

Then
$$\begin{array}{ll} Ax_1+By_1+Cz_1+D=0 \quad ...(2) & \Big| \; m_2 \\ Ax_2+By_2+Cz_2+D=0 \quad ...(3) & \Big| \; m_1 \end{array}$$

Multiplying (2) by m_2, (3) by m_1, and adding,

$$A(m_2x_1+m_1x_2)+B(m_2y_1+m_1y_2)+C(m_2z_1+m_1z_2)+D(m_2+m_1)=0.$$

Dividing thro' out by m_1+m_2,

$$A\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}\right)+B\left(\frac{m_1y_2+m_2y_1}{m_1+m_2}\right)+C\left(\frac{m_1z_2+m_2z_1}{m_1+m_2}\right)+D=0,$$

which shows that the pt.

$$\left(\frac{m_1x_2+m_2x_1}{m_1+m_2}, \frac{m_1y_2+m_2y_1}{m_1+m_2}, \frac{m_1z_2+m_2z_1}{m_1+m_2}\right)$$

for *all* values of $m_1 : m_2$, lies on the locus of (1), *i.e.*, if *any* two pts. (x_1, y_1, z_1), (x_2, y_2, z_2) are taken on the locus of (1), then *any other* pt. on the st. line joining them lies on the locus,

∴ (1) represents a *plane*. [Def.†]

Note 1. General form. Since the equation $Ax+By+Cz+D=0$ can, by a proper choice of A, B, C, D, be made to represent *any* plane, this form of the equation of a plane may be called the **general form.** It is also called the **general equation** of a plane. Notice its

characteristic. *Right Hand Side (R.H.S.) is zero.*

Note 2. Important. The student should notice the close analogy between the equation of a straight line $(Ax+By+C=0)$ in Analytical Plane Geometry and that of a plane $(Ax+By+Cz+D=0)$

---

*Called the *general equation* of the first degree in x, y, z because it contains all possible terms of the first and lower degrees in x, y, z, *i.e.*, terms of the first degree in x, y, z and constant term.

†**Plane. Euclid's Def.** A plane is a surface such that, if *any two* points are taken in it, the straight line joining them, or this straight line produced however far both ways, lies *wholly* in the surface. (See the author's *New Elementary Geometry* (Thirteenth Edition), Art. 3, (*a*).)

in Analytical Solid Geometry. He will constantly come across examples of this kind. **He should make use of this analogy as an aid to memory for standard results in plane.**

Cor. One-point form. *The equation of any plane through* (x_1, y_1, z_1) *is*

$$\mathbf{A(x-x_1)+B(y-y_1)+C(z-z_1)=0.}$$

Proof. Let the required equation of the plane be

$$Ax+By+Cz+D=0 \quad ...(1) \qquad [\text{ Art. 21 }]$$

$\because$ it passes thro' (x_1, y_1, z_1),

$$\therefore \quad Ax_1+By_1+Cz_1+D=0 \quad ...(2)$$

Subtracting (2) from (1), [To eliminate D]

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0,$$

which is the required equation.

Note. One-point form. Since in the equation

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0,$$

(x_1, y_1, z_1) is *one point* through which the plane passes, this form of the equation of a plane may be called the **one-point form.**

EXAMPLES

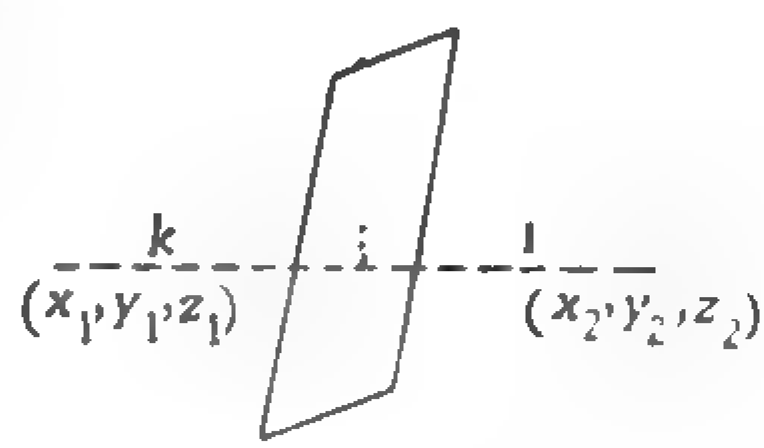
1. (*a*) Show that the surface represented by the general equation of the first degree is a plane.

(*b*) **Show that the plane ax+by+cz+d=0 divides the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio** $-\dfrac{\mathbf{ax_1+by_1+cz_1+d}}{\mathbf{ax_2+by_2+cz_2+d}}$.

(*b*) The equation of the plane is

$$ax+by+cz+d=0 \quad ...(1)$$

Let the required ratio be $k:1$. The pt. which divides the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $k:1$ is



$$\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right).$$

$\because$ it lies on the plane (1),

$$\therefore \; a\left(\frac{kx_2+x_1}{k+1}\right)+b\left(\frac{ky_2+y_1}{k+1}\right)+c\left(\frac{kz_2+z_1}{k+1}\right)+d=0$$

or, multiplying thro' out by $(k+1)$,

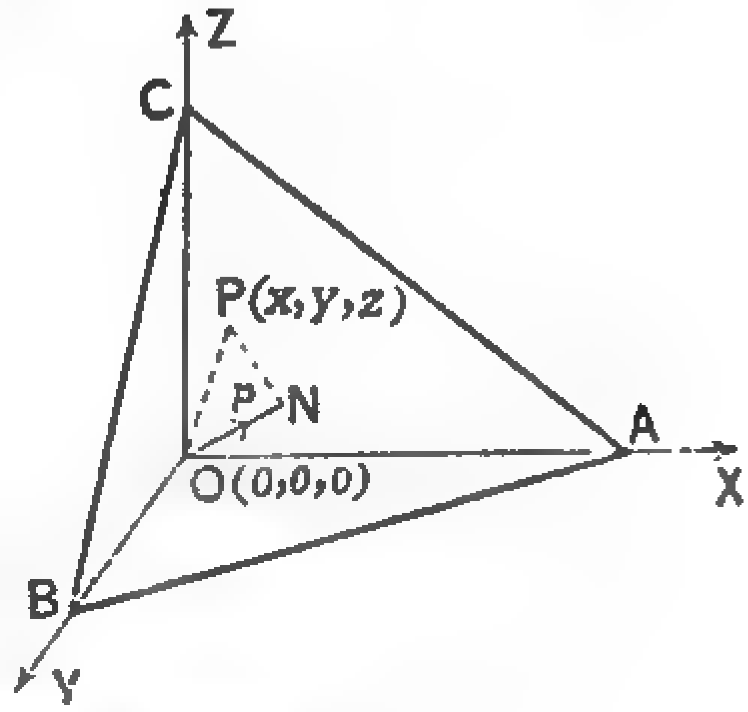
$$a(kx_2+x_1)+b(ky_2+y_1)+c(kz_2+z_1)+(k+1)d=0$$

or $\quad k\,(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0$

$$\therefore \; k=-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d}.$$

2. Find the ratios in which the co-ordinate planes divide the line joining (−2, 4, 7), (3, −5, 8). [*P(P). U. 1957*]

22. *Normal form.* **To find the equation of a plane in terms of p, the length of the perpendicular from the origin on the plane, and l, m, n, the direction-cosines of this perpendicular.**



Let ABC be the plane, and ON the ⊥ from O on it, so that $ON=p$, and its direction-cosines are l, m, n.

Let $P(x, y, z)$ be *any* pt. on the plane. Join OP, PN.

Then ON = projection of OP on ON

∴ $p=(x-0)\, l+(y-0)\, m+(z-0)\, n$

$[(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$ (Art. 14)]

or $\quad \mathbf{lx+my+nz=p},$

which is the required equation.

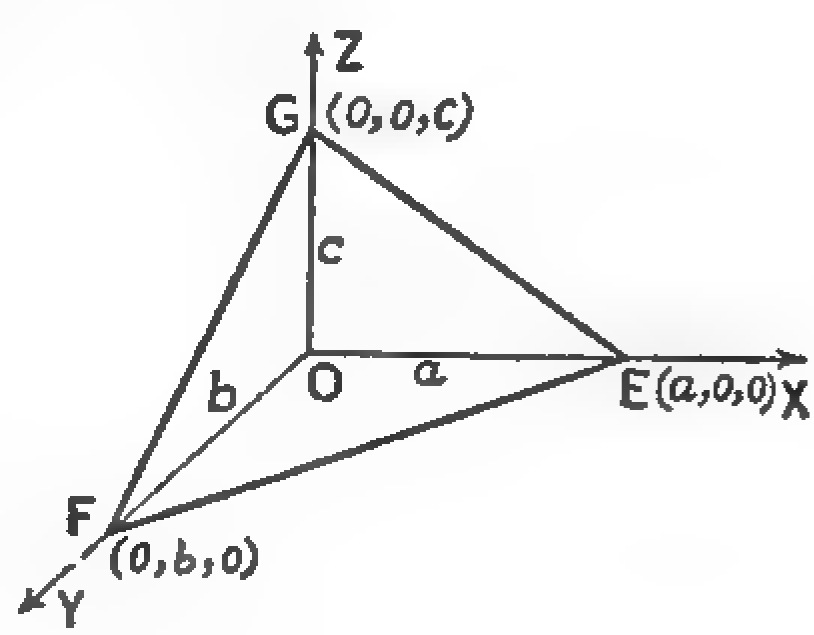
Cor. *The equation of any plane is of the first degree in x, y, z.*

The converse of this is also true and has been proved in Art. 21.

Note 1. Normal form. Since in the equation $lx+my+nz=p$, p is the length of the *normal from* the origin *to* the plane, this form of the equation of a plane may be called the **normal form.**

Note 2. Important. p is always positive.

23. *Intercept form.* **To find the equation of a plane in terms of a, b, c, the intercepts of the plane on the axes.**



Let EFG* be the plane meeting the axes in E, F, G, so that

$OE=a, OF=b, OG=c.$

Let the required equation of the plane be

$Ax+By+Cz+D=0$ ...(1) [Art. 21]

∵ it passes thro' the pts. $E(a, 0, 0)$, $F(0, b, 0)$, $G(0, 0, c)$,

∴ $Aa+D=0$ ...(2)

$Bb+D=0$ ...(3)

$Cc+D=0$ ...(4)

Solving (2), (3), (4) for A, B, C (in terms of D),

$$A=-\frac{D}{a}, B=-\frac{D}{b}, C=-\frac{D}{c}.$$

*And not ABC to avoid confusion with the coefficients A, B, C in the required equation of the plane $Ax+By+Cz+D=0$ supposed later in (1).

Substituting these values of A, B, C in (1),

$$-\frac{D}{a}x-\frac{D}{b}y-\frac{D}{c}z+D=0.$$

Dividing thro' out by D, $-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$

or, transposing,

$$\frac{\mathbf{x}}{\mathbf{a}}+\frac{\mathbf{y}}{\mathbf{b}}+\frac{\mathbf{z}}{\mathbf{c}}=\mathbf{1}, \text{ which is the required equation.}$$

Note. Intercept form. Since in the equation $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, a, b, c are the *intercepts* of the plane on the axes, this form of the equation of a plane may be called the **intercept form.**

24. Reduction of the general equation to the normal form. *To reduce the equation $Ax+By+Cz+D=0$ to the normal form*

$$lx+my+nz=p.$$

The given equation is $\quad Ax+By+Cz+D=0 \quad \ldots (1)$

The normal form is $\quad lx+my+nz=p \quad \ldots (2)$

or $\quad lx+my+nz-p=0 \quad \ldots (3)$

[To find the values of l, m, n, p in terms of A, B, C, D.]

If (1) is the same as (3), then comparing coeffs. in (3) and (1),

$$\frac{l}{A}=\frac{m}{B}=\frac{n}{C}=-\frac{p}{D} \quad \ldots (4)$$

[Take each member with the last member]

$$\therefore \quad l=-\frac{A}{D}p,\ m=-\frac{B}{D}p,\ n=-\frac{C}{D}p \quad \ldots (5)$$

Squaring and adding,

$$l^2+m^2+n^2=\frac{A^2}{D^2}p^2+\frac{B^2}{D^2}p^2+\frac{C^2}{D^2}p^2$$

or

$$1=p^2\left[\frac{A^2+B^2+C^2}{D^2}\right], \text{ or } p^2=\frac{D^2}{A^2+B^2+C^2}$$

$$\therefore \quad p=\pm\frac{D}{\sqrt{A^2+B^2+C^2}} \quad \ldots (6)$$

Case I. If D is +ve, then from (6), $\because$ p is always +ve,

$$p=\frac{D}{\sqrt{A^2+B^2+C^2}}.$$

Substituting this value of p in (5),

$$l=-\frac{A}{\sqrt{A^2+B^2+C^2}},\ m=-\frac{B}{\sqrt{A^2+B^2+C^2}},\ n=-\frac{C}{\sqrt{A^2+B^2+C^2}}.$$

Substituting these values of l, m, n, p in (2),

$$-\frac{A}{\sqrt{A^2+B^2+C^2}}x-\frac{B}{\sqrt{A^2+B^2+C^2}}y-\frac{C}{\sqrt{A^2+B^2+C^2}}z=\frac{D}{\sqrt{A^2+B^2+C^2}},$$

which is of the required form.

Case II. If D is $-$ve, then from (6), $\because p$ is always $+$ve,

$$p=-\frac{D}{\sqrt{A^2+B^2+C^2}}.$$

Substituting this value of p in (5),

$$l=\frac{A}{\sqrt{A^2+B^2+C^2}},\ m=\frac{B}{\sqrt{A^2+B^2+C^2}},\ n=\frac{C}{\sqrt{A^2+B^2+C^2}}.$$

Substituting these values of l, m, n, p in (2),

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y+\frac{C}{\sqrt{A^2+B^2+C^2}}z=-\frac{D}{\sqrt{A^2+B^2+C^2}},$$

which is of the required form.

[**Rule to reduce the general equation of a plane,**

$$\mathbf{Ax+By+Cz+D=0},$$

to the normal form, $\mathbf{lx+my+nz=p}$:

(*i*) *Divide the equation by* $\sqrt{A^2+B^2+C^2}$,

i.e., by $\sqrt{(\text{coeff. of } x)^2+(\text{coeff. of } y)^2+(\text{coeff. of } z)^2}$.

(*ii*) *Transpose the constant term to the R.H.S., and make it positive (by changing the signs throughout, if necessary).*]

Cor. Important. Direction-cosines of the normal. *The direction-cosines of the normal from the origin to the plane*

$$Ax+By+Cz+D=0$$

are proportional to A, B, C, i.e., proportional to the **coefficients of x, y, z.**

For, from (4), l, m, n are proportional to A, B, C.

EXAMPLES

1. Find the intercepts made on the co-ordinate axes by the plane $2x+y-2z=3$. Find also the direction-cosines of the normal to the plane.

[**Rule to find the intercepts of a plane on the axes :**

The plane meets the x-axis where, putting $y=0$, $z=0$ in the equation of the plane (Art. 19, Cor. 1), $x=?$ This is the intercept on the x-axis.

Similarly for the intercept on the y-axis and for the intercept on the z-axis.]

2. Find the direction-cosines of the perpendicular drawn from the origin to the plane $2x+3y-z+1=0$. [*P. U. 1941*]

25. A plane can be found to satisfy three conditions.

The general equation of a plane is $Ax+By+Cz+D=0$.

It contains four constants A, B, C, D.

Dividing thro' out by any one of them, say D, [provided $D \neq 0$]

$$\frac{A}{D}x+\frac{B}{D}y+\frac{C}{D}z+1=0$$

or, putting $\frac{A}{D}=A'$, $\frac{B}{D}=B'$, $\frac{C}{D}=C'$,

$$A'x+B'y+C'z+1=0,$$

which contains *three* arbitrary constants A', B', C'.

These constants can be determined from the three equations obtained by using the *three* conditions satisfied by the plane, each condition giving rise to one equation.

$\therefore$ a plane can be found to satisfy three conditions (*e.g.*, a plane can be found to pass through any three non-collinear points).

26. Three-point form. *To find the equation of the plane through* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

Let the required equation of the plane be

$$Ax+By+Cz+D=0 \quad \ldots(1)$$

$\because$ it passes thro' (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3),

$$\therefore \quad Ax_1+By_1+Cz_1+D=0 \quad \ldots(2)$$

$$Ax_2+By_2+Cz_2+D=0 \quad \ldots(3)$$

$$Ax_3+By_3+Cz_3+D=0 \quad \ldots(4)$$

Eliminating A, B, C, D from (1), (2), (3), (4) [by means of a determinant],

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix}=0,$$

which is the required equation.

Note 1. Three-point form. Since in the equation

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix}=0,$$

(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are the *three points* through which the plane passes, this form of the equation of a plane may be called the **three-point form.**

Note 2. Short cut for numerical examples. In numerical examples it is shorter to proceed as in Ex. 1, (*b*) following.

Note 3. It is assumed that the three points are not collinear.

EXAMPLES

1. (*a*) Find the equation of the plane through three given points.

(*b*) **Find the equation of the plane through the three points (0, 1, 1), (1, 1, 2), (−1, 2, −2).** [*P(P). U. 1957*]

(*b*) The equation of *any* plane thro' (0, 1, 1) is

$$A(x-0)+B(y-1)+C(z-1)=0$$

$$[\ A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \ \text{(Art. 21, Cor.)}\ ]$$

or $$Ax+B(y-1)+C(z-1)=0 \quad ...(1)$$

If it passes thro' (1, 1, 2), (−1, 2, −2), then

$$A(1)+B(0)+C(1)=0 \quad ...(2)$$

$$A(-1)+B(1)+C(-3)=0 \quad ...(3)$$

Solving (2) and (3) for A, B, C by cross-multiplication,

$$\frac{A}{0-1}=\frac{B}{-1-(-3)}=\frac{C}{1-0}, \text{ or } \frac{A}{-1}=\frac{B}{2}=\frac{C}{1}.$$

Substituting these values of A, B, C in (1),

$$(-1)x+2(y-1)+1(z-1)=0, \text{ or } -x+2y+z-3=0,$$

or $x-2y-z+3=0$, which is the required equation.

[**Check.** This equation of the plane, $x-2y-z+3=0$, is satisfied by the co-ordinates of the three given pts. (0,1,1), (1,1,2), (−1, 2, −2) thus,

$$0-2(1)-1+3=0, \quad \text{or} \quad 0=0,$$

$$1-2(1)-2+3=0, \quad \text{or} \quad 0=0,$$

$$-1-2(2)-(-2)+3=0, \quad \text{or} \quad 0=0. \ ]$$

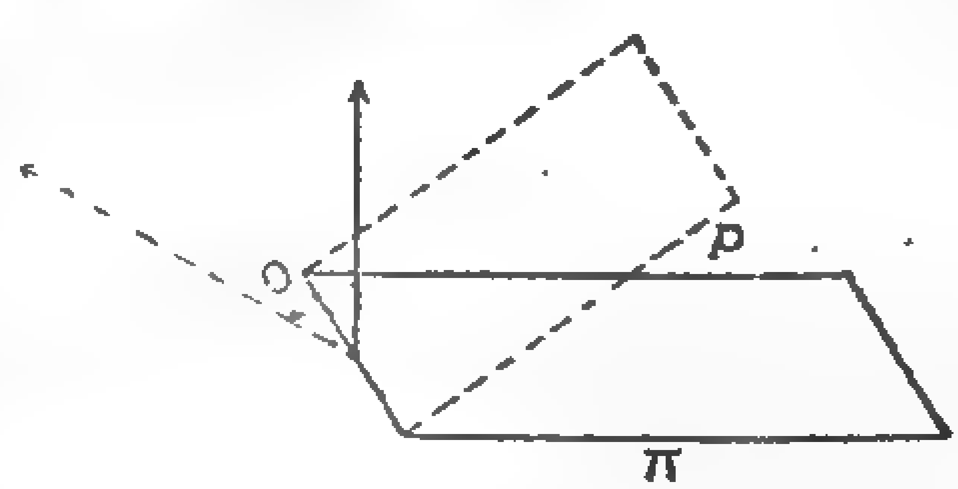
2. (*a*) **Show that the four points (0, −1, −1), (4, 5, 1), (3, 9, 4) and (−4, 4, 4) lie on a plane.** [*P. U. 1937 S*]

(*b*) **Show that the four points (0, −1, 0), (2, 1, −1), (1,1,1), (3, 3, 0) are coplanar.** [*D. U. H. 1950*]

3. **Prove that the six planes, each passing through one edge of a tetrahedron and bisecting the opposite edge, meet in a point.**

SECTION II

ANGLE BETWEEN TWO PLANES



[**Angle between two planes. Def.** The **angle between two planes** p and π (or the angle which a plane p makes with another plane π) is the angle which the positive direction of a normal to the plane p makes with the positive direction of a normal to the plane π.

Thus in the adjoining Fig. the angle between the planes p and π is θ.]

27. Angle formula for two planes. *To find the angle between two planes whose equations are given.*

Let the equations of the planes be

$$Ax+By+Cz+D=0 \quad ...(1)$$

$$A'x+B'y+C'z+D'=0 \quad ...(2)$$

The angle between two planes = the angle between their normals.

Now the direction-cosines of the normals to the planes (1) and (2) are proportional to A, B, C ; A′, B′, C′. [Art. 24, Cor.]

$\therefore$ if θ is the angle between the planes, then

$$\cos\theta=\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}} \qquad [\text{ Art. } 13, (b)\ ]$$

or

$$\theta=\cos^{-1}\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}}.$$

****Complete angle formula for two planes.** If θ is the angle between the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$, then

$$\cos\theta=\pm\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}}.$$

The equations of the planes are

$$Ax+By+Cz+D=0 \quad ...(1)$$

$$A'x+B'y+C'z+D'=0 \quad ...(2)$$

The angle between two planes = the angle between their normals.

Now the direction-cosines of the normals to the planes (1) and (2) are proportional to A, B, C ; A′, B′, C′. [Art. 24, Cor.]

$$\therefore \quad \cos\theta=\pm\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}}.$$

[Complete angle formula (Art. 13, (*b*))]

Note. $\theta=\cos^{-1}\left[\pm\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}}\right].$

Cor. 1. Condition of perpendicularity of two planes. *The condition that the planes*

$$Ax+By+Cz+D=0,$$

$$A'x+B'y+C'z+D'=0,$$

may be perpendicular, is

$$\mathbf{AA'+BB'+CC'=0.}$$

[In words : *product of coefficients of x + product of coefficients of y + product of coefficients of z = 0.*]

Proof. The equations of the planes are

$$Ax+By+Cz+D=0 \quad ...(1)$$

$$A'x+B'y+C'z+D'=0 \quad ...(2)$$

If two planes are $\perp$, their normals are also $\perp$.

Now the direction-cosines of the normals to the planes (1) and (2) are proportional to A, B, C ; A′, B′, C′. [Art. 24, Cor.]

$\therefore$ if the planes are $\perp$, $AA'+BB'+CC'=0$, [Art. 13, (*b*), Cor. 3] which is the required condition.

Note. The converse is also true, *i.e.*, if $AA'+BB'+CC'=0$, the planes are perpendicular.

For the order of the steps in the proof of Cor. 1 can be reversed.

Cor. 2. Conditions of parallelism of two planes. *The conditions that the planes*

$$Ax+By+Cz+D=0,$$
$$A'x+B'y+C'z+D'=0,$$

may be parallel, are

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}.$$

[In words : *ratio of coefficients of x=ratio of coefficients of y =ratio of coefficients of z.*]

Proof. The equations of the planes are

$$Ax+By+Cz+D=0 \quad ...(1)$$
$$A'x+B'y+C'z+D'=0 \quad ...(2)$$

If two planes are ||, their normals are also ||.

Now the direction-cosines of the normals to the planes (1) and (2) are proportional to A, B, C ; A′, B′, C′. [Art. 24, Cor.]

$\therefore$ if the planes are ||,

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}, \qquad \text{[Art. 13, (b), Cor. 4]}$$

which are the required conditions.

Note. The converse is also true, *i.e.*, if $\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}$, the planes are parallel.

For the order of the steps in the proof of Cor. 2 can be reversed.

Cor. 3. Equation of any plane parallel to a given plane. *The equation of any plane parallel to the plane*

$$Ax+By+Cz+D=0$$

is $\mathbf{Ax+By+Cz+k=0}$, *where k is any constant.*

Proof. The equations of the planes are

$$Ax+By+Cz+D=0 \quad ... (1)$$
$$Ax+By+Cz+k=0 \quad ... (2)$$

Here $\frac{\text{“A”}}{A'}=\frac{A}{A}=1,\ \frac{\text{“B”}}{B'}=\frac{B}{B}=1,\ \frac{\text{“C”}}{C'}=\frac{C}{C}=1$

$\therefore \quad \frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}, \quad \therefore$ the planes are $\parallel$. [Cor. 2]

[**Rule to write down the equation of** *any* **plane parallel to a given plane (equation in the general form) :**

In the equation of the given plane, change **only** *the constant term to a new constant k.*]

Note. The value of k is found from the second condition satisfied by the plane.

EXAMPLES

1. If the axes are rectangular, find the angle between the planes

(i) $2x-y+z=6,\ x+y+2z=3.$ [*D. U. H. 1945*]

(ii) $2x+3y-5z=6,\ 3x+8y+6z=9.$

2. **Find the equation of the plane through the points $(-1, 1, 1)$ and $(1, -1, 1)$ perpendicular to the plane $x+2y+2z=5.$** [*L. U.*]

3. **Prove that the equation to the plane through (x_1, y_1, z_1) parallel to $ax+by+cz+d=0$, is**

$$ax+by+cz = ax_1+by_1+cz_1.$$

The equation of *any* plane $\parallel$ to the plane $ax+by+cz+d=0$, is

$$ax+by+cz+k = 0 \quad \dots (1) \qquad \text{[Rule (Art. 27, Cor. 3)]}$$

If it passes thro' (x_1, y_1, z_1), then $ax_1+by_1+cz_1+k = 0$

$\therefore \quad k=-(ax_1+by_1+cz_1).$

Substituting this value of k in (1),

$$ax+by+cz-(ax_1+by_1+cz_1) = 0$$

or

$$ax+by+cz=ax_1+by_1+cz_1,$$

which is the required equation.

4. **Find the equation of the plane through $(0,1,-2)$ parallel to**

$$2x-3y+4z = 0.$$

5. **Angle formula (tangent form) for two planes.** *If θ is the angle between the planes $Ax+By+Cz+D=0,\ A'x+B'y+C'z+D'=0,$ then*

$$\tan\theta = \frac{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}}{AA'+BB'+CC'}.$$

The equations of the planes are

$$Ax+By+Cz+D = 0 \quad \dots (1)$$

$$A'x+B'y+C'z+D' = 0 \quad \dots (2)$$

The angle between two planes = the angle between their normals.

Now the direction-cosines of the normals to the planes (1) and (2) are proportional to A, B, C; A', B', C'. [Art. 24, Cor.]

$$\therefore \tan\theta = \frac{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}}{AA'+BB'+CC'} \quad \dots (3)$$

[Art. 13, (*b*), Cor. 2]

****Complete angle formula (tangent form) for two planes.** *If θ is the angle between the planes*

$$Ax+By+Cz+D=0, \ A'x+B'y+C'z+D'=0,$$

then $$\tan\theta = \pm\frac{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}}{AA'+BB'+CC'}.$$

The equations of the planes are

$$Ax+By+Cz+D=0 \ \ldots (1)$$
$$A'x+B'y+C'z+D'=0 \ \ldots (2)$$

The angle between two planes = the angle between their normals.

Now the direction-cosines of the normals to the planes (1) and (2) are proportional to A, B, C ; A′, B′, C′. [Art. 24, Cor.]

$$\therefore \ \tan\theta = \pm\frac{\sqrt{(BC'-B'C)^2+(CA'-C'A)^2+(AB'-A'B)^2}}{AA'+BB'+CC'}.$$

[Complete angle formula (tangent form), Art. 13, (*b*), Cor. 2]

6. **Position of the origin with respect to the angle between two planes.** *The quantity $AA'+BB'+CC'$ is negative or positive according as the origin is in the acute angle or obtuse angle between the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$, D, D' being both positive.*

Proof. The equations of the planes are

$$Ax+By+Cz+D=0 \ \ldots (1)$$
$$A'x+B'y+C'z+D'=0 \ \ldots (2)$$

[To reduce (1) and (2) to the normal form.]

Dividing (1) thro' out by $\sqrt{A^2+B^2+C^2}$,

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y+\frac{C}{\sqrt{A^2+B^2+C^2}}z+\frac{D}{\sqrt{A^2+B^2+C^2}}=0.$$

Transposing,

$$-\frac{A}{\sqrt{A^2+B^2+C^2}}x-\frac{B}{\sqrt{A^2+B^2+C^2}}y-\frac{C}{\sqrt{A^2+B^2+C^2}}z=\frac{D}{\sqrt{A^2+B^2+C^2}},$$

[Constant term on R. H. S. and +ve ($\because$ D is +ve)]

which is the equation of the plane (1) in the normal form.

Similarly the equation of the plane (2), in the normal form, is

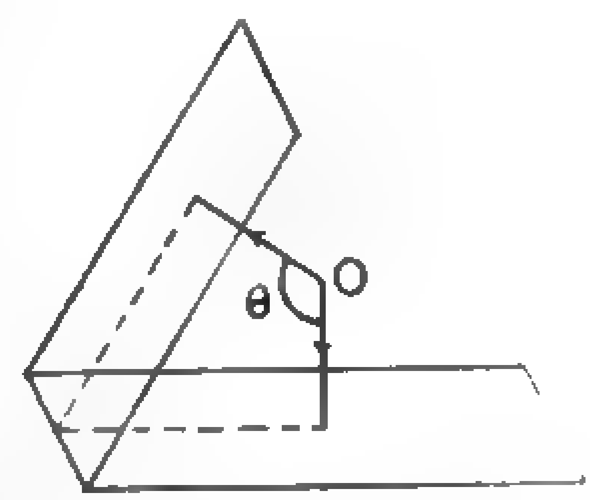
$$-\frac{A'}{\sqrt{A'^2+B'^2+C'^2}}x-\frac{B'}{\sqrt{A'^2+B'^2+C'^2}}y-\frac{C'}{\sqrt{A'^2+B'^2+C'^2}}z$$
$$=\frac{D'}{\sqrt{A'^2+B'^2+C'^2}}.$$

$\therefore$ if θ is the angle between the normals from the origin to the

planes, then

$$\cos\theta = \frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\ \sqrt{A'^2+B'^2+C'^2}} \quad \ldots (1)$$

[$ll'+mm'+nn'$ (Art. 13, (*a*))]



If the origin is in the acute angle between the planes as in the Fig., θ, the angle between the normals *from* the origin *to* the planes is *obtuse*,

$\therefore$ $\cos\theta$ is $-$ve,

$\therefore$ from (1), $AA'+BB'+CC'$ is $-$ve.

Similarly if the origin is in the obtuse angle between the planes, $AA'+BB'+CC'$ is $+$ve.

Note. The converse is also true, *i.e.*, if $AA'+BB'+CC'$ is **negative** (D, D′ being both positive), the origin is in the **acute** angle between the planes, and if $AA'+BB'+CC'$ is positive (D, D′ being both positive), the origin is in the obtuse angle between them.

For the order of the steps in the above proof can be reversed.

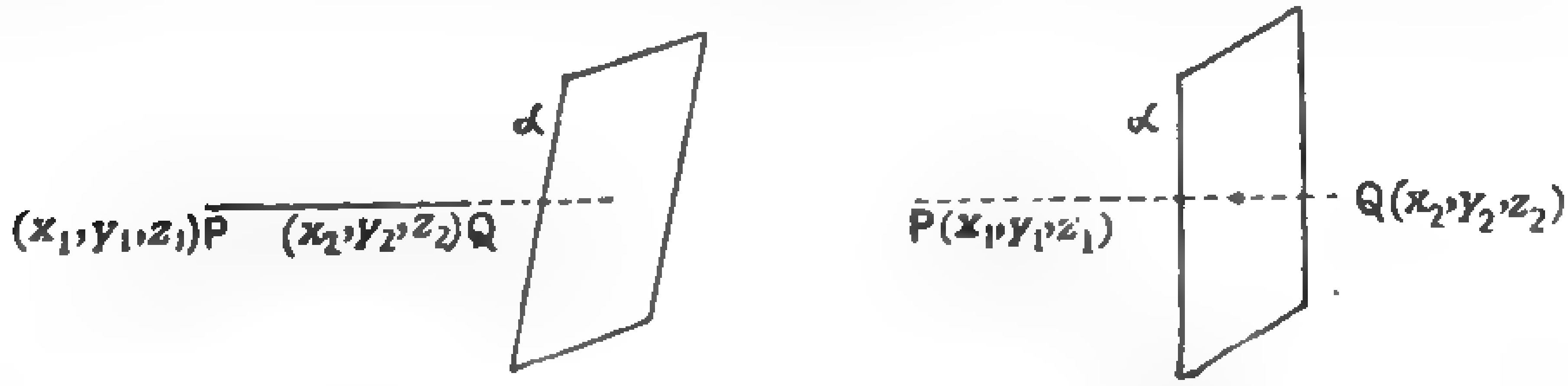
7. At what angle do the planes $x+y-z=3$, $x-2y+z=3$ cut? Is the origin in the acute angle or in the obtuse?

SECTION III

A PLANE AND A POINT

Two sides of a plane.

28. To prove that two points (x_1, y_1, z_1) and (x_2, y_2, z_2) are on the same side or on opposite sides of the plane $ax+by+cz+d=0$ according as $ax_1+by_1+cz_1+d$ and $ax_2+by_2+cz_2+d$ are of the same sign or of opposite signs.



(*i*) (*ii*)

Let P, Q be the pts. (x_1, y_1, z_1), (x_2, y_2, z_2) and α the plane

$$ax+by+cz+d=0 \quad \ldots (1)$$

Let PQ be divided by the plane α in the ratio $k:1$.

The pt. which divides PQ in the ratio $k:1$ is

$$\left(\frac{kx_2+x_1}{k+1}, \frac{ky_2+y_1}{k+1}, \frac{kz_2+z_1}{k+1}\right).$$

$\because$ it lies on the plane (1),

$$\therefore \quad a\left(\frac{kx_2+x_1}{k+1}\right)+b\left(\frac{ky_2+y_1}{k+1}\right)+c\left(\frac{kz_2+z_1}{k+1}\right)+d=0$$

or, multiplying thro' out by $k+1$,

$$a(kx_2+x_1)+b(ky_2+y_1)+c(kz_2+z_1)+(k+1)d=0$$

or $$k(ax_2+by_2+cz_2+d)+(ax_1+by_1+cz_1+d)=0$$

or $$k=-\frac{ax_1+by_1+cz_1+d}{ax_2+by_2+cz_2+d} \quad ...(2)$$

(*i*) If $ax_1+by_1+cz_1+d$, $ax_2+by_2+cz_2+d$ are of the same sign, then from (2), k is $-$ve,

$\therefore$ PQ is divided externally by the plane α as in Fig. (*i*), *i.e.*, P,Q are on the same side of the plane α.

(*ii*) If $ax_1+by_1+cz_1+d$, $ax_2+by_2+cz_2+d$ are of opposite signs, then from (2), k is $+$ve,

$\therefore$ PQ is divided internally by the plane α as in Fig. (*ii*), *i.e.*, P,Q are on opposite sides of the plane α.

[**Rule to find whether two points are on the same side or on opposite sides of a plane :**

In the L. H. S. of the equation of the plane (R. H. S. being zero) substitute in succession the co-ordinates of the two points ; if the results are of the same sign, the points are on the same side of the plane ; if the results are of opposite signs, the points are on opposite sides.]

Cor. *If D is positive, the quantity $Ax_1+By_1+Cz_1+D$ is positive if (x_1, y_1, z_1) and the origin are on the same side of the plane $Ax+By+Cz+D=0$, and negative if they are on opposite sides.*

If (x_1, y_1, z_1) and the origin (0, 0, 0) are on the same side of the plane $Ax+By+Cz+D=0$, then $Ax_1+By_1+Cz_1+D$ and $A(0)+B(0)+C(0)+D$, *i.e.*, D are of the same sign (Art. 28).

But D is $+$ve (Given), $\therefore$ $Ax_1+By_1+Cz_1+D$ is also $+$ve.

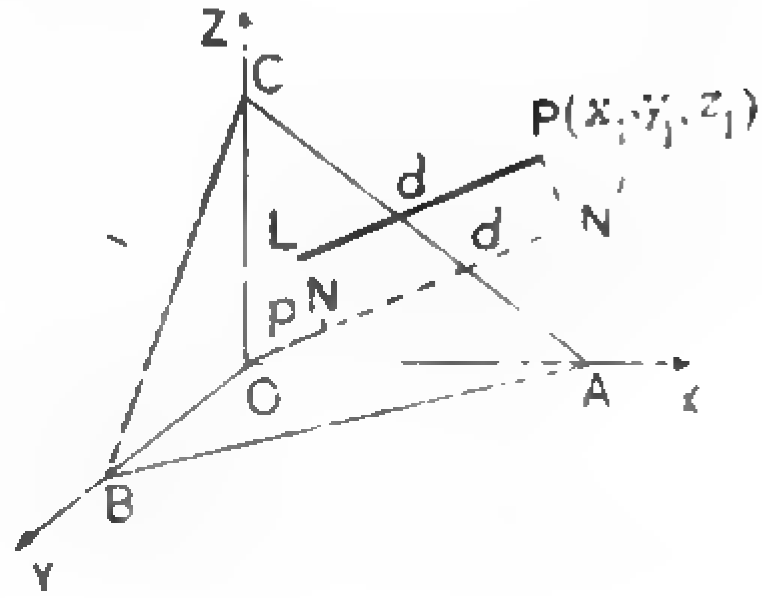
Similarly if (x_1, y_1, z_1) and the origin are on opposite sides of the plane $Ax+By+Cz+D=0$, then $Ax_1+By_1+Cz_1+D$ is $-$ve.

EXAMPLE

Are the points (2, 1, 1) and (2, 5, -1) on the same or on opposite sides of the plane $x-2y-3z+4=0$?

Perpendicular distance of a point from a plane.

29. (*a*) *Perpendicular distance formula for the plane (equation in the normal form).* **To find the perpendicular distance of the point (x_1, y_1, z_1) from the plane $lx+my+nz=p$.**



Let P be the pt. (x_1, y_1, z_1), and ABC the plane $lx+my+nz=p$.

From P draw PL $\perp$ on the plane ABC.

Let d be the required $\perp$ distance LP.

Let ON be the $\perp$ from O on the plane ABC, so that ON$=p$, and its direction-cosines are l, m, n.

Thro' P draw a plane (not shown in the Fig.) $\parallel$ to the plane ABC to meet ON produced in N'.

Then ON'$=$ON$+$NN'$=$ON$+$LP$=p+d$, and its direction-cosines are l, m, n.

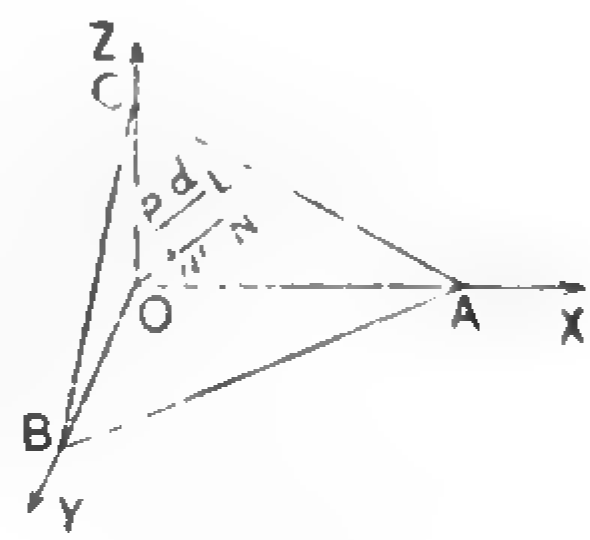
$\therefore$ the equation of this $\parallel$ plane is $lx+my+nz=p+d$.　　[Art. 22]

$\because$ it passes thro' P (x_1, y_1, z_1),

$\therefore$ $$lx_1+my_1+nz_1=p+d$$

$\therefore$ $$\mathbf{d=lx_1+my_1+nz_1-p}.$$

****Complete perpendicular distance formula for the plane (equation in the normal form).** In the Fig. of Art. 29, (*a*), $P(x_1, y_1, z_1)$ and the origin O are on opposite sides of the plane. If, however, they are on the *same* side of the plane as in the adjoining Fig., then proceeding as in Art. 29, (*a*), it will be found that



$$d=p-lx_1-my_1-nz_1$$

$\therefore$ $$d=-(lx_1+my_1+nz_1-p).$$

Combining this with the formula of Art. 29, (*a*),

$$\mathbf{d=\pm(lx_1+my_1+nz_1-p)},$$

that sign being taken on the R.H.S. which gives a *positive* result for d.

[**Rule to find the perpendicular distance of a point from a plane (equation in the normal form) :**

In the L. H. S. of the equation of the plane (R. H. S. being zero), substitute the co-ordinates of the point. The result gives the $\perp$ distance.]

EXAMPLES

1. Find the distance of the point $P(x', y', z')$ from the plane
$$p=x\cos\alpha+y\cos\beta+z\cos\gamma.$$

2. Prove that the locus of a point which moves so that its distances from two given planes are in a constant ratio, is a plane.

[Note. Important. For problems relating to perpendicular distances of points from planes, take the equations of the planes in the normal form.]

3. Prove that the locus of a point, the sum of whose distances from any number of fixed planes is constant, is a plane.

29. (b) *Perpendicular distance formula for the plane (equation in the general form).* **To find the perpendicular distance of the point (x_1, y_1, z_1) from the plane $Ax+By+Cz+D=0$.**

The equation of the plane is $Ax+By+Cz+D=0$.

[To reduce it to the normal form.]

Dividing thro' out by $\sqrt{A^2+B^2+C^2}$,

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y+\frac{C}{\sqrt{A^2+B^2+C^2}}z+\frac{D}{\sqrt{A^2+B^2+C^2}}=0.$$

Transposing,

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y+\frac{C}{\sqrt{A^2+B^2+C^2}}z=-\frac{D}{\sqrt{A^2+B^2+C^2}},$$

which is of the normal form

[Taking D to be $-$ve, so that the R. H. S. is $+$ve]

or $$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y+\frac{C}{\sqrt{A^2+B^2+C^2}}z+\frac{D}{\sqrt{A^2+B^2+C^2}}=0.$$

[Normal form with R. H. S. zero]

$\therefore$ the $\perp$ distance of (x_1, y_1, z_1) is

$$d=\frac{A}{\sqrt{A^2+B^2+C^2}}x_1+\frac{B}{\sqrt{A^2+B^2+C^2}}y_1+\frac{C}{\sqrt{A^2+B^2+C^2}}z_1+\frac{D}{\sqrt{A^2+B^2+C^2}} \quad \dots (1)$$

[Rule (Art. 29, (a))]

$$=\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}.$$

****Complete perpendicular distance formula for the plane (equation in the general form) :**

In the above work, the ordinary perpendicular distance formula for the plane (equation in the normal form) has been used in (1). If, however, the *complete* perpendicular distance formula (Art. 29, (*a*)) is used, it will be found that

$$\mathbf{d} = \pm \frac{\mathbf{Ax_1 + By_1 + Cz_1 + D}^*}{\sqrt{\mathbf{A^2+B^2+C^2}}},$$

that sign being taken on the R. H. S. which gives a positive result for *d*.

[**Aid to memory.** For the complete perpendicular distance formula for the plane (like the complete angle formula for two planes), take the double sign ($\pm$) with the ordinary perpendicular distance formula.]

[**Rule to find the perpendicular distance of a point from a plane (equation in the general form) :**

In the L. H. S. of the equation of the plane (R. H. S. being zero), substitute the co-ordinates of the point, and divide the result by

$$\sqrt{(\textit{coeff. of } x)^2 + (\textit{coeff. of } y)^2 + (\textit{coeff. of } z)^2}\,.$$

The result gives the perpendicular distance.

Note 1. This Rule also gives the result of Art. 29, (*a*), and, therefore, covers the Rule of that Art.]

Note 2. When to use the complete perpendicular distance formula for the plane. The complete perpendicular distance formula for the plane is used when the perpendicular distance of the point from the plane is *given.*

EXAMPLES

1. (*a*) **Find the perpendicular distance of a given point from a given plane.** [*D. U. H. 1939*]

(*b*) If the axes are rectangular, find the distance of the origin from the plane $2x-y+2z-6=0$.

2. *Show that the two points (1, 1, 1) and (−3, 0, 1) are equidistant from the plane* $3x+4y-12z+13=0$, *and on opposite sides of it.*

3. *Find the distances of the points (1, −2, 3), (2, 3, 3) from the plane* $x-2y+2z=5$. *Are the points on the same side of the plane ?*

4. Find the locus of a point whose distance from the origin is 7 times its distance from the plane $2x+3y-6z=2$. [*P.U. Eng. 2, 1942*]

5. Find the locus of a point the sum of the squares of whose distances from the planes $x+y+z=0$, $x-y=0$, $x+y-2z=0$, is 4.

6. *A variable plane is at a constant distance p from the origin and meets the axes in A, B, C. Through A, B, C planes are drawn parallel to the co-ordinate planes. Prove that the locus of their point of intersection is* $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

[Note. Important. For problems relating to a plane meeting the axes in A, B, C, let OA=a, OB=b, OC=c. Then the equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1. \quad \textbf{(Art. 23)]}$$

30. Equations of planes bisecting the angles between two given planes. *To find the equations of the planes bisecting the angles between the planes $Ax+By+Cz+D=0$, $A'x+B'y+C'z+D'=0$.*

Writing the equations of the planes so that the *constant terms D, D' are both +ve,* let the equations be

$$Ax+By+Cz+D=0 \quad \dots (1)$$

$$A'x+B'y+C'z+D'=0 \quad \dots (2)$$

Let (x, y, z) be *any* pt. on either of the planes bisecting the angles between the two planes.

Then the $\perp$ distance of (x, y, z) from the plane (1)

= the $\perp$ distance of (x, y, z) from the plane (2)

$$\therefore \quad \frac{Ax+By+Cz+D^*}{\sqrt{A^2+B^2+C^2}}=\pm\frac{A'x+B'y+C'z+D'}{\sqrt{A'^2+B'^2+C'^2}},$$

[Complete $\perp$ distance formula (Art. 29, (*b*))]

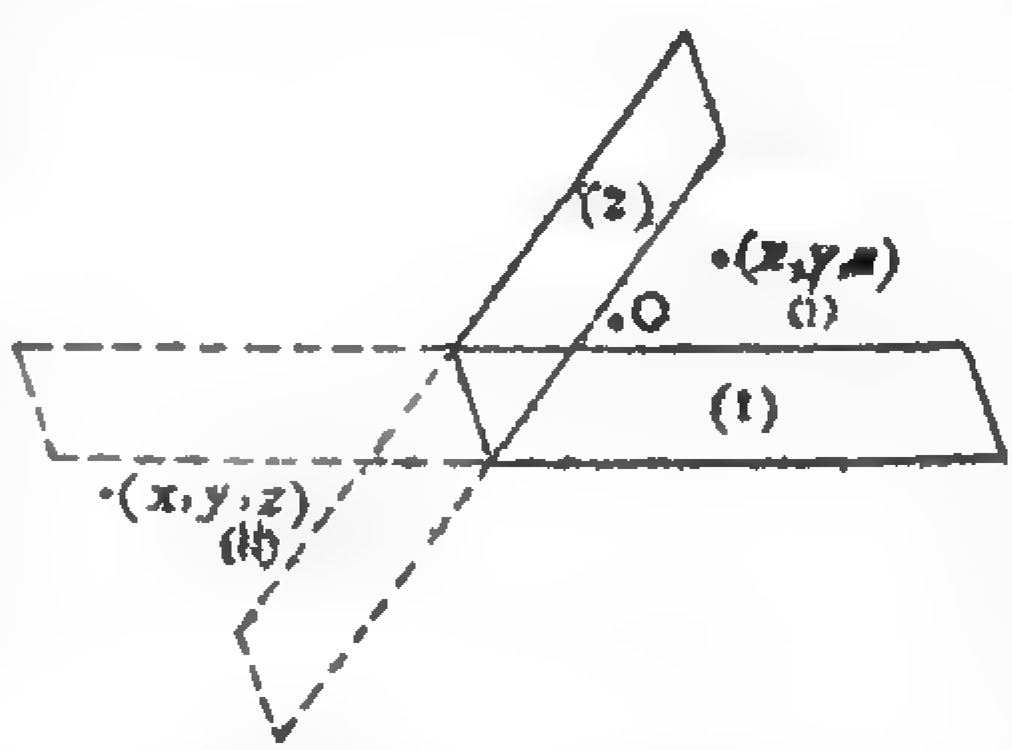
which are the required equations of the bisecting planes.

To distinguish between the two bisecting planes.

To find the equation of the plane bisecting that angle between the two planes in which the origin lies.

Let (x, y, z) be *any* pt. on the plane bisecting that angle between the two planes in which the origin lies.

Then (x, y, z) and the origin lie on the same side of *both* the planes as in Fig. (i), or (x, y, z) and the origin lie on opposite sides of *both* the planes as in Fig. (ii).



(i) If (x, y, z) and the origin lie on the same side of the plane (1), then $Ax+By+Cz+D$ and $(A.0+B.0+C.0+D$, *i.e.,*) D are of the same sign. [Art. 28]

*Note. It is superfluous to take the double sign (±) with *both* sides of the equation.

For, if $\pm A=\pm B$, then either (*i*) $A=\pm B$,
or (*ii*) $-A=\pm B$, $\therefore A=\mp B$, *i.e.,* $A=\pm B$, which is the same as (*i*).

But D is +ve, $\therefore$ $Ax+By+Cz+D$ is also +ve.

$\therefore$ the $\perp$ distance of (x, y, z) from the plane (1)

$$=\frac{Ax+By+Cz+D}{\sqrt{A^2+B^2+C^2}}.$$

Similarly the $\perp$ distance of (x, y, z) from the plane (2)

$$=\frac{A'x+B'y+C'z+D'}{\sqrt{A'^2+B'^2+C'^2}}.$$

$$\therefore\quad \frac{Ax+By+Cz+D}{\sqrt{A^2+B^2+C^2}}=\frac{A'x+B'y+C'z+D'}{\sqrt{A'^2+B'^2+C'^2}}.$$

(ii) If (x, y, z) and the origin lie on opposite sides of the plane (1), then $Ax+By+Cz+D$ and $(A.0+B.0+C.0+D$, *i.e.,*) D are of opposite signs.

But D is +ve, $\therefore$ $Ax+By+Cz+D$ is $-$ve. [Art. 28]

$\therefore$ the $\perp$ distance of (x, y, z) from the plane (1)

$$=-\frac{Ax+By+Cz+D}{\sqrt{A^2+B^2+C^2}}.$$

Similarly the $\perp$ distance of (x, y, z) from the plane (2)

$$=-\frac{A'x+B'y+C'z+D'}{\sqrt{A'^2+B'^2+C'^2}}.$$

$$\therefore\quad -\frac{Ax+By+Cz+D}{\sqrt{A^2+B^2+C^2}}=-\frac{A'x+B'y+C'z+D'}{\sqrt{A'^2+B'^2+C'^2}}$$

or
$$\frac{\mathbf{Ax+By+Cz+D}}{\sqrt{\mathbf{A^2+B^2+C^2}}}=\frac{\mathbf{A'x+B'y+C'z+D'}}{\sqrt{\mathbf{A'^2+B'^2+C'^2}}},$$

which is, therefore, the required equation of the plane bisecting that angle between the two planes in which the origin lies.

Similarly *the equation of the plane bisecting that angle between the two planes in which the origin does not lie, is*

$$\frac{\mathbf{Ax+By+Cz+D}}{\sqrt{\mathbf{A^2+B^2+C^2}}}=-\frac{\mathbf{A'x+B'y+C'z+D'}}{\sqrt{\mathbf{A'^2+B'^2+C'^2}}}.$$

EXAMPLES

1. Find the planes bisecting the angles between the planes $\mathbf{ax+by+cz+d=0}$, $\mathbf{a'x+b'y+c'z+d'=0}$, the axes being rectangular. [*P(P). U. 1956 S*]

2. Show that the origin lies in the acute angle between the planes $\mathbf{x+2y+2z=9}$, $\mathbf{4x-3y+12z+13=0}$. Find the planes bisecting the angles between them, and point out which bisects the acute angle. [*P(P). U. 1956 S*]

(*i*) The equations of the planes are

$$x+2y+2z-9=0,\ 4x-3y+12z+13=0.$$

Writing the equations of the planes so that the *constant terms are both +ve,*

$$-x-2y-2z+9=0 \quad ...(1)$$
$$4x-3y+12z+13=0 \quad ...(2)$$

[Compare (1) with $Ax+By+Cz+D=0$,
and (2) with $A'x+B'y+C'z+D'=0$]

Here $AA'+BB'+CC'=(-1)(4)+(-2)(-3)+(-2)(12)$
$=-4+6-24=-22$, which is $-$ve.

$\therefore$ the origin lies in the acute angle between the planes.
[Ex. 6, Art. 27]

(*ii*) From (1) and (2), the equations of the planes bisecting the angles between the given planes are

$$\frac{-x-2y-2z+9}{\sqrt{(-1)^2+(-2)^2+(-2)^2}}=\pm\frac{4x-3y+12z+13}{\sqrt{(4)^2+(-3)^2+(12)^2}} \quad \text{[Art. 30]}$$

or $$\frac{-x-2y-2z+9}{3}=\pm\frac{4x-3y+12z+13}{13} \quad ...(3)$$

Taking +ve sign on the R.H.S. of (3), the equation of the plane bisecting that angle between the two planes in which the origin lies, *i.e.*, from part (*i*), the equation of the plane, which bisects the acute angle, is

$$\frac{-x-2y-2z+9}{3}=\frac{4x-3y+12z+13}{13}$$

or $-13x-26y-26z+117=12x-9y+36z+39$

or $25x+17y+62z-78=0 \quad ...(4)$

Taking $-$ve sign on the R.H.S. of (3), the equation of the other bisecting plane is

$$\frac{-x-2y-2z+9}{3}=-\frac{4x-3y+12z+13}{13}$$

or $-13x-26y-26z+117=-12x+9y-36z-39$

or $x+35y-10z-156=0 \quad ...(5)$

[**Check.** *The bisecting planes are perpendicular.*

Thus from (4) and (5),

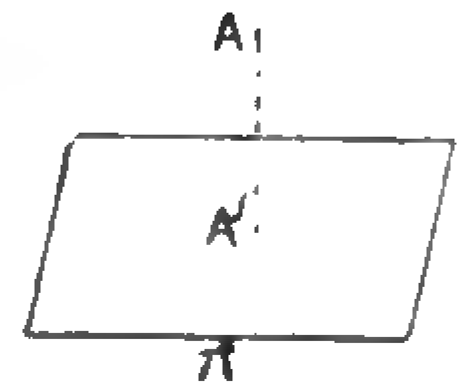
here $AA'+BB'+CC'=(25)(1)+(17)(35)+(62)(-10)$
$=25+595-620=620-620=0$,

$\therefore$ the bisecting planes are $\perp$.]

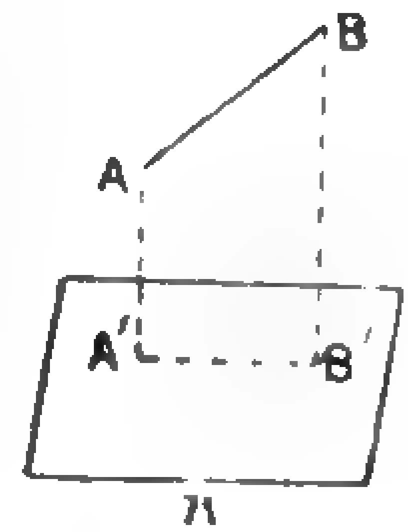
3. Find the bisector of the acute angle between the planes $2x-y+2z+3=0$, $3x-2y+6z+8=0$. [*P. U. B.Sc. H. 1943*]

SECTION IV

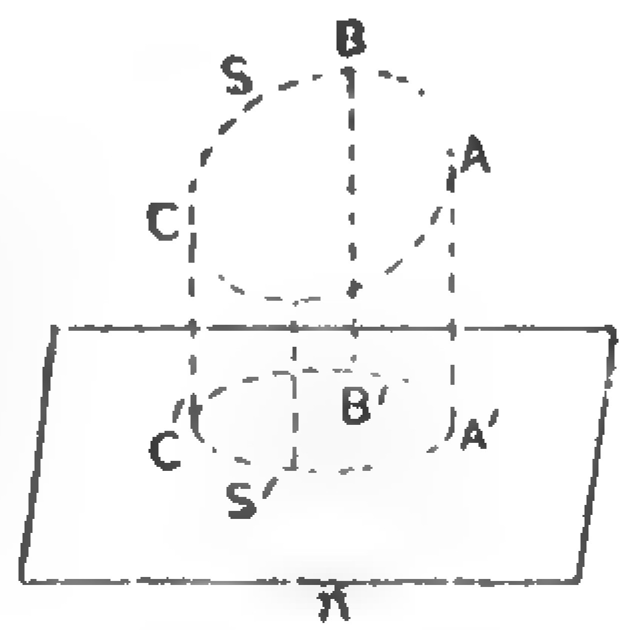
PROJECTION ON A PLANE



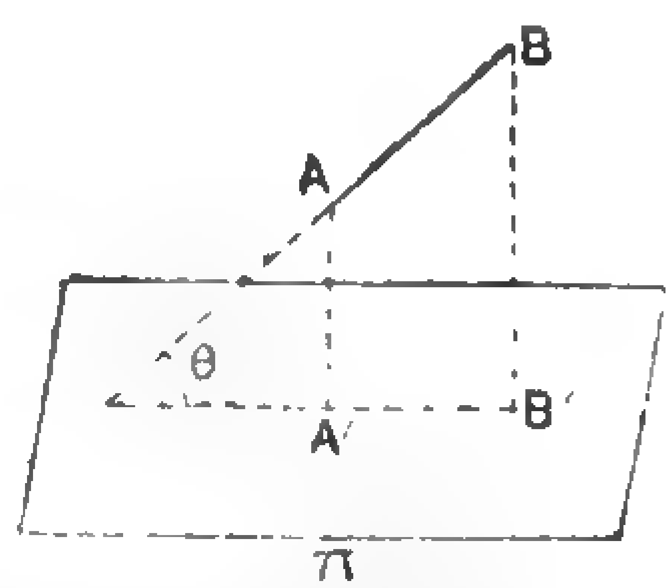
31. (*a*) **Projection of a point. Def.** *The* **projection* of a point A on a plane** π *is* A', *the foot of the perpendicular from* A *on the plane* π.



(*b*) **Projection of a segment. Def.** *The* **projection of a segment AB on a plane** π *is the segment* $A'B'$, *where* A', B' *are the feet of the perpendiculars from* A, B *on the plane* π.



(*c*) **Projection of a plane area. Def.** *The* **projection of a plane area S** *enclosed by a curve* ABC ... **on a plane** π *is the area* S' *enclosed by the curve* $A'B'C'$..., *where* A', B', C', ... *are the feet of the perpendiculars from* $A, B, C,$ ... *on the plane* π.



32. (*a*) **Length of the projection.** *The projection of a segment* AB *on a plane* π *is* $\mathbf{A'B'} = \mathbf{AB} \cos\theta$, *where* θ *is the angle which* AB *makes with the plane* π.

[From Elementary Geometry †]

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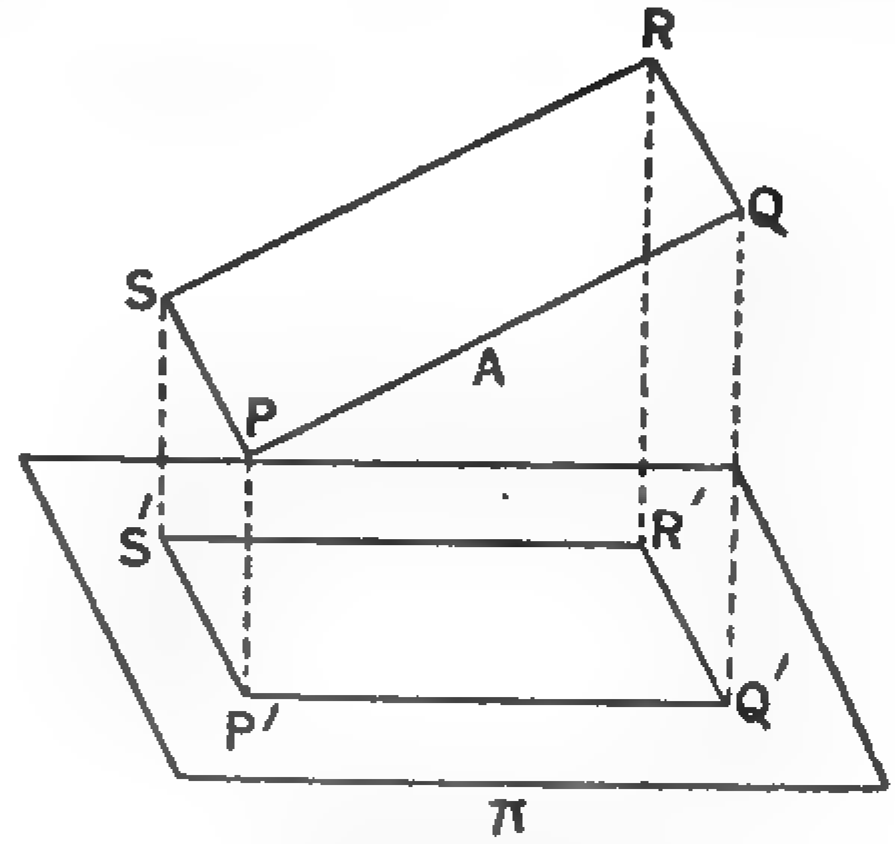
* Or, more fully, the orthogonal projection.

† See the author's *New Elementary Geometry*, Thirteenth Edition, Prop. 15, Cor. 2.

(b) **Area of the projection.** *The projection of a plane area A on a plane π is* $\mathbf{A'=A\cos\theta}$, *where θ is the angle which the plane of the area A makes with the plane π.*



****Proof.** Divide the area A into a very great number of narrow rectangles like PQRS by two sets of lines, one like PS ‖ to the line of intersection of the plane of the area A and the plane π, and the other set like PQ $\perp$ to this line of intersection. Let P′, Q′, R′, S′ be the feet of the $\perp$s from P, Q, R, S on the plane π.

Then the area of the projection, rectangle P′ Q′ R′ S′

$$= P'Q'.P'S' = PQ\cos\theta . PS$$

$$[\because P'S' = PS\cos\theta, \text{ here } \theta=0, \because PS \text{ is } \parallel \text{ to the plane } \pi]$$

$$= PQ.PS\cos\theta = \text{rect. PQRS}.\cos\theta$$

$$\therefore \quad \Sigma \text{ rect. } P'Q'R'S' = \Sigma \text{ rect. PQRS}.\cos\theta$$

$$= \cos\theta\, \Sigma \text{ rect. PQRS}.$$

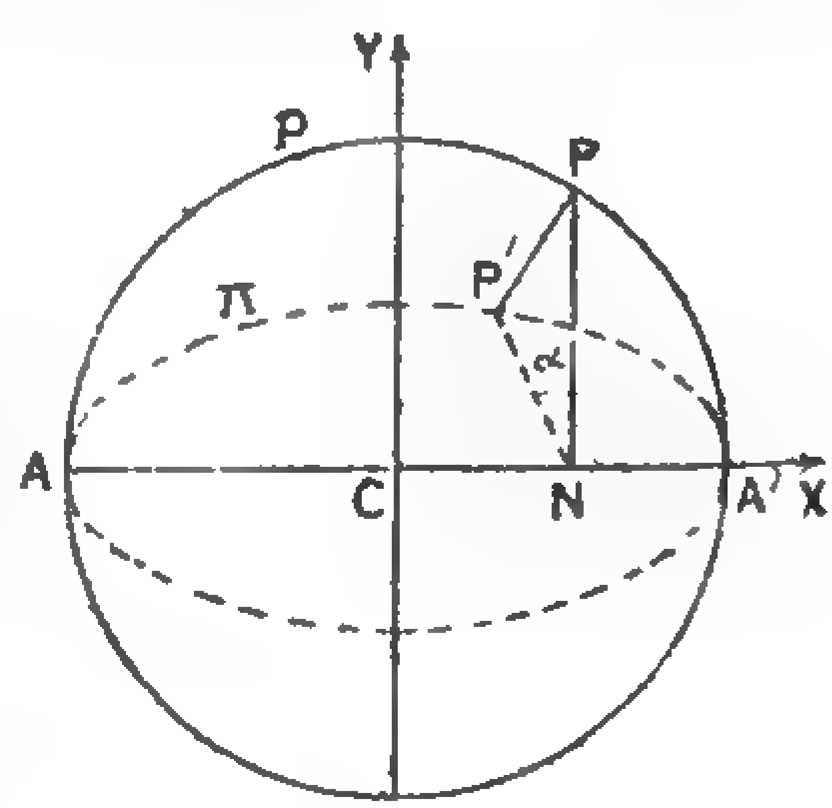
Taking the limits when the number of rects. $\to\infty$, so that the breadth of each rect. $\to 0$,

area A′ = $\cos\theta$. area A.

EXAMPLES

**1. *Prove, by projecting a circle, that the area of an ellipse whose major and minor axes are 2a, 2b, is πab.*

Let AA′ ($=2a$) be a diameter of the circle in the plane p and C its centre.



Let P($a\cos\theta$, $a\sin\theta$) be *any* pt. on the circle and PN $\perp$ on AA′, so that CN = $a\cos\theta$, NP = $a\sin\theta$.

Thro′ AA′ draw a plane π making an angle α with the plane p, such that $\cos\alpha = \dfrac{b}{a}$.

Let P′ be the foot of the $\perp$ from P on the plane π. Join P′N.

Then the co-ordinates of P′ *in the plane π* are

[Axes CA′ and the line thro′ C, in the plane π, $\perp$ to CA′]

$x = CN = a\cos\theta,\ y = NP' = NP\cos\alpha = a\sin\theta\cdot\dfrac{b}{a} = b\sin\theta$

$\therefore$ the locus of P′ *in the plane* π is an ellipse whose major axis $=2a$, and minor axis $=2b$.

Now area of ellipse = area of circle. $\cos\alpha$

[$A' = A\cos\theta$ (Art. 32, (*b*))]

$$=\pi a^2\left(\frac{b}{a}\right)=\pi ab.$$

****2.** Find the area of the section of the cylinder $9x^2+4y^2=36$ by a plane whose normal makes an angle of 60° with OZ.

33. *If A_x, A_y, A_z are the projections of a plane area A on the co-ordinate planes, then* $A=\sqrt{A_x^2+A_y^2+A_z^2}$.

Proof. Let l, m, n be the direction-cosines of the +ve direction of the normal from O to the plane of the area A.

Then $A_x = A\cos\alpha$, where α is the angle which the plane of the area A makes with the *yz*-plane, *i.e.*, the angle which the normal to the plane of the area A makes with the *x*-axis

$= A.l.$

Similarly $A_y = A.m,\ A_z = A.n.$

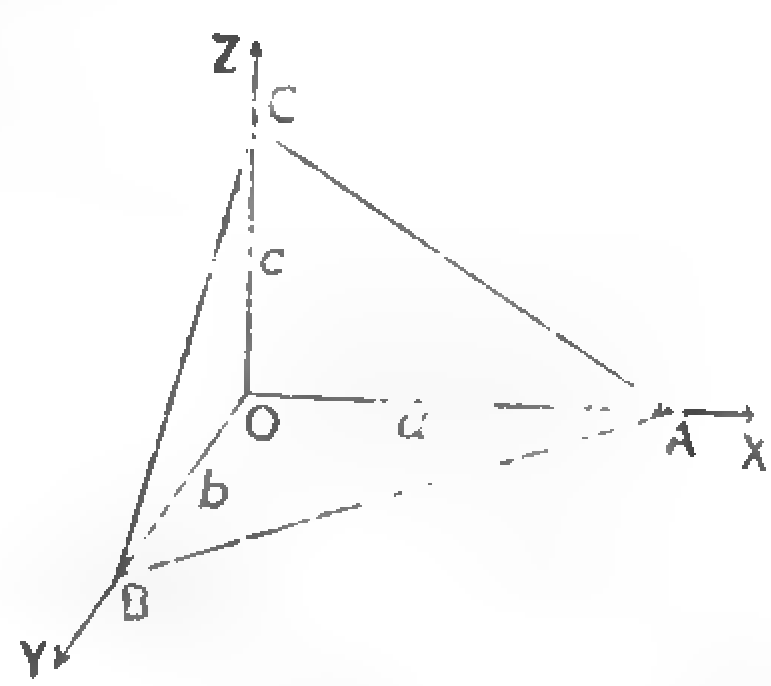
Squaring and adding,

$$A_x^2+A_y^2+A_z^2=A^2(l^2+m^2+n^2)=A^2.1$$

$$\therefore\quad A=\sqrt{A_x^2+A_y^2+A_z^2}.$$

EXAMPLES

1. A plane makes intercepts OA = a, OB = b, OC = c on the axes. Find the area of the triangle ABC. [*B. H. U. 1932*]



Here A_x = projection of area of $\triangle ABC$ on the *yz*-plane

$=\triangle OBC$

$=\frac{1}{2}bc.$

Similarly $A_y=\frac{1}{2}ca,$

$A_z=\frac{1}{2}ab.$

$\therefore$ "A" $=\sqrt{A_x^2+A_y^2+A_z^2}$ [Art. 33]

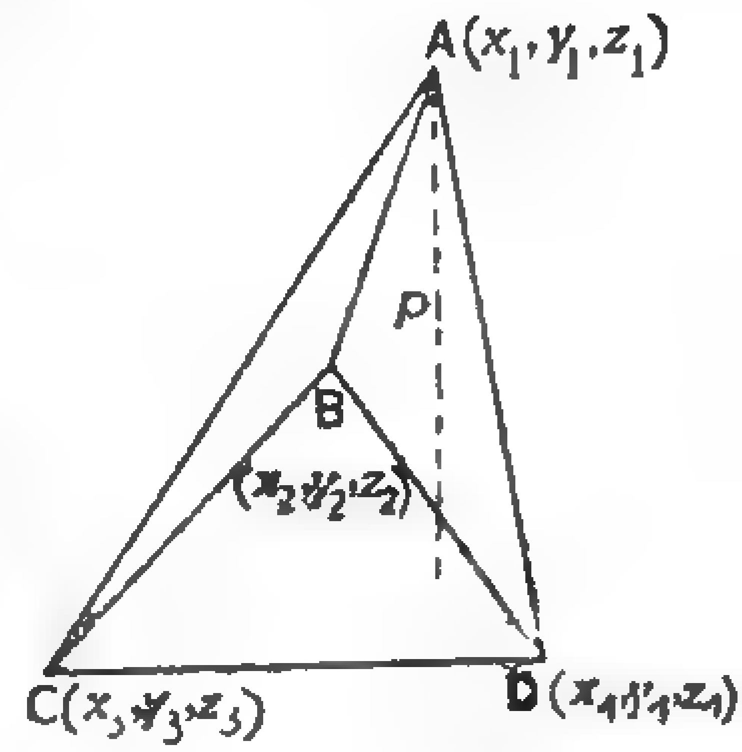
$=\sqrt{(\frac{1}{2}bc)^2+(\frac{1}{2}ca)^2+(\frac{1}{2}ab)^2}$

$=\frac{1}{2}\sqrt{b^2c^2+c^2a^2+a^2b^2}.$

2. *Area of a triangle.* **Find the area of the triangle whose vertices are (1, 2, 3), (−2, 1, −4), (3, 4, −2).** [*D. U. H. 1947*]

Volume of a tetrahedron.

34. *Volume formula.* To find the volume of a tetrahedron in terms of the co-ordinates of its angular points.



Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the co-ordinates of the vertices of the tetrahedron ABCD.

Let V be the required volume of the tetrahedron.

Then $V=\frac{1}{3}\,\triangle BCD.p$ ...(1)

[$\frac{1}{3}$ (area of base) $\times$ height (From Mensuration)]

where p is the $\perp$ distance of A from the plane BCD.

[To find p.]

The equation of the plane BCD is

$$\begin{vmatrix} x, & y, & z, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}=0.$$

[Three-pt. form (Art. 26)]

$$\therefore \quad p=\frac{\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}}{\left\{\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix}^2+\begin{vmatrix} z_2, & x_2, & 1 \\ z_3, & x_3, & 1 \\ z_4, & x_4, & 1 \end{vmatrix}^2+\begin{vmatrix} x_2, & y_2, & 1 \\ x_3, & y_3, & 1 \\ x_4, & y_4, & 1 \end{vmatrix}^2\right\}^{\frac{1}{2}}} \quad ...(2)$$

[Rule ($\perp$ distance formula, Art. 29, (*b*))]

Now if $\triangle$ is the area of the $\triangle$BCD, and $\triangle_x$, $\triangle_y$, $\triangle_z$ its projections on the co-ordinate planes, then

$$\triangle_x=\frac{1}{2}\begin{vmatrix} y_2, & z_2, & 1 \\ y_3, & z_3, & 1 \\ y_4, & z_4, & 1 \end{vmatrix},$$

[Art. 3, (*b*) and area formula (Analytical Plane Geometry)]

and so on.

$\therefore$ denominator of R.H.S. of (2)

$$=[(2\triangle_x)^2+(2\triangle_y)^2+(2\triangle_z)^2]^{\frac{1}{2}}=2\triangle \quad \text{(Art. 33)}$$
$$=2\,\triangle BCD.$$

$$\therefore \text{ from (2)}, \quad p=\frac{\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}}{2\triangle BCD}.$$

Substituting this value of p in (1),

$$V=\tfrac{1}{3}\triangle BCD.\frac{\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}}{2\triangle BCD}$$

$$=\tfrac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}.$$

****Complete volume formula for the tetrahedron.** In the above work, we have used the ordinary perpendicular distance formula for the plane. If, however, we use the *complete* perpendicular distance formula (Art. 29, (*b*)), it will be found that

$$V=\pm\tfrac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix},$$

that sign being taken on the R.H.S. which gives a positive result for V.

Note. When to use the complete volume formula for the tetrahedron. The complete volume formula is used when the volume of the tetrahedron is *given*. (See Ex. 2, following.)

EXAMPLES

1. (*a*) **Find the volume of a tetrahedron in terms of the co-ordinates of the vertices, the axes being rectangular.**

[*P(P). U. 1956 S*]

(*b*) A tetrahedron has its vertices at the points (1, 0, 0), (0, 0, 1), (0, 0, 2), (1, 2, 3) respectively. Find its volume. [*L.U.*]

[**Note on simplification of the determinant in numerical examples.**

$$V=\tfrac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$$

[Subtract each row from that immediately above it, leaving last row unaltered]

$$=\tfrac{1}{6}\begin{vmatrix} x_1-x_2, & y_1-y_2, & z_1-z_2, & 0 \\ x_2-x_3, & y_2-y_3, & z_2-z_3, & 0 \\ x_3-x_4, & y_3-y_4, & z_3-z_4, & 0 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$$

[Expand by means of the last column]

$$=\tfrac{1}{6}\begin{vmatrix} x_1-x_2, & y_1-y_2, & z_1-z_2 \\ x_2-x_3, & y_2-y_3, & z_2-z_3 \\ x_3-x_4, & y_3-y_4, & z_3-z_4 \end{vmatrix}.$$

Now expand the determinant.

Rule to evaluate a determinant whose one column has all its constituents = 1 :
Subtract each row from that immediately **above** *it, leaving the last row unaltered, and expand the resulting determinant by means of the column which has all its constituents but one = 0.*]

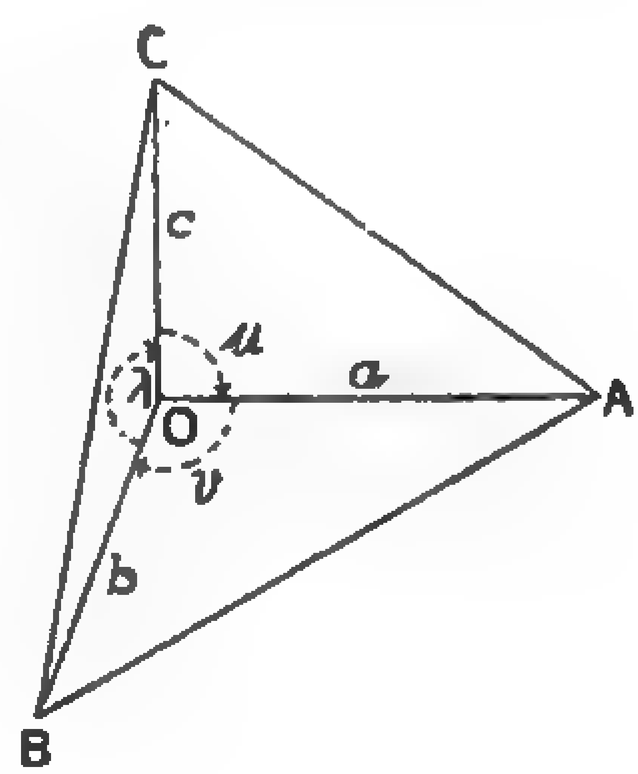
2. A, B, C are the points (1, 2, 3), (−1, 0, −2), (0, 0, −1). Find the locus of P if the volume PABC=2.

3. A, B, C, D are four coplanar points and A′, B′, C′, D′ their projections on any plane, prove that

Vol. AB′C′D′ = −Vol. A′BCD. [*Frost*]

[*Take the plane of projection as the xy-plane,* and let A, B, C, D be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4). Then their projections on the xy-plane are $A'(x_1, y_1, 0)$, $B'(x_2, y_2, 0)$, $C'(x_3, y_3, 0)$, $D'(x_4, y_4, 0)$.]

4. *Volume of a tetrahedron in terms of three coterminous edges and the angles between them.* **The lengths of the edges OA, OB, OC of a tetrahedron OABC are a, b, c, and the angles BOC, COA, AOB are λ, μ, ν; find the volume.** [*P(P). U. 1953*]



Take O as origin, and let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the direction-cosines of OA, OB, OC.

∵ OA = a,

∴ the co-ordinates of A are

$$(l_1a,\ m_1a,\ n_1a).$$

[(lr, mr, nr) (Art. 8)]

Similarly the co-ordinates of B are

$$(l_2b,\ m_2b,\ n_2b),$$

and the co-ordinates of C are

$$(l_3c,\ m_3c,\ n_3c).$$

∴ vol. of tetrahedron OABC is

$$V = \frac{1}{6}\begin{vmatrix} 0, & 0, & 0, & 1 \\ l_1a, & m_1a, & n_1a, & 1 \\ l_2b, & m_2b, & n_2b, & 1 \\ l_3c, & m_3c, & n_3c, & 1 \end{vmatrix}$$

[Expand the determinant by means of first row]

$$= -\frac{1}{6}\begin{vmatrix} l_1a, & m_1a, & n_1a \\ l_2b, & m_2b, & n_2b \\ l_3c, & m_3c, & n_3c \end{vmatrix}$$ [Take a common from the first row, b from the second row, and c from the third row]

$$=-\tfrac{1}{6}abc\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \quad \dots (1)$$

[To find value of determinant on R.H.S.]

Now $\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \times \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$

[Multiply by Product Rule for determinants (Higher Algebra)]

$$=\begin{vmatrix} l_1^2+m_1^2+n_1^2, & l_1l_2+m_1m_2+n_1n_2, & l_1l_3+m_1m_3+n_1n_3 \\ l_2l_1+m_2m_1+n_2n_1, & l_2^2+m_2^2+n_2^2, & l_2l_3+m_2m_3+n_2n_3 \\ l_3l_1+m_3m_1+n_3n_1, & l_3l_2+m_3m_2+n_3n_2, & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$=\begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}$$

$$\therefore \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = \pm\begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}^{\frac{1}{2}}.$$

Substituting this value in (1),

$$V = \pm\tfrac{1}{6}abc\begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}^{\frac{1}{2}}.$$

$\therefore$ volume (*in magnitude*)

$$=\tfrac{1}{6}abc\begin{vmatrix} 1, & \cos\nu, & \cos\mu \\ \cos\nu, & 1, & \cos\lambda \\ \cos\mu, & \cos\lambda, & 1 \end{vmatrix}^{\frac{1}{2}}.$$

SECTION V

PAIR OF PLANES REPRESENTED BY A HOMOGENEOUS EQUATION OF THE SECOND DEGREE IN x, y, z

35. To find the condition that the equation

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$$

may represent a pair of planes, and supposing it satisfied, to find the angle between the planes.

(*a*) If the equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, represents the pair of planes $lx+my+nz=0$, $l'x+m'y+n'z=0$, then $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=(lx+my+nz)(l'x+m'y+n'z)$.

Equating coefficients of like terms on both sides,

$$\left.\begin{aligned} ll'=a,\ mm'=b,\ nn'=c, \\ mn'+m'n=2f,\ ln'+l'n=2g,\ lm'+l'm=2h \end{aligned}\right] \dots (1)$$

Now $\begin{vmatrix} l, & l', & 0 \\ m, & m', & 0 \\ n, & n', & 0 \end{vmatrix}$* $\times \begin{vmatrix} l', & l, & 0 \\ m', & m, & 0 \\ n', & n, & 0 \end{vmatrix} = 0 \times 0$

[Expanding each determinant on L.H.S. by means of last column]

$$=0$$

or $\begin{vmatrix} 2ll', & lm'+l'm, & ln'+l'n \\ ml'+m'l, & 2mm', & mn'+m'n \\ nl'+n'l, & nm'+n'm, & 2nn' \end{vmatrix} = 0$ [Product Rule for determinants (Higher Algebra)]

$\therefore$ from (1), $\begin{vmatrix} 2a, & 2h, & 2g \\ 2h, & 2b, & 2f \\ 2g, & 2f, & 2c \end{vmatrix} = 0$, or $8 \begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix} = 0$, [Cancel 8]

or $$a(bc-f^2)-h(ch-fg)+g(hf-bg)=0,$$

or $$abc-af^2-ch^2+fgh+fgh-bg^2=0,$$

or $$abc+2fgh-af^2-bg^2-ch^2=0,$$

which is the required condition.

Note. The converse is also true, *i.e.*, if $abc+2fgh-af^2-bg^2-ch^2=0$, the equation represents a pair of planes.

For the order of the steps in the above proof can be reversed.

(*b*) If θ is the angle between the planes

$$lx+my+nz=0,$$
$$l'x+m'y+n'z=0,$$

then $$\tan\theta \left(=\frac{\sin\theta}{\cos\theta}\right) = \frac{\sqrt{(mn'-m'n)^2+(nl'-n'l)^2+(lm'-l'm)^2}}{ll'+mm'+nn'} \quad ...(2)$$

[Ex. 5, Art. 27]

Now $(mn'-m'n)^2=(mn'+m'n)^2-4mm'nn'$ which, from (1),

$$=4f^2-4bc=4(f^2-bc), \text{ and so on.}$$

$\therefore$ from (2), $$\tan\theta = \frac{\sqrt{4(f^2-bc)+4(g^2-ca)+4(h^2-ab)}}{a+b+c}$$

$$=\frac{2\sqrt{f^2+g^2+h^2-bc-ca-ab}}{a+b+c} \quad ...(3)$$

---

*How to write this step. Write the first determinant whose columns are

$$\begin{matrix} l, & l', & 0 \\ m, & m', & 0 \\ n, & n', & 0 \end{matrix}$$

and for the second determinant interchange the first two columns of the first determinant, thus getting

$$\begin{matrix} l', & l, & 0 \\ m', & m, & 0 \\ n', & n, & 0 \end{matrix}$$

Cor. Condition of perpendicularity of a pair of planes. *The condition, that the pair of planes* $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ *may be perpendicular, is*

$$\mathbf{a+b+c=0.}$$

[In words : *coefficient of* x^2+*coefficient of* y^2+*coefficient of* $z^2=0$.]

If the planes are $\perp$, θ, the angle between them, $=90°$

$$\therefore \qquad \tan\theta=\tan 90°=\infty$$

$\therefore$ from (3), the denominator $a+b+c=0$, which is the required condition.

EXAMPLES

1. Prove that the equation

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$$

represents a pair of planes if

$$\mathbf{abc+2fgh-af^2-bg^2-ch^2=0.}$$

Prove that the angle between the planes is

$$\tan^{-1}\left[\frac{2(f^2+g^2+h^2-bc-ca-ab)^{\frac{1}{2}}}{a+b+c}\right]. \qquad [\textit{P. U. 1960 S}]$$

2. Find the condition that

$$\mathbf{ax^2+by^2+cz^2+2a'yz+2b'zx+2c'xy=0}$$

may represent a pair of planes ; and supposing it satisfied, if θ be the angle between the planes, prove that

$$\tan\theta=\frac{2\sqrt{a'^2+b'^2+c'^2-bc-ca-ab}}{a+b+c}. \qquad [\textit{Bar. U. 1954}]$$

3. Prove that the equation $2x^2-6y^2-12z^2+18yz+2zx+xy=0$ represents a pair of planes, and find the angle between them.

[*P. U. 1957 S*]

[**Rule to prove that a given numerical equation represents a pair of planes, and to find the separate equations of the planes :**

Write the equation as a quadratic in x(or y or z), and solve it. The two values of x (in terms of y and z) give, by transposition, the separate equations of the planes.

Note 1. The above method enables us not only to prove that the given equation represents a pair of planes but also to find *their separate equations.*]

Note 2. Important. *"The angle between two planes"* means, for definiteness, "the *acute* angle between two planes".

MISCELLANEOUS EXAMPLES ON CHAPTER IV

1. E, F, G, H are four coplanar points on the sides AB, BC, CD, DA of a skew quadrilateral. Prove that

$$\frac{AE}{EB}\cdot\frac{BF}{FC}\cdot\frac{CG}{GD}\cdot\frac{DH}{HA}=1.$$

$\therefore \quad x-x_1=lr.$

Similarly $y-y_1=mr,\ z-z_1=nr.$

$$\therefore \quad \frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{l}}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{m}}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{n}}(=\mathbf{r}),$$

which are the required equations.

Cor. 1. Symmetrical form of the equations of a straight line whose proportional direction-cosines are given. *The equations of the straight line passing through* (x_1, y_1, z_1) *and having direction-cosines proportional to* a, b, c, *are*

$$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{a}}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{b}}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{c}}.$$

Proof. The direction-cosines of the line are proportional to a, b, c. Dividing by $\sqrt{a^2+b^2+c^2}$, the actual direction-cosines of the line are

$$\frac{a}{\sqrt{a^2+b^2+c^2}},\ \frac{b}{\sqrt{a^2+b^2+c^2}},\ \frac{c}{\sqrt{a^2+b^2+c^2}}, \text{ [Rule (Art. 10)]}$$

and it passes thro' (x_1, y_1, z_1)

$\therefore$ the equations of the line are

$$\frac{x-x_1}{\dfrac{a}{\sqrt{a^2+b^2+c^2}}}=\frac{y-y_1}{\dfrac{b}{\sqrt{a^2+b^2+c^2}}}=\frac{z-z_1}{\dfrac{c}{\sqrt{a^2+b^2+c^2}}}$$

$$\left[\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \text{ (Art. 37)}\right]$$

or, cancelling $\sqrt{a^2+b^2+c^2}$ from the denoms.,

$$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{a}}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{b}}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{c}},$$

which are the required equations.

Note 1. Symmetrical form. The form

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}, \text{ or } \frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}$$

of the equations of a straight line is called the **symmetrical form.**

Note 2. The symmetrical equations of a straight line are of the same form whether the actual direction-cosines are used or their proportionals.

Caution. It is only when l, m, n are the *actual* (not proportional) direction-cosines of a line that each member of the equations of the line in the symmetrical form $=r$, the *distance* of the point (x, y, z) from (x_1, y_1, z_1). Then the symmetrical form may also be called the **distance form.**

Note 3. What is the use of symmetrical form ? The use of the symmetrical form of the equations of a line is that it enables us to find the co-ordinates of any point on the line in terms of a single variable r, called the *parameter*.

Cor. 2. Any point on the line. *Any point on the line*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

is $(\mathbf{x_1+lr, y_1+mr, z_1+nr})$.

Proof. Putting $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}=r$,

$$x-x_1=lr,\ y-y_1=mr,\ z-z_1=nr,$$

$\therefore$ $x=x_1+lr,\ y=y_1+mr,\ z=z_1+nr.$

$\therefore$ any pt. on the line is (x_1+lr, y_1+mr, z_1+nr).

[**Rule to find any point on the line (equations in the symmetrical form :** $\dfrac{\mathbf{x-x_1}}{\mathbf{l}}=\dfrac{\mathbf{y-y_1}}{\mathbf{m}}=\dfrac{\mathbf{z-z_1}}{\mathbf{n}}$) :

Put each member of the equations of the line=r (mentally), and find the values of x, y, z. These are the co-ordinates of any point on the line.]

EXAMPLES

1. (*a*) Obtain the equations of a straight line in the form

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$ [*P (P). U. 1957*]

(*b*) Find the equations to the straight line through (a, b, c) parallel to OZ.

2. (*a*) Write down the equations of the straight line through (1, 2, 3) equally inclined to the axes.

(*b*) Find the angle between the lines

$$\frac{x}{1}=\frac{y}{0}=\frac{z}{-1} \text{ and } \frac{x}{3}=\frac{y}{4}=\frac{z}{5}.$$

3. (*a*) Show that the lines $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{1}$ and $\dfrac{x}{1}=\dfrac{y}{-1}=\dfrac{z}{1}$ are at right angles. [*P. U.*]

(*b*) Show that the line $2x=3y=-z$ is perpendicular to the line $6x=-y=-4z$.

4. **Show that, if the axes are rectangular, the equations of the line through $(\mathbf{x_1, y_1, z_1})$ perpendicular to the lines**

$$\frac{\mathbf{x}}{\mathbf{l_1}}=\frac{\mathbf{y}}{\mathbf{m_1}}=\frac{\mathbf{z}}{\mathbf{n_1}},\ \frac{\mathbf{x}}{\mathbf{l_2}}=\frac{\mathbf{y}}{\mathbf{m_2}}=\frac{\mathbf{z}}{\mathbf{n_2}}$$

are $$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{m}_1\mathbf{n}_2-\mathbf{m}_2\mathbf{n}_1}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{n}_1\mathbf{l}_2-\mathbf{n}_2\mathbf{l}_1}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{l}_1\mathbf{m}_2-\mathbf{l}_2\mathbf{m}_1}.\quad [\textit{P. U. 1960}]$$

38. *Reduction of the general equations to the symmetrical form.* **To reduce the equations**

$$\mathbf{ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0}$$

to the symmetrical form.

The given equations are

$$\left.\begin{aligned} ax+by+cz+d&=0\\ a'x+b'y+c'z+d'&=0\end{aligned}\right]\quad ...(1)$$

(*i*) [To find the direction-cosines of the line.]

The equations of the line thro' the origin ‖ to the given line are [omitting the constant terms from (1)],

$$ax+by+cz=0,$$
$$a'x+b'y+c'z=0$$

[∵ these are the equations of the planes thro' the origin ‖ to the planes (1)]

or $$\frac{x}{bc'-b'c}=\frac{y}{ca'-c'a}=\frac{z}{ab'-a'b}$$

whose direction-cosines and ∴ the direction-cosines of the given line are proportional to $bc'-b'c,\ ca'-c'a,\ ab'-a'b$ ...(2)

(*ii*) [To find one pt. on the line.]

In the equations (1), putting $z=0$,

$$ax+by+d=0,$$
$$a'x+b'y+d'=0$$

or $$\frac{x}{bd'-b'd}=\frac{y}{da'-d'a}=\frac{1}{ab'-a'b}$$

or $$x=\frac{bd'-b'd}{ab'-a'b},\ y=\frac{da'-d'a}{ab'-a'b},\ \text{also } z=0$$

∴ one pt. on the line is $\left(\dfrac{bd'-b'd}{ab'-a'b},\ \dfrac{da'-d'a}{ab'-a'b},\ 0\right)$ ...(3)

(*iii*) From (2) and (3), the equations of the line in the symmetrical form are

$$\frac{x-\dfrac{bd'-b'd}{ab'-a'b}}{bc'-b'c}=\frac{y-\dfrac{da'-d'a}{ab'-a'b}}{ca'-c'a}=\frac{z}{ab'-a'b}.$$

$$\left[\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}\quad \text{(Art. 37, Cor. 1)}\right]$$

[Rule to reduce the general equations of a line to the symmetrical form :

(*i*) [*To find the direction-cosines of the line.*] *Find the equations*

of the line thro' the origin $\parallel$ *to the given line [by omitting the constant terms from the given equations and solving the resulting equations for x, y, z]. These give the direction-cosines of the* $\parallel$ *line and* $\therefore$ *of the given line.*

(ii) [To find one pt. on the line.] In the given equations, put $z=0$, *and solve the resulting equations for x, y. These give the x-, y- coordinates of one pt. on the line and the z-coordinate is zero.*

(iii) From the results of steps (i) and (ii), find the equations of the line in the symmetrical form.]

EXAMPLES

1. Find in a symmetrical form the equations of the line $x+y+z+1=0$, $4x+y-2z+2=0$ and find its direction-cosines.

[*P.U. 1937*]

(*a*) The given equations are

$$\left.\begin{array}{r} x+y+z+1=0 \\ 4x+y-2z+2=0 \end{array}\right] \quad ...(1)$$

(*i*) [To find the direction-cosines of the line.]

The equations of the line thro' the origin $\parallel$ to the given line are [omitting the constant terms from (1)],

$$x+y+z=0,$$
$$4x+y-2z=0$$

$$\therefore \quad \frac{x}{-2-1}=\frac{y}{4-(-2)}=\frac{z}{1-4}, \text{ or } \frac{x}{-3}=\frac{y}{6}=\frac{z}{-3},$$

or $\dfrac{x}{-1}=\dfrac{y}{2}=\dfrac{z}{-1}$,

whose direction-cosines and $\therefore$ the direction-cosines of the given line are proportional to $-1, 2, -1$ ...(2)

(*ii*) [To find one pt. on the line.]

In the equations (1), putting $z=0$,

$$x+y+1=0,$$
$$4x+y+2=0$$

$$\therefore \quad \frac{x}{2-1}=\frac{y}{4-2}=\frac{1}{1-4}, \text{ or } \frac{x}{1}=\frac{y}{2}=\frac{1}{-3},$$

or $\quad x=-\dfrac{1}{3},\ y=-\dfrac{2}{3}$, also $z=0$

$\therefore$ one pt. on the line is $\left(-\dfrac{1}{3}, -\dfrac{2}{3}, 0\right)$ ...(3)

(*iii*) From (2) and (3), the equations of the line in the symmetrical form are $\dfrac{x-(-\frac{1}{3})}{-1}=\dfrac{y-(-\frac{2}{3})}{2}=\dfrac{z-0}{-1}$ (Art. 37, Cor. 1)

or $$\frac{x+\frac{1}{3}}{-1}=\frac{y+\frac{2}{3}}{2}=\frac{z}{-1}.$$

(*b*) From (2), dividing by $\sqrt{(-1)^2+(2)^2+(-1)^2}=\sqrt{6}$, the direction-cosines of the line are $-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}$.

2. Prove that the equations to the line of intersection of the planes $4x+4y-5z=12$, $8x+12y-13z=32$ can be written

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z}{4}.$$ [*P(P). U. 1953*]

3. Find in a symmetrical form the equations of the line

$$x-2y=3,\ 2x+y-5z=0.$$

4. Find the direction-cosines of the line whose equations are

$$x+y-z+1=0,\ 4x+y-2z+2=0.$$ [*P.U. Eng. 2, 1937*]

5. Find the angle between the lines

$$x-2y+z=0=x+2y-2z,\ x+2y+z=0=3x+9y+5z$$

(rectangular axes). [*L.U.*]

6. (*a*) **Prove that the lines $2x+3y-4z=0=3x-4y+z$, $5x-y-3z+12=0=x-7y+5z-6$ are parallel.** [*P. U. 1947*]

(*b*) Show that the lines $2x+3y+z-4=0=x+y-2z-3$, and $5x+8y-7z=0=10x-2y-2z$ are at right angles.

7. Find the equations to the line through the point (1, 2, 3) parallel to the line $x-y+2z=5$, $3x+y+z=6$. [*P. U. 1936*]

39. Number of constants in the equations of a line. *The general equations of a straight line contain four arbitrary constants.*

Proof. The equations of a st. line, in the symmetrical form, are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(1)$$ [Art. 37]

[To reduce them to the form $x=Ay+B$, $y=Cz+D$.]

From the first and second members of (1),

$$x-x_1=\frac{l}{m}(y-y_1)=\frac{l}{m}y-\frac{l}{m}y_1$$

or $$x=\frac{l}{m}y+\left(x_1-\frac{l}{m}y_1\right)$$

or $$x=Ay+B, \text{ where } A=\frac{l}{m},\ B=x_1-\frac{l}{m}y_1.$$

Again, from the second and third members of (1),

$$y-y_1=\frac{m}{n}(z-z_1)=\frac{m}{n}z-\frac{m}{n}z_1$$

or $$y=\frac{m}{n}z+\left(y_1-\frac{m}{n}z_1\right)$$

or $$y=Cz+D, \text{ where } C=\frac{m}{n},\ D=y_1-\frac{m}{n}z_1.$$

Thus the equations of a st. line are

$$x=Ay+B,$$
$$y=Cz+D,$$

which contain four arbitrary constants.

EXAMPLES

1. Prove that the symmetrical form of the equations of the line of intersection of the planes $4x-3y=1$, $2y-4z=3$ is

$$\frac{x-\frac{1}{4}}{3}=\frac{y}{4}=\frac{z+\frac{3}{4}}{2}.$$

The equations of the line are $4x-3y=1$, $2y-4z=3$.

[Here each equation contains only two variables. Solve each equation for the *common variable* y*.]

$$4x-1=3y, \therefore\ y=\frac{4x-1}{3}=\frac{4(x-\frac{1}{4})}{3}=\frac{x-\frac{1}{4}}{\frac{3}{4}},$$

$$2y=4z+3, \therefore\ y=\frac{4z+3}{2}=\frac{4(z+\frac{3}{4})}{2}=2(z+\tfrac{3}{4})=\frac{z+\frac{3}{4}}{\frac{1}{2}}.$$

$$\therefore\ \frac{x-\frac{1}{4}}{\frac{3}{4}}=\frac{y}{1}=\frac{z+\frac{3}{4}}{\frac{1}{2}}$$

or, multiplying the denoms. by 4 (to clear off fractions from the denoms.),

$$\frac{x-\frac{1}{4}}{3}=\frac{y}{4}=\frac{z+\frac{3}{4}}{2},$$

which are the required equations.

2. Show that the symmetrical form of the equations of the line $x=ay+b$, $z=cy+d$ is $\frac{x-b}{a}=\frac{y}{1}=\frac{z-d}{c}$.

3. Find the condition that the lines $x=ay+b$, $z=cy+d$, $x=a'y+b'$, $z=c'y+d'$ may be perpendicular.

4. Find a, b, c, d so that the line $x=ay+b$, $z=cy+d$ may pass through the points $(3, 1, -3)$, $(4, 2, -4)$, and hence show that the given points and $(5, 3, -5)$ are collinear.

40. *Two-point form.* **To find the equations of a straight line through two given points.**

Let (x_1, y_1, z_1), (x_2, y_2, z_2) be the two pts.

Then the direction-cosines of the line thro' them are proportional to x_2-x_1, y_2-y_1, z_2-z_1 (Art. 12), and it passes thro' (x_1, y_1, z_1)

---

*Also suggested by the second member $\frac{y}{4}$ of the required symmetrical form

$$\frac{x-\frac{1}{4}}{3}=\frac{y}{4}=\frac{z+\frac{3}{4}}{2}.$$

$\therefore$ its equations are

$$\frac{\mathbf{x}-\mathbf{x}_1}{\mathbf{x}_2-\mathbf{x}_1}=\frac{\mathbf{y}-\mathbf{y}_1}{\mathbf{y}_2-\mathbf{y}_1}=\frac{\mathbf{z}-\mathbf{z}_1}{\mathbf{z}_2-\mathbf{z}_1}.$$

$$\left[\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \text{ (Art. 37, Cor. 1)}\right]$$

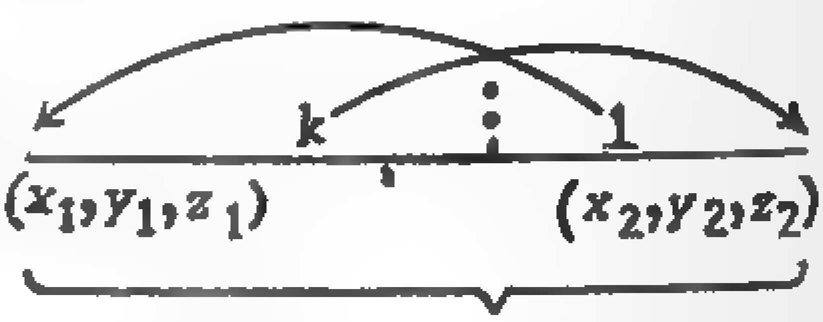
Note. Two-point form. Since in the equations

$$\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1},$$

(x_1, y_1, z_1), (x_2, y_2, z_2) are the *two points* thro' which the line passes, this form of the equations of a line may be called the **two-point form.**

Cor. Any point on the line through two points. *Any point on the line through* (x_1, y_1, z_1), (x_2, y_2, z_2) *is*

$$\left(\frac{\mathbf{kx}_2+\mathbf{x}_1}{\mathbf{k}+1}, \frac{\mathbf{ky}_2+\mathbf{y}_1}{\mathbf{k}+1}, \frac{\mathbf{kz}_2+\mathbf{z}_1}{\mathbf{k}+1}\right).$$



Let the pt. divide the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) in the ratio $k : 1$.

Then its co-ordinates are

$$x=\frac{kx_2+x_1}{k+1}, \quad y=\frac{ky_2+y_1}{k+1}, \quad z=\frac{kz_2+z_1}{k+1}. \qquad \text{[Art. 5]}$$

Note. These are the co-ordinates of any point on the line through (x_1, y_1, z_1), (x_2, y_2, z_2) in terms of the parameter k.

EXAMPLES

1. *Find the equations of the straight line joining the points* $(-2, 1, 3)$ *and* $(3, 1, -2)$. [*L. U.*]

The equations of the line are (1) $(-2, 1, 3)$ (2) $(3, 1, -2)$

$$\frac{x-(-2)}{3-(-2)}=\frac{y-1}{1-1}=\frac{z-3}{-2-3}$$

$$\left[\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \quad \text{(Art. 40)}\right]$$

or $$\frac{x+2}{5}=\frac{y-1}{0}=\frac{z-3}{-5}, \text{ or } \frac{x+2}{1}=\frac{y-1}{0}=\frac{z-3}{-1}.$$

Note. *To show that the equations of the straight line joining the points* $(-2, 1, 3)$ *and* $(3, 1, -2)$ *are* $y-1=0$, $x+z-1=0$.

The equations of the line are

$$\frac{x+2}{1}=\frac{y-1}{0}=\frac{z-3}{-1}. \qquad \text{[As proved in Ex. 1]}$$

From the first and second members, cross-multiplying, $y-1=0$; again, from the first and third members, cross-multiplying,

$$-x-2 = z-3, \quad \text{or} \quad x+z-1=0$$

$\therefore$ the equations of the line are $y-1=0$, $x+z-1=0$.

[**Check.** The equations of the line,

$$\frac{x+2}{1} = \frac{y-1}{0} = \frac{z-3}{-1},$$

are satisfied by the co-ordinates of the two given pts. $(-2, 1, 3)$, $(3, 1, -2)$, thus

$$\frac{-2+2}{1} = \frac{1-1}{0} = \frac{3-3}{-1}, \quad \text{or} \quad \frac{0}{1}\left(=\frac{0}{0}\right) = \frac{0}{-1}, \text{ or } 0=0,$$

$$\frac{3+2}{1} = \frac{1-1}{0} = \frac{-2-3}{-1}, \text{ or } \frac{5}{1}\left(=\frac{0}{0}\right) = \frac{-5}{-1}, \text{ or } 5=5. \ ]$$

2. *Prove that the points $(1, 2, 3)$, $(4, 0, 4)$, $(-2, 4, 2)$, $(7, -2, 5)$ are collinear.*

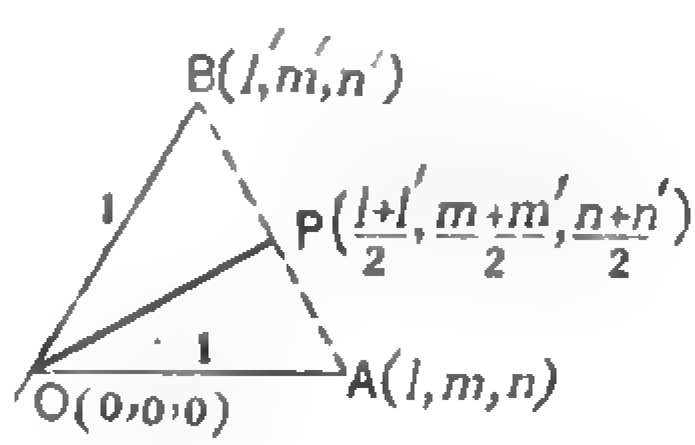
3. Find the equations of the straight lines which bisect the angles between the lines

$$\frac{\mathbf{x}}{\mathbf{l}} = \frac{\mathbf{y}}{\mathbf{m}} = \frac{\mathbf{z}}{\mathbf{n}}, \textbf{ and } \frac{\mathbf{x}}{\mathbf{l'}} = \frac{\mathbf{y}}{\mathbf{m'}} = \frac{\mathbf{z}}{\mathbf{n'}}. \quad [\textit{P.U. 1939 S}]$$

Let OA, OB be the lines

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}, \quad \frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'}.$$

Cut off OA = 1, OB = 1.



Then the co-ordinates of A, B are (l, m, n), (l', m', n').

[(lr, mr, nr) (Art. 8), here $r=1$]

$\therefore$ the co-ordinates of P, the mid-pt. of AB, are

$$\left(\frac{l+l'}{2}, \frac{m+m'}{2}, \frac{n+n'}{2}\right),$$

and the co-ordinates of O are $(0, 0, 0)$

$\therefore$ the equations of OP, the bisector of $\angle$ AOB, are

$$\frac{x-0}{\frac{l+l'}{2}-0} = \frac{y-0}{\frac{m+m'}{2}-0} = \frac{z-0}{\frac{n+n'}{2}-0}$$

$$\left[\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1} \text{ (Art. 40)}\right]$$

or $$\frac{\mathbf{x}}{\mathbf{l+l'}} = \frac{\mathbf{y}}{\mathbf{m+m'}} = \frac{\mathbf{z}}{\mathbf{n+n'}}.$$

Similarly the equations of the bisector of the exterior $\angle$ AOB$'$, *i.e.*, angle between OA (direction-cosines l, m, n) and OB$'$ (direction-cosines $-l', -m', -n'$) (Art. 7, Cor. 2) are

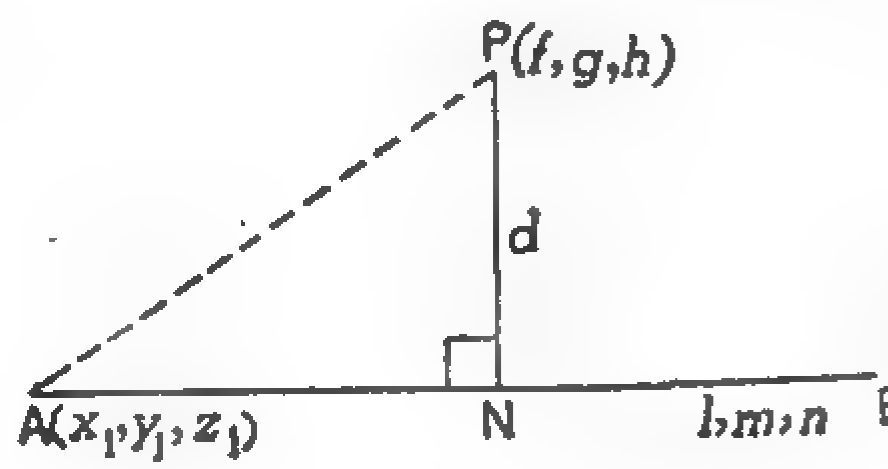
$$\frac{x}{l-l'}=\frac{y}{m-m'}=\frac{z}{n-n'}.$$

Note. It is assumed in the question that $l, m, n\,;\, l', m', n'$ are the *actual* direction-cosines of the lines.

4. If l_r, m_r, n_r $(r=1, 2)$ are two directions inclined at an angle θ, show that the actual direction-cosines of the direction bisecting them are $\frac{1}{2}(l_1+l_2)\sec\frac{\theta}{2}$, etc. [*P. U. 1955 S*]

SECTION II
A LINE AND A POINT

41. *Perpendicular distance formula for the line.* **To find the perpendicular distance of a given point from a given straight line.**



Let P be the given pt. (f, g, h), and AB the given line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Let A be the pt. (x_1, y_1, z_1). From P draw PN $\perp$ on AB. Let d be the required $\perp$ distance NP.

Join AP.

Then $NP^2=AP^2-AN^2$ ...(1)

But $AP=\sqrt{(f-x_1)^2+(g-y_1)^2+(h-z_1)^2}$,

$$[\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}\ \text{(Art. 4)}]$$

AN = projection of AP on AB

$=(f-x_1)l+(g-y_1)m+(h-z_1)n$

$$[(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n\ \text{(Art. 14)}]$$

$\therefore$ from (1),

$$d^2=(f-x_1)^2+(g-y_1)^2+(h-z_1)^2-[(f-x_1)l+(g-y_1)m+(h-z_1)n]^2$$

$$\therefore\quad d=\{(f-x_1)^2+(g-y_1)^2+(h-z_1)^2-[(f-x_1)l+(g-y_1)m+(h-z_1)n]^2\}^{\frac{1}{2}}.$$

Cor. Second form. *The perpendicular distance of the point*

(f, g, h) *from the line* $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$,

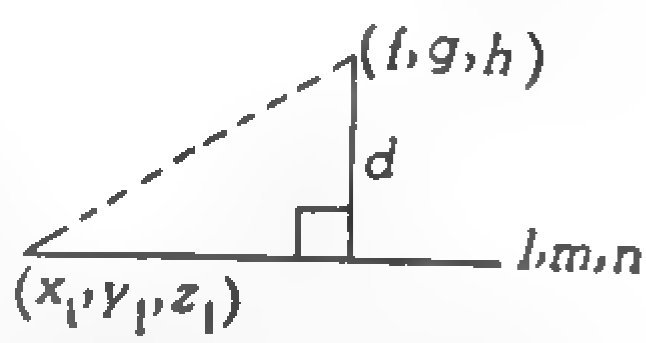
is
$$d=\{ [(g-y_1)n-(h-z_1)m]^2+[(h-z_1)l-(f-x_1)n]^2 +[(f-x_1)m-(g-y_1)l]^2 \}^{\frac{1}{2}}.$$

Proof.
$$d^2=(f-x_1)^2+(g-y_1)^2+(h-z_1)^2 -[(f-x_1)l+(g-y_1)m+(h-z_1)n]^2 \quad \text{(Art. 41)}$$
$$=[(f-x_1)^2+(g-y_1)^2+(h-z_1)^2]\,(l^2+m^2+n^2) -[(f-x_1)l+(g-y_1)m+(h-z_1)n]^2 \quad [\because l^2+m^2+n^2=1]$$
which, by Lagrange's identity (Art. 13, (*a*))
$$=[(g-y_1)n-(h-z_1)m]^2 +[(h-z_1)l-(f-x_1)n]^2 +[(f-x_1)m-(g-y_1)l]^2 \qquad \left|\begin{matrix} f-x_1, & g-y_1, & h-z_1 \\ l, & m, & n \end{matrix}\right.$$
$$\therefore \quad d=\{[(g-y_1)n-(h-z_1)m]^2+[(h-z_1)l-(f-x_1)n]^2 +[(f-x_1)m-(g-y_1)l]^2\}^{\frac{1}{2}} \quad \text{... \textbf{Second form}}$$

Note. Important. l, m, n are the *actual* direction-cosines of the line.

[**Rule to find the perpendicular distance of a given point from a given line (equations in the symmetrical (actual direction-cosines) form :**

First form.



(*i*) *Find the distance between one pt. on the line and the given pt.*

(**Rule to find one pt. on the line (equations in the symmetrical form) :** *Put each member of the equations of the line=0, and find the values of* x, y, z. *These are the co-ordinates of one pt. on the line.*)

(*ii*) *Find the projection of this distance on the given line.*

(*iii*) *Then* $\perp=\sqrt{(\text{distance})^2-(\text{projection})^2}$.]

Second form.

(*i*) *Substitute the co-ordinates of the given pt.* (f, g, h) *in the equations of the given line,* $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$,

and write in two rows the numerators and denominators, thus getting
$$\begin{matrix} f-x_1, & g-y_1, & h-z_1, \\ l, & m, & n. \end{matrix}$$

(*ii*) *Draw diagonals mentally as in cross-multiplication, thus*
$$\begin{matrix} g-y_1 & & h-z_1 & & f-x_1 & & g-y_1 \\ & \times & & \times & & \times & \\ m & & n & & l & & m \end{matrix}$$

take the square of each result, and add, thus getting
$$[(g-y_1)n-(h-z_1)m]^2+[(h-z_1)l-(f-x_1)n]^2+[(f-x_1)m-(g-y_1)l]^2.$$

(iii) Take the square root of the result. This gives the $\perp$ *distance.*]

Note 1. When to use the second form of the perpendicular distance formula for the line. The second form of the perpendicular distance formula for the line (Art. 41, Cor.) is generally used when the co-ordinates of the given point and the constants in the equations of the given line are not numerical. (See Ex. 9 following.)

****Note 2. Complete perpendicular distance formula for the line (equations in the symmetrical form).**

$$\because \quad d^2 = (f-x_1)^2 + (g-y_1)^2 + (h-z_1)^2 \\ -[(f-x_1)\,l+(g-y_1)\,m+(h-z_1)\,n]^2$$

$$\therefore \quad d = \pm\{(f-x_1)^2 + (g-y_1)^2 + (h-z_1)^2 \\ -[(f-x_1)\,l+(g-y_1)\,m+(h-z_1)\,n]^2\}^{\frac{1}{2}},$$

that sign being taken on the R. H. S. which gives a positive result for d.

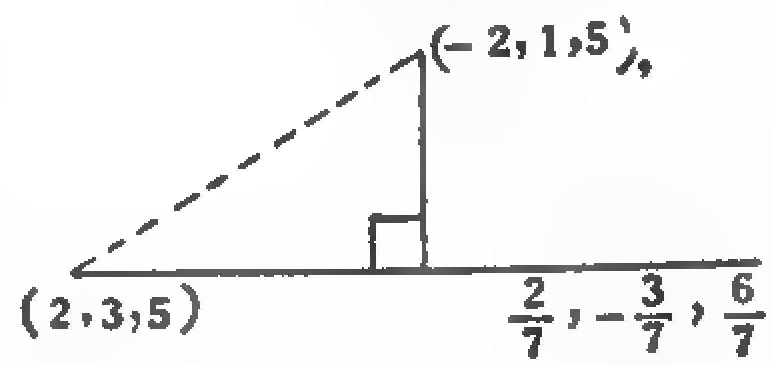
EXAMPLES

1. Find the perpendicular distance of P(x′, y′, z′) from the line through A(a, b, c) whose direction-cosines are

cos α, cos β, cos γ. [*P. U. 1960 S*]

2. Find the distance of (−2, 1, 5) from the line through (2, 3, 5) whose direction-cosines are proportional to 2, −3, 6.

[*P(P). U.*]



The direction-cosines of the line are proportional to 2, −3, 6.

Dividing by $\sqrt{(2)^2+(-3)^2+(6)^2}=7$, the actual direction-cosines are

$$\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}.$$

(*i*) The distance between one pt. on the line, (2, 3, 5), and the given pt. $(-2, 1, 5)=\sqrt{(-2-2)^2+(1-3)^2+(5-5)^2}$

$$[\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \text{ (Art. 4)}]$$

$$=\sqrt{(-4)^2+(-2)^2+(0)^2}=\sqrt{16+4}=\sqrt{20}.$$

(*ii*) The projection of this distance on the line

$$=(-2-2)\tfrac{2}{7}+(1-3)(-\tfrac{3}{7})+(5-5)\tfrac{6}{7}$$

$$[(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n \text{ (Art. 14)}]$$

$$=(-4)\tfrac{2}{7}+(-2)(-\tfrac{3}{7})+0(\tfrac{6}{7})=-\tfrac{8}{7}+\tfrac{6}{7}=-\tfrac{2}{7}.$$

(*iii*) $\therefore \perp=\sqrt{(\text{distance})^2-(\text{projection})^2}=\sqrt{(\sqrt{20})^2-(-\tfrac{2}{7})^2}=\sqrt{20-\tfrac{4}{49}}$

$$=\sqrt{\frac{980-4}{49}}=\sqrt{\frac{976}{49}}=\frac{4\sqrt{61}}{7}.$$

3. Find the distance of A, (1, −2, 3) from the line PQ, through P, (2, −3, 5), which makes equal angles with the axes.

[*P(P). U. 1953 S*]

4. How far is the point (4, 1, 1) from the line of intersection of $x+y+z=4$, $x-2y-z=4$? [*P. U. 1961*]

5. Find the distances of the point (1, 2, 3) from the co-ordinate axes.

6. Find the perpendicular distance of an angular point of a cube from a diagonal which does not pass through that angular point. [*P. U. Eng. 2, 1937*]

7. *Equations of the perpendicular and foot of the perpendicular on a line.* **Show that the equations of the perpendicular from the point (1, 6, 3) to the line** $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ **are** $\frac{x-1}{0}=\frac{y-6}{-3}=\frac{z-3}{2}$, **and the foot of the perpendicular is (1, 3, 5).** [*P(P). U. 1950*]

8. The axes being rectangular, find the equations to the perpendicular from the origin to the line

$$x+2y+3z+4=0,\ 2x+3y+4z+5=0.$$ [*P. U. 1957 S*]

Find also the co-ordinates of the foot of the perpendicular.

[*J. & K. U. 1957*]

****9.** A line through the origin makes angles α, β, γ with its projections on the co-ordinate planes, which are rectangular. The distances of any point (x, y, z) from the line and its projections are d, a, b, c. Prove that

$$d^2=(a^2-x^2)\cos^2\alpha+(b^2-y^2)\cos^2\beta+(c^2-z^2)\cos^2\gamma.$$ [*Bell*]

[Let OA be the line, and A the pt. (x_1, y_1, z_1), so that the equations of OA are $\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}$. Let L be the projection of A on the yz-plane, so that L is $(0, y_1, z_1)$. Then the equations of OL, the projection of OA on the yz-plane, are $\frac{x}{0}=\frac{y}{y_1}=\frac{z}{z_1}$. Use the $\perp$ distance formula for the line, second form (Art. 41, Cor.).]

SECTION III
A LINE AND A PLANE

42. Intersection of a straight line and a plane. *To find the point of intersection of a given line and a given plane.*

Let the equations of the line be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \dots (1)$$

and the equation of the plane be

$$Ax+By+Cz+D=0 \quad ...(2)$$

Any pt. on the line (1) is (x_1+lr, y_1+mr, z_1+nr) ... (3)

[Rule (Art. 37, Cor. 2)]

If it lies on the plane (2), then

$$A(x_1+lr)+B(y_1+mr)+C(z_1+nr)+D=0$$

or $\quad r(Al+Bm+Cn)+Ax_1+By_1+Cz_1+D=0$ ... (4)

$$\therefore \qquad r=-\frac{Ax_1+By_1+Cz_1+D}{Al+Bm+Cn} \quad ...(5)$$

Substituting this value of r in (3), we get the required pt. of intersection.

****Cor. 1. Conditions of parallelism of a line and a plane.** *To deduce the conditions, that the line* $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ *may be parallel to the plane* $Ax+By+Cz+D=0$.

If the line is ‖ to the plane, it meets the plane at ∞

$\therefore \quad r=\infty, \quad \therefore$ from (5) (Art. 42),

the *denominator* $\qquad \mathbf{Al+Bm+Cn=0,}$

and the numerator $\qquad \mathbf{Ax_1+By_1+Cz_1+D\neq 0,}$*

which are the required conditions.

****Cor. 2.** *Conditions for a line to lie in a plane.* **To deduce the conditions, that the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ may lie in the plane $Ax+By+Cz+D=0$.**

If the line lies in the plane, then (4) (Art. 42) is true for *all* values of r.

$\therefore \quad$ coeff. of $r=0$, and constant term $=0$

[$\because \; r(0)+0=0$ for *all* values of r]

i.e., $\mathbf{Al+Bm+Cn=0,}$ **and** $\mathbf{Ax_1+By_1+Cz_1+D=0,}$

which are the required conditions.

EXAMPLES

1. Find where the line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-4}$ meets the plane $x+2y-z-8=0$. [*P. U.*]

The equations of the line are

$$\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{-4} \quad ...(1)$$

* $\because$ if $Ax_1+By_1+Cz_1+D$ is also $=0$, then (4) is true for *all* values of r $\because$ $r(0)+0=0$ for *all* values of r), $\therefore$ the line *lies* in the plane. (See Cor. 2.)

and the equation of the plane is

$$x+2y-z-8=0 \quad \ldots(2)$$

Any pt. on the line (1) is $(1+2r, -2+3r, 3-4r)$ ...(3)

[Rule (Art. 37, Cor. 2)]

If it lies on the plane (2), then

$1+2r+2(-2+3r)-(3-4r)-8=0$, or $12r-14=0$, $\therefore\ r=\frac{7}{6}$.

Substituting this value of r in (3), the pt. of intersection is

$[1+2(\frac{7}{6}), -2+3(\frac{7}{6}), 3-4(\frac{7}{6})]$, or $(\frac{10}{3}, \frac{3}{2}, -\frac{5}{3})$.

2. If the axes are rectangular, find the distance from the point (3, 4, 5) to the point where the line $\dfrac{x-3}{1}=\dfrac{y-4}{2}=\dfrac{z-5}{2}$ **meets the plane** $x+y+z=2$. [*P(P). U. 1948 Em.*]

3. Find the distance of the point $(-1, -5, -10)$ **from the point of intersection of the line** $\dfrac{x-2}{3}=\dfrac{y+1}{4}=\dfrac{z-2}{12}$ **and the plane** $x-y+z=5$, **(rectangular axes).** [*P(P). U. 1947*]

4. Find the point where the line joining $(1, -2, 3)$, $(3, -4, 5)$ cuts the plane $x-2y+3z=2$.

5. Find the distance of the point $(1, -2, 3)$ **from the plane** $x-y+z=5$ **measured parallel to the line** $\dfrac{x}{2}=\dfrac{y}{3}=\dfrac{z}{-6}$, **(rectangular axes).** [*P(P). U. 1957 S*]

6. Find the conditions that the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ **may lie in the plane** $ax+by+cz+d=0$.

Interpret these conditions geometrically. [*D.U.H. 1951*]

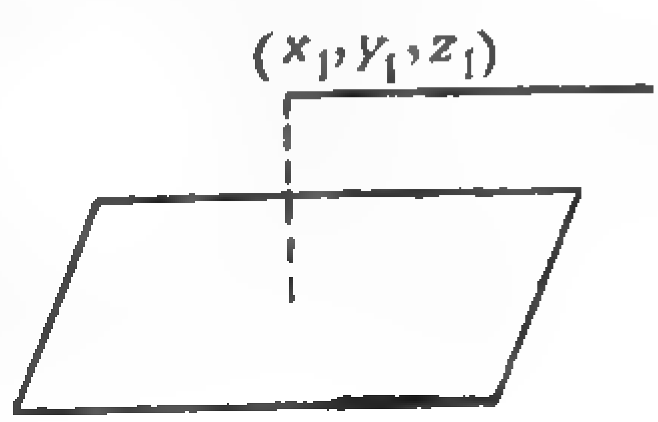
43. (*a*) **Conditions of parallelism of a line and a plane.** *To find the conditions that the line*

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

may be parallel to the plane $Ax+By+Cz+D=0$.

The equations of the line are $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ ...(1)

and the equation of the plane is $\quad Ax+By+Cz+D=0$ ...(2)



If the line is $\parallel$ *to the plane, then* (*i*) *it is* $\perp$ *to the normal to the plane, and* (*ii*) its pt. (x_1, y_1, z_1) does not lie on the plane.

Now the direction-cosines of the line (1) and the normal to the plane (2) are proportional to l, m, n ; A, B, C

$\therefore$ from (*i*), $\mathbf{Al+Bm+Cn=0}$,

[$aa'+bb'+cc' = 0$ (Art. 13, (*b*), Cor. 3)]

and from (*ii*), $\mathbf{Ax_1+By_1+Cz_1+D \neq 0}$,
which are the required conditions.

[**Aid to memory.** *If a line is* $\parallel$ *to a plane, its direction-cosines satisfy the first degree terms* (in the equation of the plane) $=0$, *i.e.*, $Al+Bm+Cn = 0$.]

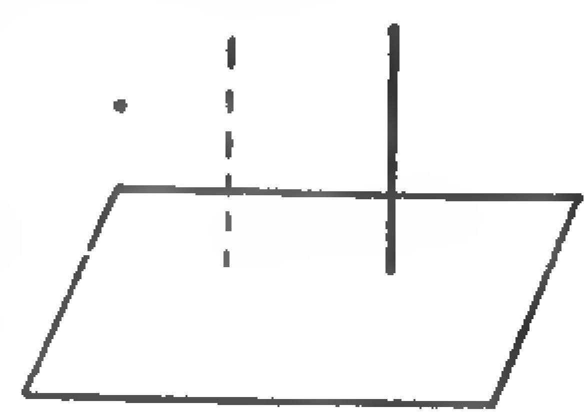
43. (*b*) Conditions of perpendicularity of a line and a plane.

To find the conditions that the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ *may be normal to the plane* $Ax+By+Cz+D = 0$.

The equations of the line are $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ ...(1)

and the equation of the plane is $Ax+By+Cz+D=0$ ...(2)

If the line is normal to the plane, it is $\parallel$ *to the normal to the plane* ... (*i*)



Now the direction-cosines of the line (1) and the normal to the plane (2) are proportional to l, m, n ; A, B, C

$\therefore$ from (*i*),

$$\frac{\mathbf{l}}{\mathbf{A}} = \frac{\mathbf{m}}{\mathbf{B}} = \frac{\mathbf{n}}{\mathbf{C}}, \left[\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} \text{ (Art. 13, (b), Cor. 4)} \right]$$

which are the required conditions.

43. (*c*) *Conditions for a line to lie in a plane.* **To find the conditions that the line** $\frac{\mathbf{x-x_1}}{\mathbf{l}} = \frac{\mathbf{y-y_1}}{\mathbf{m}} = \frac{\mathbf{z-z_1}}{\mathbf{n}}$ **may lie in the plane** $\mathbf{Ax+By+Cz+D=0}$**.**

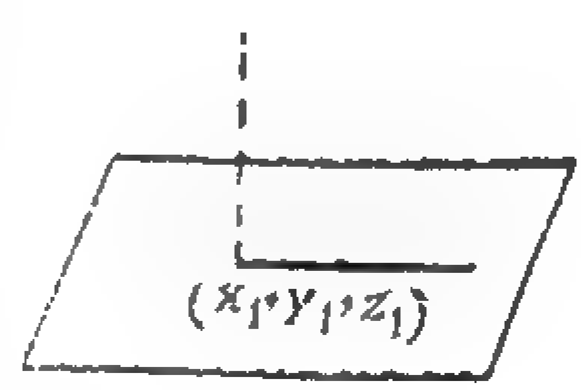
The equations of the line are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad ...(1)$$

and the equation of the plane is

$$Ax + By + Cz + D=0 \quad ...(2)$$

If the line lies in the plane, then (*i*) *it is* $\perp$ *to the normal to the plane, and* (*ii*) *its pt.* (x_1, y_1, z_1) *lies on the plane.*



Now the direction-cosines of the line (1) and the normal to the plane (2) are proportional to l, m, n ; A, B, C

$\therefore$ from (*i*), $\mathbf{Al+Bm+Cn=0}$.

[$aa'+bb'+cc'=0$ (Art. 13, (*b*), Cor. 3)]

and from (*ii*), $\mathbf{Ax_1+By_1+Cz_1+D=0}$,
which are the required conditions.

EXAMPLES

1. (*a*) Find the conditions that the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$
may be (*i*) parallel to,
or (*ii*) perpendicular to the plane $ax+by+cz+d=0$.
[*B. U. 1930 S*]

(*b*) **Find the conditions that the line** $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$
may lie in the plane $\mathbf{ax+by+cz+d=0}$. [*P. U. 1951*]

2. Prove that the line $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ is parallel to the plane $3x+2y-3z=0$.

3. (*a*) Show that the equations

$$by+cz+d=0,\ cz+ax+d=0,\ ax+by+d=0$$

represent planes parallel to OX, OY, OZ respectively.

(*b*) Find the equations of the planes through the points (1, 2, 0), (3, -4, 2) parallel to the co-ordinate axes.

4. (*a*) Find the equations to the line joining (1, 2, 3), (-3, 4, 3), and show that it is parallel to the plane XOY.

(*b*) Find the equations of a line through (x_1, y_1, z_1) parallel to the plane XOY.

(*b*) The equations of *any* line thro' (x_1, y_1, z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(1)$$

If it is $\parallel$ to the plane XOY, it is $\perp$ to the normal OZ to the plane, whose direction-cosines are 0, 0, 1

$\therefore$ $l(0)+m(0)+n(1)=0$, or $n=0$.

Substituting this value of n in (1),

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{0} \quad ...(2),$$

which are the required equations.

Note 1. *To show that the equations of a line through* (x_1, y_1, z_1) *parallel to the plane XOY are* $m(x-x_1)=l(y-y_1)$, $z=z_1$.

From the first and second members of the equations (2), cross-multiplying, $m(x-x_1)=l(y-y_1)$,
and from the second and third members, cross-multiplying, $z-z_1=0$, or $z=z_1$.

$\therefore$ $m(x-x_1)=l(y-y_1)$, $z=z_1$ ...(3)

which are the required equations.

Note 2. (2) is the symmetrical form of (3).

5. Show that the line $x+2y-z-3=0=x+3y-z-4$ is parallel to the plane $y=0$, and find the co-ordinates of the point where it meets the plane $x=0$.

6. **Find the equation to the plane through the points $(2, -1, 0)$, $(3, -4, 5)$ parallel to the line $2x=3y=4z$.** [*P. U. 1951*]

7. Prove that the join of $(1, 2, 3)$, $(2, 1, 4)$ is normal to the plane through $(5, -1, -9)$, $(1, 0, -4)$, $(-1, 2, 0)$, the axes being rectangular.

8. **If the axes are rectangular and P is the point $(2, -3, 1)$, find the equation of the plane through P perpendicular to OP.**

9. **Find the equation to the plane through $(2, -3, 1)$ normal to the line joining $(3, 4, -1)$, $(2, -1, 5)$, (axes rectangular).**

[*P(P). U. 1954*]

10. **Find the equation of the plane through $(3, 1, -1)$ perpendicular to the line of intersection of the planes**

$3x+4y+7z+4=0$, **and** $x-y+2z+3=0$; [*D.U.H. 1933*]

also of the plane through $(2, 1, -1)$ perpendicular to the line of intersection of the planes $2x+y-z=0$, $3x-3y+2z=0$.

11. **If the axes are rectangular, and if l_1, m_1, n_1 ; l_2, m_2, n_2 are direction-cosines, show that the equations to the planes through the lines which bisect the angles between**

$$x/l_1=y/m_1=z/n_1 \text{ and } x/l_2=y/m_2=z/n_2$$

and at right angles to the plane containing them, are

$$(l_1 \pm l_2)x+(m_1 \pm m_2)y+(n_1 \pm n_2)z=0. \qquad [\textit{P. U. 1945}]$$

[**Note.** It is assumed that the ambiguous signs are taken *all* positive or *all* negative.]

12. **Through a point P, (x', y', z') a plane is drawn at right angles to OP to meet the axes (rectangular) in A, B, C. Prove that the area of the triangle ABC is $\dfrac{r^5}{2x'y'z'}$, where r is the measure of OP.** [*P. U. 1953*]

13. **The equations to AB are $\dfrac{x}{1}=\dfrac{y}{-2}=\dfrac{z}{3}$. Through a point P(1, 2, 3), PN is drawn perpendicular to AB, and PQ is drawn parallel to the plane $2x+3y+4z=0$ to meet AB in Q. Find the equations of PN and PQ and the co-ordinates of N and Q.** [*L. U.*]

14. Show that if the axes are rectangular, the equations to the perpendicular from the point (x_1, y_1, z_1) to the plane

$$Ax+By+Cz+D=0$$

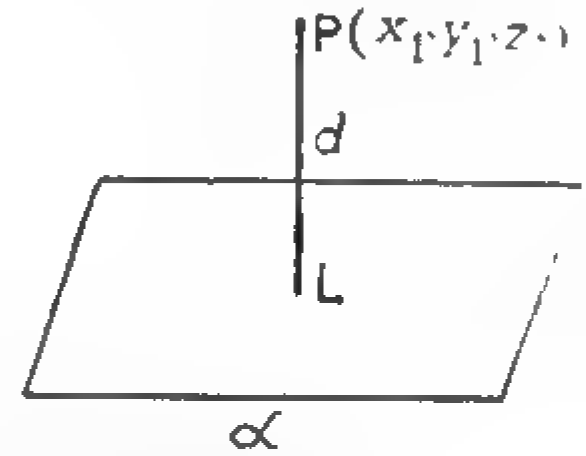
are
$$\frac{x-x_1}{A}=\frac{y-y_1}{B}=\frac{z-z_1}{C},$$

and deduce the perpendicular distance of the point (x_1, y_1, z_1) from the plane. [*D. U. H. 1953*]

Let P be the pt. (x_1, y_1, z_1), and α the plane

$$Ax+By+Cz+D=0 \quad ...(1)$$

From P draw PL $\perp$ on the plane α.



(*i*) To find the equations of PL.

The equations of *any* line thro' P are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(2)$$

If it is $\perp$ to the plane (1), it is $\parallel$ to the normal to the plane,

$\therefore \quad \dfrac{l}{A}=\dfrac{m}{B}=\dfrac{n}{C}$. [Art. 13, (*b*), Cor. 4]

Substituting these values of l, m, n in (2), the equations of PL are

$$\frac{x-x_1}{A}=\frac{y-y_1}{B}=\frac{z-z_1}{C}.$$

(*ii*) [To find its pt. of intersection with the plane (1).]

Let the pt. L on PL be (x_1+Ar, y_1+Br, z_1+Cr) ...(3)

$\because$ it lies on the plane (1),

$\therefore \quad A(x_1+Ar)+B(y_1+Br)+C(z_1+Cr)+D=0$

or $\quad r(A^2+B^2+C^2)+Ax_1+By_1+Cz_1+D=0$

or $\quad r=-\dfrac{Ax_1+By_1+Cz_1+D}{A^2+B^2+C^2} \quad ...(4)$

Now $\quad PL^2=(x_1+Ar-x_1)^2+(y_1+Br-y_1)^2+(z_1+Cr-z_1)^2$

[$(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2$ (Art. 4)]

or $\quad d^2=r^2(A^2+B^2+C^2)$ [Substitute from (4)]

$$=\frac{(Ax_1+By_1+Cz_1+D)^2}{(A^2+B^2+C^2)^2}(A^2+B^2+C^2)$$

$$=\frac{(Ax_1+By_1+Cz_1+D)^2}{A^2+B^2+C^2} \quad ...(5)$$

$$\therefore \quad d=\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}.$$

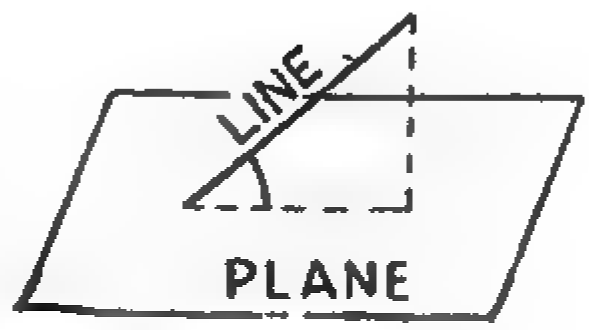
****Complete perpendicular distance formula for the plane (equation in the general form).**

From (5), $d=\pm\dfrac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}$,

that sign being taken on the R.H.S. which gives a +ve result for d.

Note. Foot of the perpendicular. Substituting the value of r $\left(=-\dfrac{Ax_1+By_1+Cz_1+D}{A^2+B^2+C^2}\right)$ from (4) in (3), we get the foot of the $\perp$.

[**Angle between a line and a plane.** **Def.** The **angle between a straight line and a plane** is the angle between the line and its projection on the plane. (Generally the *acute* angle is taken.)]



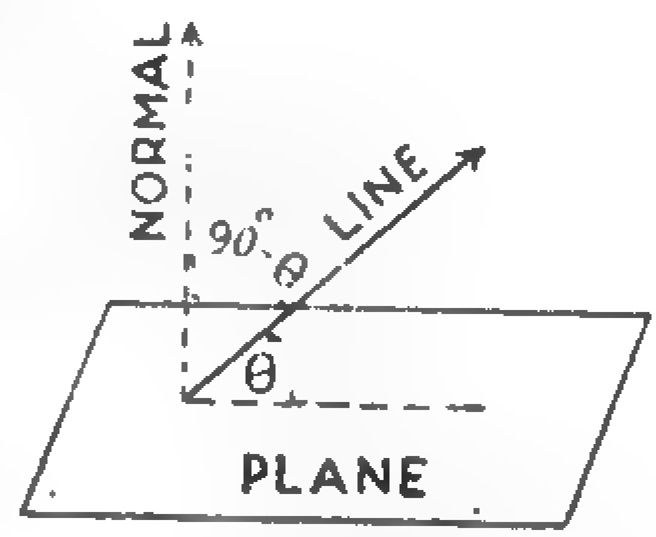
15. *Angle between a line and a plane.* **Find the angle between the line** $\frac{\mathbf{x-x_1}}{\mathbf{l}}=\frac{\mathbf{y-y_1}}{\mathbf{m}}=\frac{\mathbf{z-z_1}}{\mathbf{n}}$ **and the plane** $\mathbf{ax+by+cz+d=0}$**.**

[*P. U. 1934 S*]

The equations of the line are $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ... (1)

and the equation of the plane is $ax+by+cz+d=0$ ... (2)

If θ is the angle between the line and the plane, then $(90°-\theta)$ is the angle between the line and the normal to the plane ... (*i*)



Now the direction-cosines of the line (1) and the normal to the plane (2) are proportional to l, m, n* ; a, b, c

∴ from (*i*),

$$\cos(90°-\theta)=\frac{la+mb+nc}{\sqrt{l^2+m^2+n^2}\sqrt{a^2+b^2+c^2}} \quad \text{[Art. 13, (}b\text{)]}$$

or
$$\sin\theta=\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}}$$

or
$$\theta=\sin^{-1}\frac{al+bm+cn}{\sqrt{a^2+b^2+c^2}\sqrt{l^2+m^2+n^2}},$$

which is the required angle.

***Note. Important.** We have *no* right to assume that l, m, n are the *actual* direction-cosines of the line unless they are given in the question as such.

Thus, here we have *not* assumed that $l^2+m^2+n^2=1$.

44. (*a*) **Any plane through the line of intersection of two given planes (equations of both planes in the general form).** *The equation of* **any** *plane through the line of intersection of the planes*

$$Ax+By+Cz+D=0,\ A'x+B'y+C'z+D'=0$$

is
$$\mathbf{Ax+By+Cz+D+k(A'x+B'y+C'z+D')=0},$$
where k is any constant.

[In words : **one plane + k (other plane)=0,** *where 'one plane' stands for the 'L.H.S. of the equation of one plane (R.H.S. being zero)', and so for the 'other plane'.*]

Proof. The equations of the planes are

$$Ax + By + Cz + D = 0 \quad \dots (1)$$

$$A'x + B'y + C'z + D' = 0 \quad \dots (2)$$

Consider the equation

$$Ax + By + Cz + D + k(A'x + B'y + C'z + D') = 0 \quad \dots (3)$$

where k is any constant.

(*i*) It is an equation of the first degree in x, y, z

$\therefore$ it represents a plane. [Art. 21]

(*ii*) The co-ordinates of the pts. which satisfy both (1) and (2), also satisfy (3) [$\because$ substituting from (1) and (2) in (3), we get $0 + k(0) = 0$, or $0 = 0$, which is true]

$\therefore$ the pts. of intersection of the planes (1) and (2) lie on the plane (3)

$\therefore$ (3) is the equation of any plane thro' the line of intersection of the planes (1) and (2).

Note. The value of k is found from the second condition satisfied by the plane.

Abridged notation.

If $u = Ax + By + Cz + D$, $v = A'x + B'y + C'z + D'$, *the equation of* **any** *plane through the line of intersection of the planes* $u = 0, v = 0$ *is* $\mathbf{u + kv = 0}$, *where* k *is any constant.*

EXAMPLES

1. (*a*) If $S = 0$ and $S' = 0$ are the equations of two planes, prove that $S - \lambda S' = 0$ is the general equation of a plane through their intersection.

(*b*) **Find the equation of the plane through the origin and through the intersection of the two planes**

$$\mathbf{x - 3y + 2z + 3 = 0 \text{ and } 3x - y - 2z - 5 = 0.}$$ [*P. U.*]

(*b*) The equations of the planes are

$$x - 3y + 2z + 3 = 0,$$

and

$$3x - y - 2z - 5 = 0.$$

The equation of *any* plane thro' their line of intersection is

$$x - 3y + 2z + 3 + k(3x - y - 2z - 5) = 0 \quad \dots (1)$$

[one plane $+k$ (other plane) $= 0$ (Art. 44, (*a*))]

If it passes thro' (0, 0, 0), then $3 + k(-5) = 0$, $\therefore$ $k = \frac{3}{5}$.

Substituting this value of k in (1),

$$x - 3y + 2z + 3 + \tfrac{3}{5}(3x - y - 2z - 5) = 0$$

or $5(x - 3y + 2z + 3) + 3(3x - y - 2z - 5) = 0$, or $14x - 18y + 4z = 0$,

or $$7x - 9y + 2z = 0,$$

which is the required equation.

2. Find the equation of the plane containing the line

$$x+y+z = 1, \; 2x+3y+4z=5,$$

and perpendicular to the plane $x-y+z = 0$. [*B.H.U. 1941*]

3. Find the equation of the plane through the line

$$u \equiv ax+by+cz+d=0, \; v \equiv a'x+b'y+c'z+d'=0$$

parallel to the line $x/l = y/m = z/n$. [*Pesh. U. 1955*]

4. The plane $ax+by=0$ is rotated about its line of intersection with the plane $z=0$ through an angle α. Prove that the equation of the plane in its new position is

$$ax+by\pm z\sqrt{a^2+b^2}\tan\alpha=0.$$

**5. Find the equations to the line through (f, g, h) which is parallel to the plane $lx+my+nz=0$ and intersects the line

$$ax+by+cz+d=0, \; a'x+b'y+c'z+d'=0. \text{ [D.U.H. 1954]}$$

[The required line (*i*) lies in the plane thro' (f, g, h) and $\parallel$ to the plane $lx+my+nz = 0$, and (*ii*) lies in the plane thro' (f, g, h) and the line $ax+by+cz+d=0$, $a'x+b'y+c'z+d'=0$.]

44. (*b*) Any plane through a given line (equations in the symmetrical form). *The equation of* **any** *plane through the line*

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \text{ is}$$

$$A(x-x_1)+B(y-y_1)+C(z-z_1) = 0 \;...\; (1)$$

$$\text{where} \quad Al+Bm+Cn = 0 \;...\; (2)$$

Proof. The equations of the line are $\dfrac{x-x_1}{l} = \dfrac{y-y_1}{m} = \dfrac{z-z_1}{n}$.

(*i*) The equation of *any* plane thro' (x_1, y_1, z_1) is

$$A(x-x_1)+B(y-y_1)+C(z-z_1) = 0 \;...\; (1) \quad \text{[Art. 21, Cor.]}$$

(*ii*) If it passes thro' the line, its normal is $\perp$ to the line,

$\therefore \; Al+Bm+Cn = 0 \;...\; (2)$ [Art. 13, (*b*), Cor. 3]

Cor. Plane through one line and parallel to another line (equations of both lines in the symmetrical form). *The equation of the plane through the line* $\dfrac{x-x_1}{l_1} = \dfrac{y-y_1}{m_1} = \dfrac{z-z_1}{n_1}$, *and parallel to the line* $\dfrac{x}{l_2} = \dfrac{y}{m_2} = \dfrac{z}{n_2}$ *is*

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix} = 0.$$

Proof. The equation of *any* plane thro' the line

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \text{ is}$$

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \quad \dots (1)$$

where $\quad Al_1 + Bm_1 + Cn_1 = 0 \quad \dots (2)$ [Art. 44, (*b*)]

If it is ∥ to the line $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$, its normal is ⊥ to this line

$\therefore \quad Al_2+Bm_2+Cn_2=0 \quad \dots (3)$ [Art. 13, (*b*), Cor. 3]

Eliminating A, B, C from (1), (2), (3),

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0,$$

which is the required equation.

[**Rule to write down the equation of the plane through one line and parallel to another line (equations of both lines in the symmetrical form) :**

(*i*) *Write down in two rows the numerators and denominators in the equations of the first line and in the third row the (proportional) direction-cosines of the second line.*

(*ii*) *Equate to zero the determinant so formed. This is the required equation.*

Note. *To simplify the equation, expand the determinant by means of the first row.*]

EXAMPLES

1. Find the equation of the plane through the point $(\alpha', \beta', \gamma')$ **and through the line whose equations are** $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$.

[*P*(*P*). *U*. *1953 S*]

The equation of *any* plane thro' the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ is

$$A(x-\alpha)+B(y-\beta)+C(z-\gamma)=0 \quad \dots (1)$$

where $\quad Al+Bm+Cn=0 \quad \dots (2)$ [Art. 44, (*b*)]

If it passes thro' $(\alpha', \beta', \gamma')$, then

$$A(\alpha'-\alpha)+B(\beta'-\beta)+C(\gamma'-\gamma)=0 \quad \dots (3)$$

Eliminating A, B, C from (1), (3), (2),

$$\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ \alpha'-\alpha, & \beta'-\beta, & \gamma'-\gamma \\ l, & m, & n \end{vmatrix}=0,$$

which is the required equation.

2. Prove that the plane through the point (x_0, y_0, z_0) and the

line $x=py+q=rz+s$ is given by

$$\begin{vmatrix} x, & py+q, & rz+s \\ x_0, & py_0+q, & rz_0+s \\ 1, & 1, & 1 \end{vmatrix}=0. \qquad [\ P(P).\ U.\ 1952\ S\ ]$$

3. **Find the equation to the plane through the line**

$$\frac{\mathbf{x}-\alpha}{\mathbf{l}}=\frac{\mathbf{y}-\beta}{\mathbf{m}}=\frac{\mathbf{z}-\gamma}{\mathbf{n}}$$

parallel to the line $\dfrac{\mathbf{x}}{\mathbf{l'}}=\dfrac{\mathbf{y}}{\mathbf{m'}}=\dfrac{\mathbf{z}}{\mathbf{n'}}$. [*P. U. 1947*]

4. Find the equations to the planes through the lines

(*i*) $2x+3y-5z-4=0=3x-4y+5z-6$, (*ii*) $\dfrac{x-1}{2}=\dfrac{y-2}{3}=\dfrac{z-3}{4}$

parallel to the co-ordinate axes. [*P. U. 1944 S*]

SECTION IV
TWO LINES
Intersecting lines.

45. (*a*) *Condition of intersection of two lines (equations of both lines in the symmetrical form).* **To find the condition that two given lines may intersect.**

Let the equations of the lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad ...(1)$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad ...(2)$$

[**Method of plane.**] If the lines intersect, they lie in a plane.

The equation of *any* plane thro' the line (1) is

$$A(x-x_1)+B(y-y_1)+C(z-z_1)=0 \quad ...(3)$$

where $Al_1+Bm_1+Cn_1=0$ ...(4) [Art. 44, (*b*)]

If the line (2) lies in this plane, it is $\perp$ to the normal to the plane, and its pt. (x_2, y_2, z_2) lies on this plane.

$$\therefore \quad Al_2+Bm_2+Cn_2=0 \quad ...(5)$$

and $A(x_2-x_1)+B(y_2-y_1)+C(z_2-z_1)=0$ ...(6) [Art. 43, (*c*)]

Eliminating A, B, C from (6), (4), (5),

$$\begin{vmatrix} \mathbf{x_2-x_1}, & \mathbf{y_2-y_1}, & \mathbf{z_2-z_1} \\ \mathbf{l_1}, & \mathbf{m_1}, & \mathbf{n_1} \\ \mathbf{l_2}, & \mathbf{m_2}, & \mathbf{n_2} \end{vmatrix}=\mathbf{0},$$

which is the required condition.

[**Rule to prove that two given lines intersect (or are co-planar) (equations of both lines in the symmetrical form) :**

(*i*) *Write down the equation of the plane through one line and parallel to the other.* [Rule (Art. 44, (*b*), Cor.)]

(*ii*) *Show that this plane passes through one point of the other line.*

For the equation of the plane thro' the line (1) and ∥ to the line (2) is

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0. \text{ [Rule (Art. 44, (b), Cor.)]}$$

If it passes thro' one pt. (x_2, y_2, z_2) of the line (2), then

$$\begin{vmatrix} x_2-x_1, & y_2-y_1, & z_2-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0,$$

which is the condition of Art. 45, (*a*).]

Cor. To find the equation of the plane in which two intersecting lines lie (equations of both lines in the symmetrical form) :

Eliminating A, B, C from (3), (4), (5),

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0,$$

which is the required equation.

[**Rule to find the equation of the plane in which two intersecting (or two coplanar) lines lie (equations of both lines in the symmetrical form) :** *Same as the Rule to find the equation of the plane through one line and parallel to the other* [Rule (Art. 44, (*b*), Cor.)].

For the equation of Art. 45, (*a*), Cor. is the same as that of Art. 44, (*b*), Cor.

Note. To simplify the equation, expand the determinant by means of the first row.]

EXAMPLES

1. *Coplanar lines.* **Find the condition that two given lines may be coplanar.** [*P. U. 1960*]

Assuming this condition to be satisfied, find the equation of the plane containing them. [*P. U. 1956 S*]

2. Prove that the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4};\ \frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$$

are coplanar. [*P(P). U. 1959*]

The equations of the lines are

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad ...(1)$$

$$\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5} \quad ...(2)$$

(*i*) The equation of the plane thro' the line (1) and || to the line (2) is

$$\begin{vmatrix} x-1, & y-2, & z-3 \\ 2, & 3, & 4 \\ 3, & 4, & 5 \end{vmatrix}=0 \quad \text{[Rule (Art. 44, (b), Cor.)]}$$

[Expand the determinant by means of first row]

or $(x-1)(15-16)-(y-2)(10-12)+(z-3)(8-9)=0$

or $(x-1)(-1)-(y-2)(-2)+(z-3)(-1)=0$

or $-x+2y-z=0$, or $x-2y+z=0$ ... (3)

(*ii*) It passes thro' the pt. (2, 3, 4) of the line (2),

if $2-2(3)+4=0$, or $0=0$, which is true

∴ the lines are coplanar.

Note 1. To find the equation of the plane in which the lines lie. From (3), the equation of the plane, in which the lines lie, is

$$x-2y+z=0.$$

Note 2. The method of Ex. 2 enables us not only to prove that two given lines are coplanar, but also to find the *equation of the plane in which they lie.*

3. Show that the two lines $x-1=2y-4=3z$, $3x-5=4y-9=3z$ meet in a point, and that the equation of the plane on which they lie is $3x-8y+3z+13=0$. [*P. U.*]

4. Prove that the lines

$$\frac{x-a+d}{\alpha-\delta}=\frac{y-a}{\alpha}=\frac{z-a-d}{\alpha+\delta};$$

$$\frac{x-b+c}{\beta-\gamma}=\frac{y-b}{\beta}=\frac{z-b-c}{\beta+\gamma}$$

are coplanar, and find the equation to the plane in which they lie. [*P(P). U. 1955*]

5. Prove that the lines $x=ay+b=cz+d$, $x=\alpha y+\beta=\gamma z+\delta$ are coplanar if $(\gamma-c)(a\beta-b\alpha)-(\alpha-a)(c\delta-d\gamma)=0$. [*B. U. 1953*]

6. Prove that the lines

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n},$$

$$ax+by+cz+d=0=a'x+b'y+c'z+d'$$

are coplanar if $\dfrac{ax_1+by_1+cz_1+d}{al+bm+cn}=\dfrac{a'x_1+b'y_1+c'z_1+d'}{a'l+b'm+c'n}$.

7. Show that the straight lines

$$\frac{x}{\alpha}=\frac{y}{\beta}=\frac{z}{\gamma},\ \frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma},\ \frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

will lie in one plane, if $\frac{l}{\alpha}(b-c)+\frac{m}{\beta}(c-a)+\frac{n}{\gamma}(a-b)=0.$

[*P. U. 1960*]

8. Prove that the three lines drawn from O with direction-cosines l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are coplanar if

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=0.$$ [*P(P). U. 1954 S*]

The three lines drawn from O with direction-cosines l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are coplanar if they are all $\perp$ to the same line thro' O*. Let l, m, n be the direction-cosines of this line. Then

$$ll_1+mm_1+nn_1=0 \quad ...(1)$$
$$ll_2+mm_2+nn_2=0 \quad ...(2)$$
$$ll_3+mm_3+nn_3=0 \quad ...(3)$$

Eliminating l, m, n from (1), (2), (3),

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}=0,$$

which is the required condition.

9. Prove that the three lines drawn from O with direction-cosines proportional to (1, −1, 1), (2, −3, 0), (1, 0, 3) lie in one plane. [*P(P). U. 1952 S*]

10. Show that the equation of the plane through

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n},$$

and which is perpendicular to the plane containing

$$\frac{x}{m}=\frac{y}{n}=\frac{z}{l} \text{ and } \frac{x}{n}=\frac{y}{l}=\frac{z}{m}$$

is $x(m-n)+y(n-l)+z(l-m)=0.$ [*P(P). U. 1954 S*]

11. *Point of intersection of two coplanar lines.* **Show that the lines**

$$\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5}\ ;\ \frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3}$$

are coplanar ; find their common point and the equation of the plane in which they lie. [*P. U. 1949*]

[Method of intersection.]

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*All straight lines drawn perpendicular to a given straight line at a given point are coplanar. (See the author's *New Elementary Geometry*, Thirteenth Edition, Page 54, Prop. 4.)

(a) The equations of the lines are

$$\frac{x-5}{4}=\frac{y-7}{4}=\frac{z+3}{-5} \quad \ldots (1)$$

$$\frac{x-8}{7}=\frac{y-4}{1}=\frac{z-5}{3} \quad \ldots (2)$$

Any pt. on the line (1) is $(5+4r, 7+4r, -3-5r)$ ... (3)

It lies on the line (2), if $\dfrac{5+4r-8}{7}=\dfrac{7+4r-4}{1}=\dfrac{-3-5r-5}{3}$

or $$\frac{-3+4r}{7}=\frac{3+4r}{1}=\frac{-8-5r}{3} \quad \ldots (4)$$

From the first and second members,

$$-3+4r=21+28r, \text{ or } -24r=24, \quad \therefore \ r=-1.$$

Substituting this value of r $(=-1)$ in (4), we get

$$\frac{-3+4(-1)}{7}=\frac{3+4(-1)}{1}=\frac{-8-5(-1)}{3}, \text{ or } \frac{-7}{7}=\frac{-1}{1}=\frac{-3}{3},$$

or $-1=-1=-1$,

which is true,

$\therefore$ the lines intersect, and are $\therefore$ coplanar.

(b) Substituting the value of $r(=-1)$ in (3), the pt. of intersection is $[5+4(-1), 7+4(-1), -3-5(-1)]$, or $(1, 3, 2)$.

[**Check.** The co-ordinates of the pt. of intersection $(1, 3, 2)$ satisfy the equations of both the lines (1) and (2) thus,

$$\frac{1-5}{4}=\frac{3-7}{4}=\frac{2+3}{-5}, \text{ or } \frac{-4}{4}=\frac{-4}{4}=\frac{5}{-5}, \text{ or } -1=-1=-1,$$

$$\frac{1-8}{7}=\frac{3-4}{1}=\frac{2-5}{3}, \text{ or } \frac{-7}{7}=\frac{-1}{1}=\frac{-3}{3}, \text{ or } -1=-1=-1.]$$

Note. The above method enables us not only to prove that two given lines intersect (or are coplanar), but also to find the *co-ordinates of their point of intersection.*

(c) The equation of the plane, in which the lines lie, is

$$\begin{vmatrix} x-5, & y-7, & z+3 \\ 4, & 4, & -5 \\ 7, & 1, & 3 \end{vmatrix}=0 \qquad \text{[Rule (Art. 45, (a), Cor.)]}$$

or $(x-5)[12-(-5)]-(y-7)[12-(-35)]+(z+3)(4-28)=0$

or $(x-5)(17)-(y-7)(47)+(z+3)(-24)=0$,

or $17x-47y-24z+172=0$.

$$\begin{array}{r} -85+329-72 \\ -72 \\ \hline -157 \end{array} \qquad \begin{array}{r} 329 \\ -157 \\ \hline 172 \end{array}$$

45 (b). Condition of intersection of two lines (equations of both lines in the general form). *To find the condition that the lines*

$$a_1x+b_1y+c_1z+d_1=0,\ a_2x+b_2y+c_2z+d_2=0\ ;$$

and $$a_3x+b_3y+c_3z+d_3=0,\ a_4x+b_4y+c_4z+d_4=0$$

may intersect.

The equations of the lines are

$$a_1x+b_1y+c_1z+d_1=0,$$
$$a_2x+b_2y+c_2z+d_2=0;$$

and
$$a_3x+b_3y+c_3z+d_3=0,$$
$$a_4x+b_4y+c_4z+d_4=0.$$

If the lines intersect, their pt. of intersection lies on each of the four planes represented by the above equations.

$\therefore$ eliminating x, y, z from these equations,

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix} = 0,$$

which is the required condition.

Note. In numerical examples it is better to reduce the equations of one line to the symmetrical form, and proceed as in the Rule (Art. 45, (a)). (See Misc. Ex. 19, Chap. V.)

Cor. To find the point of intersection of the two lines. Solving any three of the four equations, say, the first three, simultaneously as in Algebra, we get the values of x, y, z, which are the co-ordinates of the pt. of intersection.

EXAMPLES

1. Prove that the lines

$$ax+by+cz+d=0=a'x+b'y+c'z+d'\ ;$$
$$\alpha x+\beta y+\gamma z+\delta=0=\alpha'x+\beta'y+\gamma'z+\delta'$$

are coplanar if $\begin{vmatrix} a, & a', & \alpha, & \alpha' \\ b, & b', & \beta, & \beta' \\ c, & c', & \gamma, & \gamma' \\ d, & d', & \delta, & \delta' \end{vmatrix} = 0.$

2. Find the equation of the plane through the lines

$ax+by+cz=0=a'x+b'y+c'z,\ \alpha x+\beta y+\gamma z=0=\alpha'x+\beta'y+\gamma'z.$

[*P. U. 1959 S*]

3. *A, A' ; B, B' ; C, C' are points on the axes ; show that the lines of intersection of the planes*

A'BC, AB'C' ; B'CA, BC'A' ; C'AB, CA'B'

are coplanar. [*Ag. U. 1946*]

Shortest distance between two lines.

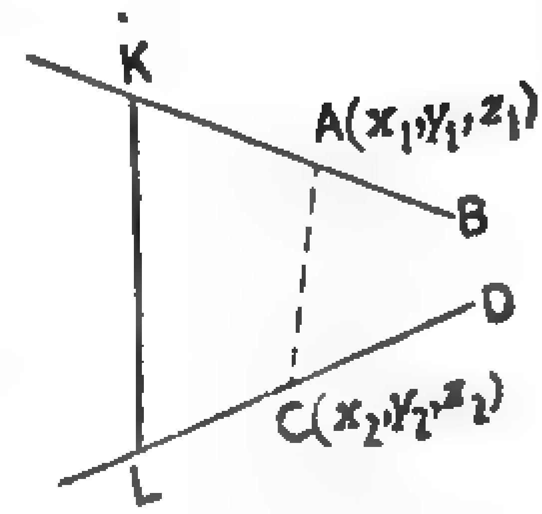
46. *Shortest distance between two lines (equations of both lines in the symmetrical form).* **To find the shortest distance between two straight lines whose equations are given.**

[Method of projection.]

Let the equations of the lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad \ldots (1)$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad \ldots (2)$$



Let AB, CD be the lines, and A, C the pts. (x_1, y_1, z_1), (x_2, y_2, z_2).

Let KL be the shortest distance* (S. D.) between AB and CD, and l, m, n its direction-cosines.

Then $\because$ KL is $\perp$ to both AB, CD

$\therefore$ $ll_1+mm_1+nn_1=0$,

$ll_2+mm_2+nn_2=0$.

$\therefore$ by cross-multiplication,

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1}$$

i.e., the direction-cosines of KL are proportional to

$$m_1n_2-m_2n_1,\ n_1l_2-n_2l_1,\ l_1m_2-l_2m_1 \quad \ldots(3)$$

Dividing by $\sqrt{\Sigma(m_1n_2-m_2n_1)^2}$, [Rule (Art. 10)]

the actual direction-cosines are

$$\frac{m_1n_2-m_2n_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}},\ \frac{n_1l_2-n_2l_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}},\ \frac{l_1m_2-l_2m_1}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}.$$

(*a*) *To find the length of the S.D.*

KL=projection of AC on KL

$$=\frac{(x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-n_2l_1)+(z_2-z_1)(l_1m_2-l_2m_1)}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}} \quad \ldots(4)$$

[$(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$ (Art. 14)]

(*b*) *To find the equations of the S.D.*

KL is the line of intersection of the planes AKL (thro' AB and KL) and CKL (thro' CD and KL)

*If two straight lines neither intersect nor are parallel, then

(*i*) there is one straight line perpendicular to both of them ;

(*ii*) this common perpendicular is the shortest distance between the given lines. (See the author's *New Elementary Geometry*, Thirteenth Edition, Prop. 17.)

$\therefore$ its equations are

$$\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ m_1n_2-m_2n_1, & n_1l_2-n_2l_1, & l_1m_2-l_2m_1 \end{vmatrix}=0,$$

$$\begin{vmatrix} x-x_2, & y-y_2, & z-z_2 \\ l_2, & m_2, & n_2 \\ m_1n_2-m_2n_1, & n_1l_2-n_2l_1, & l_1m_2-l_2m_1 \end{vmatrix}=0.$$

[Rule (Art. 45, (*a*), Cor.), from (1) and (3), and from (2) and (3)]

Cor. To deduce the condition that two given lines may be coplanar.

Let the equations of the lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1},\ \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}.$$

If the lines are coplanar, the shortest distance between them $=0$,

i.e., $$\frac{(x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-n_2l_1)+(z_2-z_1)(l_1m_2-l_2m_1)}{\sqrt{\Sigma(m_1n_2-m_2n_1)^2}}=0$$

$\therefore (x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-n_2l_1)+(z_2-z_1)(l_1m_2-l_2m_1)=0,$
which is the required condition.

Note. Abbreviation. 'Shortest distance' is sometimes abbreviated into **S.D.**

EXAMPLES

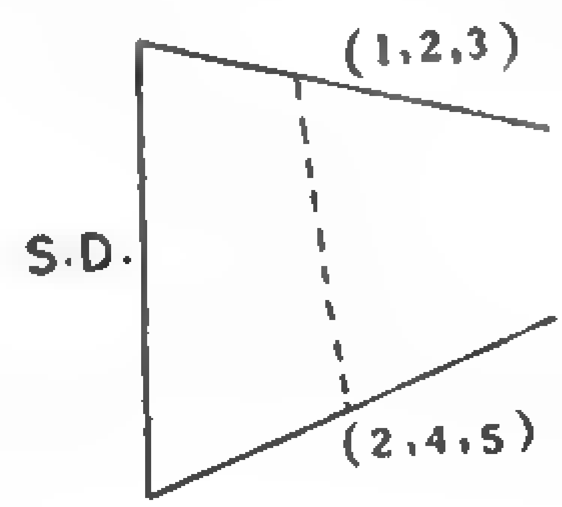
1. The axes being rectangular, find the shortest distance between the lines $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$; $\dfrac{x-\alpha'}{l'}=\dfrac{y-\beta'}{m'}=\dfrac{z-\gamma'}{n'}$.

[*P. U. 1961*]

2. Find the length and the equations of the shortest distance between the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4};\ \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}.$$ [*P. U. 1958 S*]

The equations of the lines are



$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \quad ...(1)$$

$$\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5} \quad ...(2)$$

[**Method of projection.**]

Let l,m,n be the direction-cosines of the S.D.

Then $\because$ it is $\perp$ to both the lines (1) and (2),

$\therefore \quad 2l+3m+4n=0,$

$3l+4m+5n=0.$

$\therefore$ by cross-multiplication,

$$\frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9}, \text{ or } \frac{l}{-1}=\frac{m}{2}=\frac{n}{-1}$$

i.e., the direction-cosines of the S. D. are proportional to

$$-1, 2, -1 \quad \dots (3)$$

Dividing by $\sqrt{(-1)^2+(2)^2+(-1)^2}=\sqrt{6}$,

the actual direction-cosines are $-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}$.

(*a*) *To find the length of the S. D.*

S.D. = projection of the join of (1, 2, 3) and (2, 4, 5) on the S. D.

$$=(2-1)\left(-\frac{1}{\sqrt{6}}\right)+(4-2)\frac{2}{\sqrt{6}}+(5-3)\left(-\frac{1}{\sqrt{6}}\right)$$

$$=1\left(-\frac{1}{\sqrt{6}}\right)+2\left(\frac{2}{\sqrt{6}}\right)+2\left(-\frac{1}{\sqrt{6}}\right)=\frac{1}{\sqrt{6}}.$$

(*b*) *To find the equations of the S. D.*

The S.D. is the line of intersection of the plane thro' the line (1) and the S. D., and the plane thro' the line (2) and the S. D.

$\therefore$ its equations are

$$\begin{vmatrix} x-1, & y-2, & z-3 \\ 2, & 3, & 4 \\ -1, & 2, & -1 \end{vmatrix}=0, \quad \begin{vmatrix} x-2, & y-4, & z-5 \\ 3, & 4, & 5 \\ -1, & 2, & -1 \end{vmatrix}=0$$

[Rule (Art. 45, (*a*), Cor.), from (1) and (3), and from (2) and (3)]

or $\quad (x-1)(-3-8)-(y-2)[-2-(-4)]+(z-3)[4-(-3)]=0,$

$\quad (x-2)(-4-10)-(y-4)[-3-(-5)]+(z-5)[6-(-4)]=0$

or $\quad (x-1)(-11)-(y-2)(2)+(z-3)(7)=0,$

$\quad (x-2)(-14)-(y-4)(2)+(z-5)(10)=0$

or $\quad -11x-2y+7z-6=0, \; -14x-2y+10z-14=0$

or $\quad 11x+2y-7z+6=0, \; 7x+y-5z+7=0.$

3. **Determine the length of the shortest distance between the lines $2x=y=-z$ and $x-1=-y-2=-2z+1$. Find the equations of the straight line of which the shortest distance forms a part.**

**4. If the axes are rectangular, the S. D. between the lines

$$y=az+b, \; z=\alpha x+\beta; \; y=a'z+b', \; z=\alpha' x+\beta' \text{ is}$$

$$\frac{(\alpha-\alpha')(b-b')-(\alpha'\beta-\alpha\beta')(a-a')}{\{\alpha^2\alpha'^2(a-a')^2+(\alpha-\alpha')^2+(a\alpha-a'\alpha')^2\}^{\frac{1}{2}}}. \quad \text{[P.U. 1952 S]}$$

5. *Prove that the shortest distances between the diagonal of a rectangular parallelepiped and the edges not meeting it are*

$$\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}},$$

where a, b, c are the lengths of the edges.

6. *Points of intersection of the shortest distance with the lines.* **Find the shortest distance between the lines**

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1};\ \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}.$$

[*J. & K. U. 1958*]

Find also its equations and the points in which it meets the given lines. [*Ag. U. 1953*]

[**Method of intersections.**]

The equations of the lines are

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \quad \dots (1)$$

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4} \quad \dots (2)$$

Let the S. D. meet the lines in K, L.

Let the pt. K on the line (1) be $(3+3r,\ 8-r,\ 3+r)$...(3), and the pt. L on the line (2) be $(-3-3r',\ -7+2r',\ 6+4r')$ ... (4)

[*Note the use of r' instead of r*]

Then the direction-cosines of KL are proportional to

$$-3-3r'-3-3r,\ -7+2r'-8+r,\ 6+4r'-3-r$$

[$x_2-x_1,\ y_2-y_1,\ z_2-z_1$ (Art. 12)]

i.e., proportional to $-3r-3r'-6,\ r+2r'-15,\ -r+4r'+3$.

$\because$ KL is $\perp$ to both the lines (1) and (2),

$\therefore\ \ 3(-3r-3r'-6)+(-1)(r+2r'-15)+1(-r+4r'+3)=0,$

$-3(-3r-3r'-6)+2(r+2r'-15)+4(-r+4r'+3)=0$

or $\quad -11r-7r'=0,\ 7r+29r'=0$

or $\quad 11r+7r'=0,\ 7r+29r'=0.$ Solving for r, r', $r=0,\ r'=0.$

(*a*) Substituting these values of r, r' in (3) and (4),

K is (3, 8, 3), L is (−3, −7, 6), which are the required pts. of intersection of the S. D. with the given lines.

	(1)	(2)
	(3, 8, 3)	(−3, −7, 6)

(*b*) $KL=\sqrt{(-3-3)^2+(-7-8)^2+(6-3)^2}$

[$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$ (Art. 4)]

$=\sqrt{(-6)^2+(-15)^2+(3)^2}$

$=\sqrt{36+225+9}=\sqrt{270}=3\sqrt{30}.$

(c) The equations of KL are

	(1)	(2)
	(3, 8, 3)	(−3, −7, 6)

$$\frac{x-3}{-3-3}=\frac{y-8}{-7-8}=\frac{z-3}{6-3} \qquad \left[\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \text{ (Art. 40)}\right]$$

or $\dfrac{x-3}{-6}=\dfrac{y-8}{-15}=\dfrac{z-3}{3}$ [Divide denom. by 3]

or $\dfrac{x-3}{-2}=\dfrac{y-8}{-5}=\dfrac{z-3}{1}$.

Note. The above method of finding the length and the equations of the shortest distance between two given lines (equations of both lines in the symmetrical form) is used when it is required to *find also the points of intersection of the shortest distance with the given lines.*

7. Find the length of the S. D. between the lines

$$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-7}{1};\ \frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}.$$

Find also its equations and the points where it intersects the lines. [*P. U. 1959*]

8. A line with direction-cosines proportional to 2, 7, –5 is drawn to intersect the lines

$$\frac{x-5}{3}=\frac{y-7}{-1}=\frac{z+2}{1};\ \frac{x+3}{-3}=\frac{y-3}{2}=\frac{z-6}{4}.$$

Find the co-ordinates of the points of intersection and the length intercepted on it. [*P(P). U. 1955 S*]

9. Find the S. D. between the axis of z and the line

$$\mathbf{ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0.} \quad [\textit{P.U. 1959 S}]$$

[**Method of parallel plane.**]

[*S.D.* = $\perp$ *distance of any pt. on the first line from the plane drawn thro' the second line and* $\parallel$ *to the first* ... (*i*)

The equation of *any* plane thro' the line

$$ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0 \text{ is}$$
$$ax+by+cz+d+k(a'x+b'y+c'z+d')=0 \quad ...(1) \quad [\text{Art. } 44,(a)]$$

If it is $\parallel$ to the z-axis, its normal is $\perp$ to the z-axis (direction-cosines 0, 0, 1),

$\therefore\ (a+ka')0+(b+kb')0+(c+kc')1=0$, or $k=-\dfrac{c}{c'}$.

Substituting this value of k in (1), the equation of the plane is

$$ax+by+cz+d-\frac{c}{c'}(a'x+b'y+c'z+d')=0$$

or
$$(ac'-a'c)x+(bc'-b'c)y+(dc'-d'c)=0 \quad ...(2)$$

$\therefore$ from (*i*),

S. D. $=\perp$ distance of *any* pt. on the z-axis, say, the origin (0,0,0), from the plane (2)

$$=\frac{dc'-d'c}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}}$$ [Rule (Art. 29, (*b*))]

$$=-\frac{cd'-c'd}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}}$$ [Assume $cd'-c'd$ is +ve]

$\therefore$ changing the sign, S.D. (*in magnitude*)

$$=\frac{cd'-c'd}{\sqrt{(ac'-a'c)^2+(bc'-b'c)^2}}.$$

Note. The above method of finding the length of the shortest distance between two given lines is used *when the equations of one line are given in the symmetrical form and those of the other in the general form.*

10. Find the shortest distance between the straight lines

$\frac{1}{2}(x-1)=\frac{1}{4}(y-2)=z-3$, and $y-mx=z=0$. [*P. U. 1939 S*]

11. Show that the equation to the plane containing the line $y/b+z/c=1$, $x=0$; and parallel to the line $x/a-z/c=1$, $y=0$ is $x/a-y/b-z/c+1=0$, and if 2d is the S.D. prove that

$$\frac{1}{d^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$ [*P. U. 1955 S*]

****12. Find the length and equations of the S.D. between**

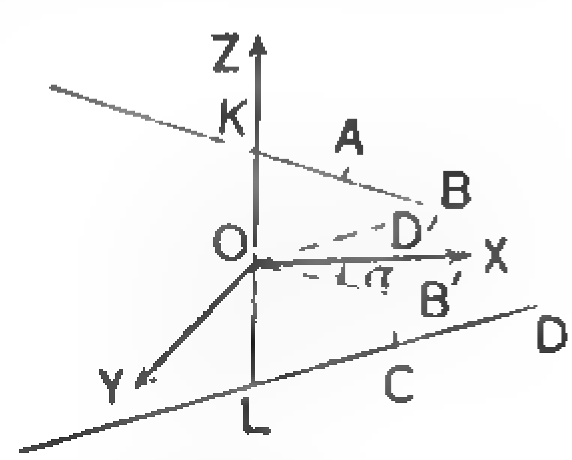
$$x-y+z=0=2x-3y+4z,$$

$$x+y+2z-3=0=2x+3y+3z-4.$$ [*P. U. Eng. 2*]

[**Note.** *When it is required to find the length and the equations of the shortest distance between two given lines (equations of both lines in the general form), first reduce the equations of both lines to the symmetrical form, and then use the method of Art. 46.*]

47. *Equations of two non-intersecting lines in the simplest form.* **To prove that by a proper choice of axes the equations of two straight lines can be put in the form**

$$y=x\tan\alpha,\ z=c;\ y=-x\tan\alpha,\ z=-c.$$



Let AB, CD be the lines, and KL$=2c$, the S.D. between them.

Take O, the mid-pt. of KL, as origin and OK as the z-axis.

Thro' O draw OB', OD' $\parallel$ to AB, CD.

Take OX, OY, the internal and external bisectors of $\angle$B'OD', as the x- and y-axes.

Then if $\angle$B'OD'$=2\alpha$, OB' makes with the axes angles

$$\alpha,\ 90^\circ-\alpha,\ 90^\circ$$

$\therefore$ the direction-cosines of OB' are

$$\cos\alpha,\ \cos(90^\circ-\alpha),\ \cos 90^\circ,\ \textit{i.e.}, \cos\alpha,\ \sin\alpha,\ 0.$$

$\therefore$ the direction-cosines of AB ($\parallel$ to OB′) are also $\cos\alpha$, $\sin\alpha$, 0, and it passes thro' K(0, 0, c),

$\therefore$ its equations are $\dfrac{x-0}{\cos\alpha}=\dfrac{y-0}{\sin\alpha}=\dfrac{z-c}{0}$

$$\left[\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}\ \text{(Art. 37)}\right]$$

or $$y=x\tan\alpha,\ z=c.$$

Again, OD′ makes with the axes angles $-\alpha$, $90°+\alpha$, $90°$

$\therefore$ the direction-cosines of OD′ are

$\cos(-\alpha)$, $\cos(90°+\alpha)$, $\cos 90°$, *i.e.*, $\cos\alpha$, $-\sin\alpha$, 0.

$\therefore$ the direction-cosines of CD ($\parallel$ to OD′) are also

$\cos\alpha$, $-\sin\alpha$, 0, and it passes thro' L(0, 0, $-c$),

$\therefore$ its equations are $\dfrac{x-0}{\cos\alpha}=\dfrac{y-0}{-\sin\alpha}=\dfrac{z-(-c)}{0}$

or $$y=-x\tan\alpha,\ z=-c.$$

Note. The angle between AB and CD $=\angle$B′OD′ [Def. Art. 6, *(b)*]

$=2\alpha$.

Cor. To prove that by a proper choice of axes the equations of any two lines can be put in the form

$$\mathbf{y=mx,\ z=c\ ;\ y=-mx,\ z=-c.}$$

[Proceed as in Art. 47, and prove that the equations of the lines are $y=x\tan\alpha,\ z=c\ ;\ y=-x\tan\alpha,\ z=-c.$

Putting $\tan\alpha=m$, the equations of the lines are

$$\mathbf{y=mx,\ z=c\ ;\ y=-mx,\ z=-c.}\ ]$$

Note. Important. For problems relating to two given lines, let the equations of the lines be

$$\mathbf{y=mx,\ z=c\ ;\ y=-mx,\ z=-c.}$$

EXAMPLES

1. A line of constant length has its extremities on two fixed straight lines ; show that the locus of its middle point is an ellipse. [*Bar. U. 1953*]

Let the equations of the fixed lines be

$y=mx,\ z=c\ ;\ y=-mx,\ z=-c$ (Art. 47, Cor.)

or, in the symmetrical form,

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0}\ \ldots(1),\quad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0}\ \ldots(2)$$

Let one extremity P on the line (1) be (r, mr, c) ...(3)

and the other extremity P′ on the line (2) be $(r', -mr', -c)$...(4)

Then $\because$ PP′=constant$=2k$ (say)

$\therefore$ $(r'-r)^2+m^2(r'+r)^2+4c^2=4k^2$...(5) [(Art. 4) From (3) and (4)]

Now the co-ordinates of the mid-pt. of PP′ are

$$x=\frac{r+r'}{2},\ y=\frac{m(r-r')}{2},\ z=0 \quad ...(6)$$

Eliminating r, r' from (5) and (6) [by substituting the values of $r+r'$ $(=2x)$, $r-r'$ $(=\frac{2y}{m})$ from (6) in (5)],

$$4\frac{y^2}{m^2}+m^2(4x^2)+4c^2=4k^2,\ z=0 \qquad \text{[Cancel 4]}$$

or $\quad m^2x^2+\dfrac{y^2}{m^2}=k^2-c^2,\ z=0,$

which is the required locus of the mid-pt.

It is an ellipse in the xy-plane.

2. Show that the locus of the mid-points of lines of constant length which have their extremities on two given lines is an ellipse whose centre bisects the S. D., and whose axes are equally inclined to the lines. [*Ag. U. 1956*]

3. **Find the locus of the mid-points of lines whose extremities are on two given lines and which are parallel to a given plane.** [*P. U. 1933 S*]

4. **Find the locus of a point which is equidistant from two given lines.**

5. **The lengths of two opposite edges of a tetrahedron are a, b, their S.D. is d, and the angle between them is θ; prove that its volume is $\dfrac{abd \sin\theta}{6}$.** [*Ag. U. 1949*]

6. AA′ is the S.D. between two given lines, and B, B′ are variable points on them such that the volume AA′BB′ is constant. Prove that the locus of the mid-point of BB′ is a hyperbola whose asymptotes are parallel to the lines. [*Ag. U. 1940*]

7. *P, P′ are variable points on two given non-intersecting lines AB, CD. Find the locus of a point Q such that QP, QP′ are perpendicular to one another and perpendicular to AB, CD respectively.*

Let the equations of AB, CD be

$y=mx,\ z=c$; $y=-mx,\ z=-c$ or, in the symmetrical form,

$$\frac{x}{1}=\frac{y}{m}=\frac{z-c}{0} \quad ...(1),\qquad \frac{x}{1}=\frac{y}{-m}=\frac{z+c}{0} \quad ...(2)$$

Let P on the line (1) be (r, mr, c), and P′ on the line (2) be $(r', -mr', -c)$. Let Q be (x, y, z).

Then the direction-cosines of PQ are proportional to

$$x-r,\ y-mr,\ z-c;$$

and those of P′Q are proportional to $x-r',\ y+mr',\ z+c$.

$\because$ PQ is $\perp$ to P′Q

$\therefore \quad (x-r)(x-r')+(y-mr)(y+mr')+(z-c)(z+c) = 0 \ldots (3)$

$\because$ PQ is $\perp$ to AB

$\therefore \quad (x-r)1+(y-mr)m = 0 \ldots (4)$

$\because$ P'Q is $\perp$ to CD

$\therefore \quad (x-r')1+(y+mr')(-m) = 0 \ldots (5)$

Eliminating r, r' from (3), (4), (5) [by substituting in (3) their values from (4), $r=\dfrac{x+my}{1+m^2}$, and from (5), $r'=\dfrac{x-my}{1+m^2}$],

$$\left(x-\frac{x+my}{1+m^2}\right)\left(x-\frac{x-my}{1+m^2}\right)+\left(y-\frac{mx+m^2y}{1+m^2}\right)\left(y+\frac{mx-m^2y}{1+m^2}\right)+z^2-c^2=0$$

or $$\frac{(m^2x-my)(m^2x+my)}{(1+m^2)^2}+\frac{(y-mx)(y+mx)}{(1+m^2)^2}+z^2-c^2=0$$

or $$\frac{m^2(m^2x^2-y^2)}{(1+m^2)^2}+\frac{y^2-m^2x^2}{(1+m^2)^2}+(z^2-c^2)=0$$

or $$\frac{(y^2-m^2x^2)(1-m^2)}{(1+m^2)^2}+(z^2-c^2)=0$$

or $$-\frac{m^2(1-m^2)}{(1+m^2)^2}x^2+\frac{1-m^2}{(1+m^2)^2}y^2+z^2=c^2,$$

which is the required locus.

$$(m^2x-my)(m^2x+my) = m^4x^2-m^2y^2 = m^2(m^2x^2-y^2)$$

48. Any line intersecting two given lines. *The equations of any line intersecting the lines $u_1=0$, $v_1=0$; $u_2=0$, $v_2=0$, are*

$\mathbf{u_1+k_1v_1=0}$, $\mathbf{u_2+k_2v_2=0}$, *where k_1, k_2 are any constants.*

Proof. The equations of the lines are

$$u_1=0, \quad v_1=0 \ldots (1)$$
$$u_2=0, \quad v_2=0 \ldots (2)$$

Consider the equations

$$u_1+k_1v_1=0, \; u_2+k_2v_2 \ldots (3)$$

These two equations together represent a st. line. [Art. 36]

It lies in the plane $u_1+k_1v_1=0$, in which plane also lies the line (1) (Art. 44, (*a*)),

$\therefore$ it intersects the line (1).

Similarly it intersects the line (2).

$\therefore$ (3) are the equations of any line intersecting the lines (1) and (2).

Note 1. The values of k_1, k_2 are found from the second condition satisfied by the line.

Note 2. Important. *If the equations of the lines are given in the symmetrical form, first reduce them to the general form, and then use the equations of Art. 48. (See Ex. 3 following.)*

EXAMPLES

1. Find the equations to the straight line drawn from the origin to intersect the lines

$$3x+2y+4z-5=0=2x-3y+4z+1,$$
$$2x-4y+z+6=0=3x-4y+z-3. \qquad [\textit{P. U. 1942}]$$

The equations of the lines are

$$3x+2y+4z-5=0,\ 2x-3y+4z+1=0 \ \dots (1)$$
$$2x-4y+z+6=0,\ 3x-4y+z-3=0 \ \dots (2)$$

The equations of *any* line intersecting the lines (1) and (2) are

$$\left.\begin{aligned} 3x+2y+4z-5+k_1(2x-3y+4z+1)=0, \\ 2x-4y+z+6+k_2(3x-4y+z-3)=0. \end{aligned}\right] \dots(3) \ [\text{Art. 48}]$$

If it passes thro' (0, 0, 0), then

$$-5+k_1(1)=0,\ 6+k_2(-3)=0, \quad \therefore \quad k_1=5,\ k_2=2.$$

Substituting these values of k_1, k_2 in (3),

$$3x+2y+4z-5+5(2x-3y+4z+1)=0,$$
$$2x-4y+z+6+2(3x-4y+z-3)=0$$

or
$$13x-13y+24z=0,\ 8x-12y+3z=0,$$

which are the required equations.

2. Find the equations to the line that intersects the lines

$$x+y+z=1,\ 2x-y-z=2;\ x-y-z=3,\ 2x+4y-z=4,$$

and passes through the point (1, 1, 1). [*P(P). U. 1956 S*]

3. Find the equations of the straight line through the origin which will intersect both the lines

$$\frac{x-1}{2}=\frac{y+3}{4}=\frac{z-5}{3};\ \frac{x-4}{2}=\frac{y+3}{3}=\frac{z-14}{4}. \qquad [\textit{P. U. Eng. 1939}]$$

[**Rule to reduce the symmetrical form of the equations of a line to the general form :** *Take the first and second members, and then the second and third members ; the two resulting equations are the equations of the line in the general form.*

Note. This is the converse of Art. 38.]

4. Find the equations to the line through (2, 2, 2) which meets both the lines $\frac{x-2}{2}=\frac{y+2}{3}=\frac{z-4}{4}$, $x=2y=3z$, and show that its intersection with the second line is $\left(\frac{15}{13}, \frac{15}{26}, \frac{5}{13}\right)$. [*P.U.*]

****5. Find the equations to the line drawn parallel to**

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{-2}$$

so as to meet the lines $z=x-2=2y+3$, $z=2x-3=3y+4$.

[The equations of the lines, in the general form, are

$$x-z-2=0,\ \ x-2y-5=0 \ \dots(1)$$
$$2x-z-3=0,\ 2x-3y-7=0 \ \dots(2)$$

The equations of *any* line intersecting the lines (1) and (2) are

$$\left.\begin{array}{r} x-z-2+k_1(\ x-2y-5)=0, \\ 2x-z-3+k_2(2x-3y-7)=0. \end{array}\right] \quad ...(3)$$

If it is ∥ to the line $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{-2}$ ...(4), then *any* plane thro' it is ∥ to the line (4)*,

∴ in particular each of the two planes, represented by the equations (3) taken separately, is ∥ to the line (4),

$$\therefore\ (1+k_1)1+(-2k_1)2+(-1)(-2)=0,$$
$$(2+2k_2)1+(-3k_2)2+(-1)(-2)=0.\ ]$$

49. Locus of a line intersecting three given lines. *To find the locus of the line intersecting the lines*

$$u_1=0,\ v_1=0\ ;\ u_2=0,\ v_2=0\ ;\ u_3=0,\ v_3=0.$$

The equations of the lines are

$$u_1=0,\ v_1=0\ ...(1)$$
$$u_2=0,\ v_2=0\ ...(2)$$
$$u_3=0,\ v_3=0\ ...(3)$$

The equations of *any* line intersecting the lines (1) and (2) are

$$u_1+k_1v_1=0,\ u_2+k_2v_2=0\ ...(4) \qquad [\text{Art. } 48\ ]$$

If it intersects the line (3), we have to eliminate x, y, z from (3) and (4). [Art. 45, (*b*)]

Then we get a relation between k_1 and k_2, say,

$$f(k_1, k_2)=0\ ...(5)$$

Eliminating k_1, k_2 from (4) and (5) [by substituting their values from (4) $\left(k_1=-\dfrac{u_1}{v_1},\ k_2=-\dfrac{u_2}{v_2}\right)$ in (5)],

$$f\left(-\frac{u_1}{v_1},\ -\frac{u_2}{v_2}\right)=0,$$

which is the required locus.

EXAMPLES

1. Find the locus of lines which intersect the three lines

$$\mathbf{y=b,\ z=-c\ ;\ z=c,\ x=-a\ ;\ x=a,\ y=-b.}$$

The equations of the lines, in the general form, are

$$y-b=0,\ z+c=0\ ...(1)$$
$$z-c=0,\ x+a=0\ ...(2)$$
$$x-a=0,\ y+b=0\ ...(3)$$

The equations of *any* line intersecting the lines (1) and (2) are

$$y-b+k_1(z+c)=0,\ z-c+k_2(x+a)=0\ ...(4) \qquad [\text{Art. } 48\ ]$$

* If one line is parallel to another line, then *any* plane through it is parallel to the other line.

If it intersects the line (3), we have to eliminate x, y, z from (3) and (4). Substituting the values of $x(=a), y(=-b)$ from (3) in (4), $-2b+k_1(z+c)=0,\ z-c+k_2(2a)=0.$

Eliminating z (by equating its values),

$$\frac{2b-k_1c}{k_1}=c-2k_2a, \text{ or } 2ak_1k_2-2ck_1+2b=0$$

or
$$ak_1k_2-ck_1+b=0 \quad ...(5)$$

Eliminating k_1, k_2 from (4) and (5) [by substituting their values from (4) $\left(k_1=-\frac{y-b}{z+c},\ k_2=-\frac{z-c}{x+a}\right)$ in (5)],

$$a\left(-\frac{y-b}{z+c}\right)\left(-\frac{z-c}{x+a}\right)-c\left(-\frac{y-b}{z+c}\right)+b=0$$

or
$$a(y-b)(z-c)+c(y-b)(x+a)+b(z+c)(x+a)=0$$

or
$$a(yz-cy-bz+bc)+c(xy+ay-bx-ab)+b(zx+az+cx+ca)=0$$

or
$$ayz+bzx+cxy+abc=0,$$

which is the required locus.

[**Check.** The equation of the locus $ayz+bzx+cxy+abc=0$.. (6) is satisfied by the equations of the three given lines thus,

putting $y=b,\ z=-c$ in (6), $a(b)(-c)+b(-c)x+cx(b)+abc=0$, or $0=0$;

putting $z=c,\ x=-a$ in (6), $ay(c)+bc(-a)+c(-a)y+abc=0$, or $0=0$;

putting $x=a,\ y=-b$ in (6), $a(-b)z+bz(a)+c(a)(-b)+abc=0$, or $0=0$.]

2. *Find the surface on which lie all lines which intersect the lines* $y=mx,\ z=c$; $y=-mx,\ z=-c$; *and the x-axis.*

3. Prove that the locus of a variable line which intersects the three given lines $y=mx,\ z=c$; $y=-mx,\ z=-c$; $y=z,\ mx=-c$; is the surface $y^2-m^2x^2=z^2-c^2$. [*Ag. U. 1954*]

4. **Prove that the locus of lines which intersect the three lines $\mathbf{y-z=1,\ x=0}$; $\mathbf{z-x=1,\ y=0}$; $\mathbf{x-y=1,\ z=0}$ is**

$$\mathbf{x^2+y^2+z^2-2yz-2zx-2xy=1.} \quad \text{[Ag. U. 1955]}$$

5. *Find the locus of the straight lines which meet the lines* $x=2,\ 4y=3z$; $x+2=0,\ 4y+3z=0$; $y=3,\ 2x+z=0$. [*P(P).U.H. 1953*]

6. **Prove that the locus of a line which meets the lines $\mathbf{y=\pm mx,\ z=\pm c}$ and the circle $\mathbf{x^2+y^2=a^2,\ z=0}$ is

$$\mathbf{c^2m^2(cy-mxz)^2+c^2(yz-cmx)^2=a^2m^2(z^2-c^2)^2.} \quad \text{[J. \& K.U. 1954]}$$

7. **Find the locus of a straight line which intersects two given lines and makes a right angle with one of them.**

[*P. U. H. 1934*]

8. **A straight line meets two given straight lines and makes the same angle with both of them. Find the surface which it generates.** [*D. U. H. 1943*]

****9. Find the surface generated by a straight line which intersects two given lines and is parallel to a given plane.**

[*D. U. H. 1931*]

10. Through two straight lines given in space two planes are drawn perpendicular to one another ; find the locus of their line of intersection.

SECTION V

INTERSECTION OF THREE PLANES

50. Notation. If the equations of three planes are

$$a_1x+b_1y+c_1z+d_1=0 \quad ...(1)$$
$$a_2x+b_2y+c_2z+d_2=0 \quad ...(2)$$
$$a_3x+b_3y+c_3z+d_3=0 \quad ...(3)$$

then writing down the coefficients in the equations (1), (2), (3), we get

$$\left\|\begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{matrix}\right\| \quad ...(4)$$

The four determinants, obtained by omitting the *first, second, third, fourth columns one by one,* are denoted respectively by $\Delta_1, \Delta_2, \Delta_3, \Delta_4$.

Thus (omitting the *first* column from (4)) $\Delta_1=\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}$,

(omitting the *second* column from (4)) $\Delta_2=\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}$,

(omitting the *third* column from (4)) $\Delta_3=\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}$,

(omitting the *fourth* column from (4)) $\Delta_4=\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}$.

51. Case I. Condition for three planes to intersect at a point. *To find the condition that three given planes may intersect at a point.*

Let the equations of the planes be

$$a_1x+b_1y+c_1z+d_1=0 \quad ...(1)$$
$$a_2x+b_2y+c_2z+d_2=0 \quad ...(2)$$
$$a_3x+b_3y+c_3z+d_3=0 \quad ...(3)$$

Solving (1), (2), (3),

$$\frac{x}{\begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}}$$

or $\dfrac{x}{\triangle_1} = \dfrac{-y}{\triangle_2} = \dfrac{z}{\triangle_3} = \dfrac{-1}{\triangle_4}$,

where $\triangle_1 = \begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}$,

i.e., the determinant obtained by writing down coeffs. in the equations (1), (2), (3), *i.e.*,

$$\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{vmatrix},$$

and omitting the *first* column, and similarly for $\triangle_2$, $\triangle_3$, $\triangle_4$.
[Notation, Art. 50]

$\therefore\ x = -\dfrac{\triangle_1}{\triangle_4},\ y = \dfrac{\triangle_2}{\triangle_4},\ z = -\dfrac{\triangle_3}{\triangle_4}$ ...(4)

If the planes intersect at a pt., the co-ordinates of their pt. of intersection are finite.

$\therefore$ from (4), the denominator $\triangle_4 \neq 0$,
which is the required condition.

[**Rule to find the point of intersection of three planes whose equations are given :**

Solve the equations of the planes simultaneously, as in Algebra. The values of x, y, z are the co-ordinates of the point of intersection.]

EXAMPLES

1. *Find the point of intersection of the planes*
$x-y+2z=3$, $x+2y+3z=5$, $3x-4y-5z+13=0$.

The equations of the planes are

$$\begin{array}{rl|r|r} x-\ y+2z-3 = 0 & \text{... (1)} & 3 & \\ x+2y+3z-5 = 0 & \text{... (2)} & -2 & 5 \\ 3x-4y-5z+13=0 & \text{... (3)} & & 3 \end{array}$$

Eliminating z from (1) and (2) [by multiplying (1) by 3, (2) by -2, and adding],

$$x-7y+1 = 0 \quad \text{... (4)}$$

Again, eliminating z from (2) and (3) [by multiplying (2) by 5, (3) by 3, and adding],

$$14x - 2y + 14 = 0$$

or, dividing thro' out by 2, $7x-y+7=0$ ... (5)

Solving (4) and (5), by cross-multiplication,

$$\frac{x}{-49-(-1)}=\frac{y}{7-7}=\frac{1}{-1-(-49)}, \text{ or } \frac{x}{-48}=\frac{y}{0}=\frac{1}{48}$$

$\therefore \quad x=\dfrac{-48}{48}=-1,\ y=0.$

Substituting these values of x, y in (1),

$$-1-0+2z-3=0, \text{ or } \quad 2z-4=0, \quad \therefore \quad z=2$$

$\therefore$ the pt. of intersection is $(-1, 0, 2)$.

[**Check.** The co-ordinates of the point of intersection satisfy the equations of the given planes (1), (2), (3) thus

$$-1-0+2(2)-3=0, \text{ or } -1+4-3=0, \text{ or } 0=0,$$
$$-1+2(0)+3(2)-5=0, \text{ or } -1+0+6-5=0, \text{ or } 0=0,$$
$$3(-1)-4(0)-5(2)+13=0, \text{ or } -3-10+13=0, \text{ or } 0=0.\ ]$$

[**Rule to solve the (numerical) equations of three planes :**

(i) Eliminate z from the equations (1) and (2), and get an equation in x and y. Again, eliminate z from the equations (2) and (3), and get another equation in x and y.

(ii) Solve the two equations in x and y.

(iii) Substitute these values of x and y in one of the given equations, say, (1), and find the value of z.]

2. Show that the four planes $x-2y+z=0$, $x+y-2z=3$, $3x-2y+z=2$, $4x-5y+3z=1$ meet in a point.

3. Find the volume of the tetrahedron formed by planes whose equations are

$\mathbf{y+z=0,\ z+x=0,\ x+y=0,}$ **and** $\mathbf{x+y+z=1.}$ [*P(P). U. 1954*]

4. Prove that the four planes

$$\mathbf{my+nz=0,\ nz+lx=0,\ lx+my=0,\ lx+my+nz=p}$$

form a tetrahedron whose volume is $\dfrac{\mathbf{2p^3}}{\mathbf{3lmn}}$. [*B.H.U.H. 1948*]

****5. Find the volume of the tetrahedron the equations to whose faces are** $\mathbf{a_rx+b_ry+c_rz+d_r=0,\ r=1, 2, 3, 4.}$

[*D. U. H. 1938*]

The equations of the faces are

$$a_1x+b_1y+c_1z+d_1=0 \quad \dots (1)$$
$$a_2x+b_2y+c_2z+d_2=0 \quad \dots (2)$$
$$a_3x+b_3y+c_3z+d_3=0 \quad \dots (3)$$
$$a_4x+b_4y+c_4z+d_4=0 \quad \dots (4)$$

Omitting (1), and solving (2), (3), (4),

$$\frac{x}{\begin{vmatrix} b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \\ b_4, & c_4, & d_4 \end{vmatrix}}=\frac{-y}{\begin{vmatrix} a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \\ a_4, & c_4, & d_4 \end{vmatrix}}=\frac{z}{\begin{vmatrix} a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \\ a_4, & b_4, & d_4 \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \\ a_4, & b_4, & c_4 \end{vmatrix}} \quad \ldots(5)$$

Let
$$\triangle=\begin{vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \\ a_4, & b_4, & c_4, & d_4 \end{vmatrix}.$$

Then from (5), $\dfrac{x}{A_1}=\dfrac{-y}{-B_1}=\dfrac{z}{C_1}=\dfrac{-1}{-D_1}$,

where A_1, B_1, C_1, D_1 are the co-factors (*i.e.*, minors with the proper signs) of a_1, b_1, c_1, d_1 in $\triangle$,

or
$$x=\frac{A_1}{D_1},\ y=\frac{B_1}{D_1},\ z=\frac{C_1}{D_1}$$

$\therefore$ one vertex of the tetrahedron is $\left(\dfrac{A_1}{D_1}, \dfrac{B_1}{D_1}, \dfrac{C_1}{D_1}\right)$.

Similarly the other vertices are

$$\left(\frac{A_2}{D_2}, \frac{B_2}{D_2}, \frac{C_2}{D_2}\right), \left(\frac{A_3}{D_3}, \frac{B_3}{D_3}, \frac{C_3}{D_3}\right), \left(\frac{A_4}{D_4}, \frac{B_4}{D_4}, \frac{C_4}{D_4}\right).$$

$\therefore$ volume of the tetrahedron

$$=\frac{1}{6}\begin{vmatrix} \frac{A_1}{D_1}, & \frac{B_1}{D_1}, & \frac{C_1}{D_1}, & 1 \\ \frac{A_2}{D_2}, & \frac{B_2}{D_2}, & \frac{C_2}{D_2}, & 1 \\ \frac{A_3}{D_3}, & \frac{B_3}{D_3}, & \frac{C_3}{D_3}, & 1 \\ \frac{A_4}{D_4}, & \frac{B_4}{D_4}, & \frac{C_4}{D_4}, & 1 \end{vmatrix}$$

[Volume formula (Art. 34)]

[Take $\dfrac{1}{D_1}$ common from the first row, $\dfrac{1}{D_2}$ from the second row, $\dfrac{1}{D_3}$ from the third row, $\dfrac{1}{D_4}$ from the fourth row]

$$=\frac{1}{6D_1D_2D_3D_4}\begin{vmatrix} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \\ A_4, & B_4, & C_4, & D_4 \end{vmatrix}=\frac{1}{6D_1D_2D_3D_4}\triangle^3.$$

[From Higher Algebra]

6. Find the co-ordinates of the centre of the sphere inscribed in the tetrahedron formed by the planes whose equations are

$$\mathbf{x=0,\ y=0,\ z=0 \text{ and } x+y+z=a.}$$ [*P.U. 1955*]

The equations of the planes are

$$x=0 \ldots(1),\ y=0 \ldots(2),\ z=0 \ldots(3),\ x+y+z=a \ldots(4)$$

Omitting (4), and solving (1), (2), (3), the vertex is (0, 0, 0), and the equation of the opposite face is (4), *i.e.*, $x+y+z-a=0$.

Omitting (3), and solving (1), (2), (4), the vertex is $(0, 0, a)$, and the equation of the opposite face is (3), *i.e.*, $z=0$.

Omitting (2), and solving (1), (3), (4), the vertex is $(0, a, 0)$, and the equation of the opposite face is (2), *i.e.*, $y=0$.

Omitting (1), and solving (2), (3), (4), the vertex is $(a, 0, 0)$, and the equation of the opposite face is (1), *i.e.*, $x=0$.

Writing the equation of each face (plane) (and *changing the signs thro' out,* if necessary) so that the result of substituting the co-ordinates of the opposite vertex in the L. H. S. of the equation (R.H.S. being zero) is *positive,* the equations of the faces are $a-x-y-z=0$, $z=0$, $y=0$, $x=0$, or $x=0$, $y=0$, $z=0$, $a-x-y-z=0$.

Let (x, y, z) be the required co-ordinates of the centre of the sphere inscribed in the tetrahedron.

Then its $\perp$ distances from the faces are equal

$$\therefore \quad \frac{x}{1}=\frac{y}{1}=\frac{z}{1}=\frac{a-x-y-z}{\sqrt{3}} \qquad \text{[Rule (Art. 29, (b))}$$

$$=\frac{\text{sum of nums.}}{\text{sum of denoms.}}=\frac{x+y+z+a-x-y-z}{1+1+1+\sqrt{3}}$$

$$=\frac{a}{3+\sqrt{3}}$$

$$\therefore \quad x=\frac{a}{3+\sqrt{3}},\ y=\frac{a}{3+\sqrt{3}},\ z=\frac{a}{3+\sqrt{3}}.$$

$$\therefore \quad \left(\frac{a}{3+\sqrt{3}}, \frac{a}{3+\sqrt{3}}, \frac{a}{3+\sqrt{3}}\right) \text{ are the required co-ordinates.}$$

7. Find the co-ordinates of the centre of the sphere inscribed in the tetrahedron formed by the planes whose equations are

$y+z=0$, $z+x=0$, $x+y=0$, and $x+y+z=1$. [*D.U.H. 1936*]

****52. Case II.** *Conditions for three planes to form a triangular prism.* **To find the conditions that three given planes may form a triangular prism.**

Let the equations of the planes be

$$a_1x+b_1y+c_1z+d_1=0 \quad ...(1)$$
$$a_2x+b_2y+c_2z+d_2=0 \quad ...(2)$$
$$a_3x+b_3y+c_3z+d_3=0 \quad ...(3)$$

If the planes form a triangular prism, the line of intersection of two of the planes, say (2) and (3), is $\parallel$ to the third plane (1).

[To find the equations of the line of intersection of the planes (2) and (3) in the symmetrical form.]

(To find the direction-cosines of the line.)

The equations of the line thro' the origin $\parallel$ to the line of intersection of the planes (2) and (3) are [omitting the constant terms

from (2) and (3)],

$$a_2x+b_2y+c_2z=0,$$
$$a_3x+b_3y+c_3z=0$$

or $$\frac{x}{b_2c_3-b_3c_2}=\frac{y}{c_2a_3-c_3a_2}=\frac{z}{a_2b_3-a_3b_2},$$

whose direction-cosines and ∴ the direction-cosines of the line of intersection of the planes (2) and (3) are proportional to

$$b_2c_3-b_3c_2,\ c_2a_3-c_3a_2,\ a_2b_3-a_3b_2.$$

(To find one pt. on the line.)

In the equations (2) and (3), putting $z=0$,

$$a_2x+b_2y+d_2=0,$$
$$a_3x+b_3y+d_3=0$$

or $$\frac{x}{b_2d_3-b_3d_2}=\frac{y}{d_2a_3-d_3a_2}=\frac{1}{a_2b_3-a_3b_2}$$

or $$x=\frac{b_2d_3-b_3d_2}{a_2b_3-a_3b_2},\ y=\frac{d_2a_3-d_3a_2}{a_2b_3-a_3b_2},\ \text{also } z=0.$$

∴ one pt. on the line of intersection is

$$\left(\frac{b_2d_3-b_3d_2}{a_2b_3-a_3b_2},\ \frac{d_2a_3-d_3a_2}{a_2b_3-a_3b_2},\ 0\right).$$

∴ the equations of the line of intersection of the planes (2) and (3), in the symmetrical form, are

$$\frac{x-\dfrac{b_2d_3-b_3d_2}{a_2b_3-a_3b_2}}{b_2c_3-b_3c_2}=\frac{y-\dfrac{d_2a_3-d_3a_2}{a_2b_3-a_3b_2}}{c_2a_3-c_3a_2}=\frac{z}{a_2b_3-a_3b_2}\quad\ldots(4)$$

If it is ∥ to the plane (1), then it is $\perp$ to the normal to the plane (1), and its pt. $\left(\dfrac{b_2d_3-b_3d_2}{a_2b_3-a_3b_2},\ \dfrac{d_2a_3-d_3a_2}{a_2b_3-a_3b_2},\ 0\right)$ does not lie on the plane (1). [Art. 43, (*a*)]

$$\therefore\ a_1(b_2c_3-b_3c_2)+b_1(c_2a_3-c_3a_2)+c_1(a_2b_3-a_3b_2)=0,$$

and $$a_1\left(\frac{b_2d_3-b_3d_2}{a_2b_3-a_3b_2}\right)+b_1\left(\frac{d_2a_3-d_3a_2}{a_2b_3-a_3b_2}\right)+c_1(0)+d_1\neq 0$$

or $$a_1(b_2d_3-b_3d_2)+b_1(d_2a_3-d_3a_2)+d_1(a_2b_3-a_3b_2)\neq 0,$$

i.e., $$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}=0, \text{ and } \begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}\neq 0$$

[Expanding each determinant by means of first row]

i.e., $\triangle_4=0$, and $\triangle_3\neq 0$, $\quad \begin{Vmatrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{Vmatrix}$

[Notation, Art. 50]

which are the required conditions.

53. **Case III.** *Conditions for three planes to have a common line of intersection.* **To find the conditions that three given planes may have a common line of intersection.**

Let the equations of the planes be

$$a_1x+b_1y+c_1z+d_1=0 \quad ...(1)$$
$$a_2x+b_2y+c_2z+d_2=0 \quad ...(2)$$
$$a_3x+b_3y+c_3z+d_3=0 \quad ...(3)$$

The equation of *any* plane thro' the line of intersection of the planes (1) and (2) is

$$a_1x+b_1y+c_1z+d_1+k(a_2x+b_2y+c_2z+d_2)=0 \quad ...(4) \qquad \text{[Art. 44, }(a)\text{]}$$

If the three planes have a common line of intersection, then for some value of k, (4) is the same as (3),

$\therefore$ comparing coeffs. in (4) and (3),

$$\frac{a_1+ka_2}{a_3}=\frac{b_1+kb_2}{b_3}=\frac{c_1+kc_3}{c_3}=\frac{d_1+kd_2}{d_3}=-k'^{*}, \text{ (say)}$$

$\therefore$
$$a_1+ka_2+k'a_3=0 \quad ...(5)$$
$$b_1+kb_2+k'b_3=0 \quad ...(6)$$
$$c_1+kc_2+k'c_3=0 \quad ...(7)$$
$$d_1+kd_2+k'd_3=0 \quad ...(8)$$

Omitting (8) and eliminating k, k' from (5), (6), (7),

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \end{vmatrix}=0.$$

Similarly omitting (7), (6), (5) *one by one,* and eliminating k, k' from the remaining three equations,

$$\begin{vmatrix} a_1, & a_2, & a_3 \\ b_1, & b_2, & b_3 \\ d_1, & d_2, & d_3 \end{vmatrix}=0, \quad \begin{vmatrix} a_1, & a_2, & a_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{vmatrix}=0, \quad \begin{vmatrix} b_1, & b_2, & b_3 \\ c_1, & c_2, & c_3 \\ d_1, & d_2, & d_3 \end{vmatrix}=0,$$

or, changing columns into rows in the above determinants,

$$\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}=0 \quad ...(9), \qquad \begin{vmatrix} a_1, & b_1, & d_1 \\ a_2, & b_2, & d_2 \\ a_3, & b_3, & d_3 \end{vmatrix}=0 \quad ...(10)$$

$$\begin{vmatrix} a_1, & c_1, & d_1 \\ a_2, & c_2, & d_2 \\ a_3, & c_3, & d_3 \end{vmatrix}=0 \quad ...(11), \qquad \begin{vmatrix} b_1, & c_1, & d_1 \\ b_2, & c_2, & d_2 \\ b_3, & c_3, & d_3 \end{vmatrix}=0 \quad ...(12)$$

(9), (10), (11), (12) are the required conditions, or writing down coeffs. in the equations of the planes (1), (2), (3),

$$\left\| \begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{matrix} \right\|=0 \quad ...(13)$$

---

*and *not* k' to avoid negative signs in the equations (5), (6), (7), (8) following.

the notation indicating that each of the four determinants, obtained by omitting the *fourth, third, second, first* columns one by one in (13) is zero.

Note 1. Only two of the four conditions are independent.

Geometrical proof. If the three planes have two pts. in common, they have a common line of intersection.

But the fact that three planes may have two pts. in common, requires only two conditions.

$\therefore$ only *two* of the four conditions (9), (10), (11), (12) are independent.

Hence *it is sufficient to show that* **any two** *of the four determinants obtained from (13), say, obtained by omitting the fourth and third columns one by one, are zero, i.e.,* $\triangle_4=0$, $\triangle_3=0$.

Note 2. In fact it can be proved algebraically that, if *any* two of the four determinants are zero, then the remaining two are also zero. Thus if $\triangle_4=0$, and $\triangle_3=0$, then $\triangle_2=0$, and $\triangle_1=0$.

Note 3. Short cut for numerical examples. In numerical examples it is better to proceed as follows :—

(*i*) Find the value of k from the first and second members of the equations obtained by comparing coeffs. in (4) and (3).

(*ii*) Substitute this value of k in all the members, and show that the equations are satisfied. [See Ex. 4 following.]

EXAMPLES

1. Prove that the planes

$\mathbf{x+ay+(b+c)z+d=0,\ x+by+(c+a)z+d=0,\ x+cy+(a+b)z+d=0}$ **pass through one line.** [*Ag. U. 1948*]

The equations of the planes are

$$x+ay+(b+c)z+d=0 \quad \ldots (1)$$
$$x+by+(c+a)z+d=0 \quad \ldots (2)$$
$$x+cy+(a+b)z+d=0 \quad \ldots (3)$$

Writing down coeffs., we get

$$\left\|\begin{matrix} 1, & a, & b+c, & d \\ 1, & b, & c+a, & d \\ 1, & c, & a+b, & d \end{matrix}\right\| \quad \ldots (4)$$

(Omitting the fourth column from (4))

Here $$\triangle_4=\begin{vmatrix} 1, & a, & b+c \\ 1, & b, & c+a \\ 1, & c, & a+b \end{vmatrix}$$ [Add the second column to the third]

$$=\begin{vmatrix} 1, & a, & b+c+a \\ 1, & b, & c+a+b \\ 1, & c, & a+b+c \end{vmatrix}$$ [Take out the common factor $(a+b+c)$ from the third column]

$$=(a+b+c)\begin{vmatrix} 1, & a, & 1 \\ 1, & b, & 1 \\ 1, & c, & 1 \end{vmatrix}=0. \quad [\because \text{ first and third columns are identical }]$$

(Omitting the third column from (4))

$$\triangle_3=\begin{vmatrix} 1, & a, & d \\ 1, & b, & d \\ 1, & c, & d \end{vmatrix} \quad [\text{ Take out the common factor } d \text{ from the third column }]$$

$$=d\begin{vmatrix} 1, & a, & 1 \\ 1, & b, & 1 \\ 1, & c, & 1 \end{vmatrix}=0 \quad [\because \text{ first and third columns are identical }]$$

$\therefore$ the planes have a common line of intersection. [Art. 53]

2. Prove that the planes

$$\mathbf{x=cy+bz,\ y=az+cx,\ z=bx+ay}$$

pass through one line if $\mathbf{a^2+b^2+c^2+2abc=1}$. [*B. U. 1953*]

Show that the equations of this line are

$$\frac{\mathbf{x}}{\sqrt{\mathbf{1-a^2}}}=\frac{\mathbf{y}}{\sqrt{\mathbf{1-b^2}}}=\frac{\mathbf{z}}{\sqrt{\mathbf{1-c^2}}}. \qquad [\textit{ B. U. 1948 }]$$

3. Show that the planes

$$\mathbf{ax+hy+gz=0,\ hx+by+fz=0,\ gx+fy+cz=0}$$

have a common line of intersection if

$$\triangle \equiv \begin{vmatrix} \mathbf{a}, & \mathbf{h}, & \mathbf{g} \\ \mathbf{h}, & \mathbf{b}, & \mathbf{f} \\ \mathbf{g}, & \mathbf{f}, & \mathbf{c} \end{vmatrix}=0,$$

and the direction-ratios of the line satisfy the equations

$$\frac{\mathbf{l^2}}{\dfrac{\partial\triangle}{\partial \mathbf{a}}}=\frac{\mathbf{m^2}}{\dfrac{\partial\triangle}{\partial \mathbf{b}}}=\frac{\mathbf{n^2}}{\dfrac{\partial\triangle}{\partial \mathbf{c}}}. \qquad [\textit{ P(P). U. H. 1952 }]$$

4. Show that the three planes

$$\mathbf{2x+3y+z=0,\ x-y+2z=3 \text{ and } 5y-3z+6=0}$$

intersect in a straight line.

The equations of the planes are

$$2x+3y+z=0 \quad \dots (1)$$
$$x-y+2z-3=0 \quad \dots (2)$$
$$5y-3z+6=0 \quad \dots (3)$$

The equation of *any* plane thro' the line of intersection of the planes (1) and (2) is

$$2x+3y+z+k(x-y+2z-3)=0$$

or
$$(2+k)x+(3-k)y+(1+2k)z-3k=0 \quad \dots (4)$$

If the planes have a common line of intersection, then for some

value of k, (4) is the same as (3),

$\therefore$ comparing coeffs. in (4) and (3),

$$\frac{2+k}{0}=\frac{3-k}{5}=\frac{1+2k}{-3}=\frac{-3k}{6} \quad \ldots (5)$$

From the first and second members, cross-multiplying,

$$2+k=0, \quad \therefore \quad k=-2.$$

Substituting this value of k in (5), we get

$$\left(\frac{0}{0}=\right)\frac{5}{5}=\frac{-3}{-3}=\frac{6}{6}, \quad \text{or} \quad 1=1=1,$$

which is true,

$\therefore$ the planes have a common line of intersection.

5. Prove that the planes

$$\mathbf{3x-2y-4z=0,\ 2x-4y-5z=0,\ 5x-6y-9z=0}$$

pass through one line.

6. If the axes are rectangular, find the equations of the planes through the line of intersection of two of the planes

$$a_r x+b_r y+c_r z+d_r=0, \quad r=1, 2, 3,$$

perpendicular to the third. Show that the three planes pass through one line.

7. The plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ meets the axes OX, OY, OZ, which are rectangular, in A, B, C. Find the equations of BC. Prove that the planes through the axes and perpendicular to BC, CA, AB pass through the line $ax=by=cz$. [*P.U. 1959 S*] Find also the co-ordinates of the orthocentre of the triangle ABC.

8. The plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ meets the axes in A, B, C. Prove that the planes through the axes and the internal bisectors of the angles of the triangle ABC pass through the line

$$\frac{x}{a\sqrt{b^2+c^2}}=\frac{y}{b\sqrt{c^2+a^2}}=\frac{z}{c\sqrt{a^2+b^2}}.$$

54. Rule to find the nature of the intersection of the three planes

$$\mathbf{a_1x+b_1y+c_1z+d_1=0} \ldots(1),\ \mathbf{a_2x+b_2y+c_2z+d_2=0} \ldots(2),$$
$$\mathbf{a_3x+b_3y+c_3z+d_3=0} \ldots(3):$$

Write down coefficients in the equations of the planes (1), (2), (3), thus getting

$$\left\|\begin{matrix} a_1, & b_1, & c_1, & d_1 \\ a_2, & b_2, & c_2, & d_2 \\ a_3, & b_3, & c_3, & d_3 \end{matrix}\right\| \quad \ldots(4)$$

(*i*) *If $\Delta_4 \neq 0$, the planes intersect at a point.* [*Notation, Art. 50*]
[Art. 51]

(*ii*) *If* $\Delta_4=0$, *but* $\Delta_3\neq 0$, *the planes form a prism, provided no two of the three planes are* $\|$. [Art. 52]

(*iii*) *If* $\Delta_4=0$, *also* $\Delta_3=0$, *the planes have a common line of intersection.* [Art. 53]

EXAMPLES

**1. Find the nature of the intersection of the sets of planes :

(*i*) $x-2y+2z=3,\ 2x+3y-z=5,\ 3x-4y+5z=10$;

(*ii*) $2x+4y+2z=7,\ 5x+y-z=9,\ x-y-z=6$;

(*iii*) $x+2y+z=0,\ 3x+y-2z=1,\ 3x-4y-7z=2$.

(*i*) The equations of the planes are

$$x-2y+2z-3=0 \quad ...(1)$$
$$2x+3y-z-5=0 \quad ...(2)$$
$$3x-4y+5z-10=0 \quad ...(3)$$

Writing down coeffs. in the equations of the planes (1), (2), (3), we get

$$\left\|\begin{matrix} 1, & -2, & 2, & -3 \\ 2, & 3, & -1, & -5 \\ 3, & -4, & 5, & -10 \end{matrix}\right\| \quad ...(4)$$

(Omitting the fourth column from (4))

Here $\Delta_4=\begin{vmatrix} 1, & -2, & 2 \\ 2, & 3, & -1 \\ 3, & -4, & 5 \end{vmatrix}$

$=1(15-4)-2[-10-(-8)]+3(2-6)=1(11)-2(-2)+3(-4)$
$=11+4-12=3$, which is $\neq 0$.

$\therefore$ the planes intersect at a pt. [Rule (Art. 54)]

(*ii*) The equations of the planes are

$$2x+4y+2z-7=0 \quad ...(1)$$
$$5x+y-z-9=0 \quad ...(2)$$
$$x-y-z-6=0 \quad ...(3)$$

Writing down coeffs. in the equations of the planes (1), (2), (3), we get

$$\left\|\begin{matrix} 2, & 4, & 2, & -7 \\ 5, & 1, & -1, & -9 \\ 1, & -1, & -1, & -6 \end{matrix}\right\| \quad ...(4)$$

(Omitting the fourth column from (4))

Here $\Delta_4=\begin{vmatrix} 2, & 4, & 2 \\ 5, & 1, & -1 \\ 1, & -1, & -1 \end{vmatrix}$

$=2(-1-1)-5[-4-(-2)]+1(-4-2)=2(-2)-5(-2)+1(-6)$
$=-4+10-6=0$.

(Omitting the third column from (4))

$$\Delta_3=\begin{vmatrix} 2, & 4, & -7 \\ 5, & 1, & -9 \\ 1, & -1, & -6 \end{vmatrix}$$

$=2(-6-9)-5(-24-7)+1[-36-(-7)]=2(-15)-5(-31)+1(-29)$
$=-30+155-29=96$, which is $\neq 0$.

$$\begin{array}{r} 155 \\ -\ 59 \\ \hline 96 \end{array}$$

$\therefore$ the planes form a prism, provided no two of the three planes are $\parallel$.

The planes (1) and (2) are $\parallel$

if $\dfrac{2}{5}=\dfrac{4}{1}=\dfrac{2}{-1}$, which is not true.

$\therefore$ the planes (1) and (2) are not $\parallel$.

The planes (2) and (3) are $\parallel$

if $\dfrac{5}{1}=\dfrac{1}{-1}=\dfrac{-1}{-1}$, which is not true.

$\therefore$ the planes (2) and (3) are not $\parallel$.

The planes (3) and (1) are $\parallel$

if $\dfrac{1}{2}=\dfrac{-1}{4}=\dfrac{-1}{2}$, which is not true.

$\therefore$ the planes (3) and (1) are not $\parallel$.

Hence the planes form a prism.

(*iii*) The equations of the planes are

$$x+2y+z=0 \quad ...(1)$$
$$3x+y-2z-1=0 \quad ...(2)$$
$$3x-4y-7z-2=0 \quad ...(3)$$

Writing down coeffs. in the equations of the planes (1), (2), (3), we get

$$\begin{Vmatrix} 1, & 2, & 1, & 0 \\ 3, & 1, & -2, & -1 \\ 3, & -4, & -7, & -2 \end{Vmatrix} \quad ...(4)$$

(Omitting the fourth column from (4))

Here $\Delta_4=\begin{vmatrix} 1, & 2, & 1 \\ 3, & 1, & -2 \\ 3, & -4, & -7 \end{vmatrix}$

$=1(-7-8)-3[-14-(-4)]+3(-4-1)=1(-15)-3(-10)+3(-5)$
$=-15+30-15=0.$

(Omitting the third column from (4))

$$\Delta_3=\begin{vmatrix} 1, & 2, & 0 \\ 3, & 1, & -1 \\ 3, & -4, & -2 \end{vmatrix}$$ [Expand by means of first row]

$=1(-2-4)-2[-6-(-3)]+0=1(-6)-2(-3)=-6+6=0.$

$\therefore$ the planes have a common line of intersection.

[Rule (Art. 54)]

**2. Examine the nature of the intersection of the sets of planes :

(i) $x-y+z=3,\ 2x+5y+3z=0,\ 3x-2y-6z+1=0.$

(ii) $3x+2y+z=6,\ 5x+4y+3z=4,\ 3x+4y+5z+12=0.$

(iii) $x-y+z-4=0,\ 2x-y-z+4=0,\ x+y-5z+14=0.$

[*B. U. 1941*]

**3. Examine the nature of the intersection of the sets of planes ;

(i) $x+2y-5z=1,\ 4x+y+z=2,\ 6x+y+3z=3$;

(ii) $x+4y+6z=5,\ 2x+5y+9z=10,\ x+3y+5z=5$;

(iii) $x-y+z=2,\ 2x-3y+4z=8,\ x+y+z=2$;

(iv) $x+3y-z=6,\ x+2y+4z+5=0,\ 2x+6y-2z+7=0.$

****4. Prove that the three planes $2x+y+z=3$, $x-y+2z=4$, $x+z=2$, form a triangular prism, and find the area of a normal section of the prism.** [*P(P). U. 1948*]

MISCELLANEOUS EXAMPLES ON CHAPTER V

**1. A plane triangle, sides a,b,c, is placed so that the mid-points of the sides are on the axes (rectangular). Show that the lengths intercepted on the axes are given by

$$l^2=(b^2+c^2-a^2)/8,\quad m^2=(c^2+a^2-b^2)/8,\quad n^2=(a^2+b^2-c^2)/8,$$

and that the co-ordinates of the vertices are

$$(-l,\ m,\ n),\ (l,\ -m,\ n),\ (l,\ m,\ -n).$$ [*A.U. 1938*]

[Let ABC be the triangle, and D, E, F the mid-pts. of the sides BC, CA, AB on the x-, y-, z-axes. Let A,B,C be (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3). Then the mid-pts. D,E,F are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}, \frac{z_2+z_3}{2}\right)$ etc.

$\because$ D lies on the x-axis ($y=0$, $z=0$), $\therefore$ $y_2+y_3=0$, $z_2+z_3=0$ ... (1)
Similarly $z_3+z_1=0$, $x_3+x_1=0$ ... (2), $x_1+x_2=0$, $y_1+y_2=0$ ... (3)
From (1), (2), (3), find the co-ordinates of B(x_2, y_2, z_2), C(x_3, y_3, z_3) in terms of those of A. It will be found that B is $(-x_1, -y_1, z_1)$, C is $(-x_1, y_1, -z_1)$. Also A is (x_1, y_1, z_1). [Compare the co-ordinates of A (x_1, y_1, z_1) with the given co-ordinates $(-l, m, n)$.] Put $x_1=-l$, $y_1=m$, $z_1=n$, $\therefore$ A, B, C are ? BC$=a$, $\therefore$? etc. D, the mid-pt. of BC, is $(l, 0, 0)$, $\therefore$ OD $=l$.]

2. A plane meets a set of three mutually perpendicular planes in the sides of a triangle whose angles are A, B, C. Show that the first plane makes with the other three planes angles whose cosine-squares are cot B cot C, cot C cot A, cot A cot B. [*B. U.*]

[Take the set of three mutually $\perp$ planes as co-ordinate planes,

and let the equation of the plane ABC be $\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1$, A, B, C being on the x-, y-, z-axes. Draw a rough Fig.]

3. P is a fixed point on a line through the origin equally inclined to the axes ; prove that any plane through P makes intercepts on the axes the sum of whose reciprocals is constant.

**4. Prove that $\frac{a}{y-z}+\frac{b}{z-x}+\frac{c}{x-y}=0$ represents a pair of planes whose line of intersection is equally inclined to the axes. [*Bar. U. 1954*]

[(*a*) It will be found that the equation is

$$ax^2+by^2+cz^2-(b+c-a)yz-(c+a-b)zx-(a+b-c)xy=0.$$

Here "$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$" $=\begin{vmatrix} a, & -\frac{a+b-c}{2}, & -\frac{c+a-b}{2} \\ -\frac{a+b-c}{2}, & b, & -\frac{b+c-a}{2} \\ -\frac{c+a-b}{2}, & -\frac{b+c-a}{2}, & c \end{vmatrix}$.

(Notation, Art. 35)

Add the second and third columns to the first.

(*b*) If the equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, represents the pair of planes $lx+my+nz=0$, $l'x+m'y+n'z=0$, then

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=(lx+my+nz)(l'x+m'y+n'z).$$

Equating coeffs. of like terms on both sides,

$$ll'=a,\ mm'=b,\ nn'=c\ ;\ mn'+m'n=2f,\ nl'+n'l=2g,\ lm'+l'm=2h.$$

The direction-cosines of the line of intersection of the planes

$$lx+my+nz=0,\ l'x+m'y+n'z=0,$$

are proportional to $mn'-m'n$, $nl'-n'l$, $lm'-l'm$ [as in Art. 38, (*i*)].

Now $mn'-m'n=[\,(mn'+m'n)^2-4mm'nn'\,]^{\frac{1}{2}}$

$=(4f^2-4bc)^{\frac{1}{2}}$, which is here

$=[\,(b+c-a)^2-4bc]^{\frac{1}{2}}=[a^2+b^2+c^2-2bc-2ca-2ab\,]^{\frac{1}{2}}$.

The *symmetry* of the result shows that

$$mn'-m'n=nl'-n'l=lm'-l'm.\ ]$$

5. Show that the equations of the lines bisecting the angles between the lines

$$\frac{x}{0}=\frac{y}{-1}=\frac{z}{2}\ ;\ \frac{x}{12}=\frac{y}{1}=\frac{z}{-10}\ ;$$

are $$\frac{x}{6}=\frac{y}{-3}=\frac{z}{2}\ ;\ \frac{x}{3}=\frac{y}{2}=\frac{z}{-6}.$$

6. *Foot of the perpendicular on a plane.* **Find the co-ordinates of the foot of the perpendicular drawn from the origin on the plane $2x+3y-4z+1=0$.** [*P. U. 1935 S*]

7. Show that the line $\frac{x+1}{-2}=\frac{y+2}{3}=\frac{z+5}{4}$ lies in the plane $x+2y-z=0$. [*Sind. U. 1950*]

[**Line of greatest slope. Def.** The line of greatest slope through a point on an inclined plane is the line through the point perpendicular to the line of intersection of the plane with a horizontal plane.]

8. *Equations of the line of greatest slope.* **If OZ is vertical, show that the line of greatest slope through the point (1, 1, 1) on the plane $x+2y+3z=6$ is $\frac{x-1}{3}=\frac{y-1}{6}=\frac{z-1}{-5}$. [*L. U.*]

9. A plane parallel to the line $x-1=2y-5=2z$ and also parallel to the line $3x=4y-11=3z-4$ passes through the point (2, 3, 3). Find its equation. [*B. H. U. 1944*]

**10. P is a given point and PM, PN are the perpendiculars from P to the planes ZOX, XOY. OP makes angles $\theta, \alpha, \beta, \gamma$ with the plane OMN and the rectangular co-ordinate planes. Prove that

$$\operatorname{cosec}^2\theta=\operatorname{cosec}^2\alpha+\operatorname{cosec}^2\beta+\operatorname{cosec}^2\gamma.$$ [*D. U. H. 1937*]

11. *Projection of a line on a plane.* **Obtain, in the symmetrical form, the equations of the projection of the line**

$$8x-y-7z-8=0=x+y+z-1$$

on the plane $5x-4y-z=5$. [*P. U.*]

12. Prove that the equation to the two planes inclined at an angle α to the xy-plane and containing the line $y=0$, $z\cos\beta=x\sin\beta$, is

$$(x^2+y^2)\tan^2\beta+z^2-2zx\tan\beta=y^2\tan^2\alpha.$$

13. Find the equation of the plane determined by the parallel lines

$$\frac{x+1}{3}=\frac{y-2}{2}=\frac{z}{1} \text{ and } \frac{x-3}{3}=\frac{y+4}{2}=\frac{z-1}{1}.$$ [*P.U.H.*]

14. Find the equation of the plane which contains the line $x=\frac{1}{2}(y-3)=\frac{1}{3}(z-5)$ and which is perpendicular to the plane

$$2x+7y-3z=1.$$ [*L. U.*]

15. Show that the orthogonal projection of the line $\frac{x}{1}=\frac{y-1}{2}=\frac{z-2}{3}$ on the plane $2x+3y+4z=0$ is $\frac{x}{11}=\frac{y}{2}=\frac{z}{-7}$.

16. Prove that the lines

$$\frac{x+1}{1}=\frac{y+1}{2}=\frac{z+1}{3},\quad x+2y+3z-8=0=2x+3y+4z-11$$

are coplanar, and find the co-ordinates of their point of intersection.

Find also the equation of the plane containing them.

17. Find the angle between the lines

$$\frac{x-6}{2}=\frac{y-2}{1}=\frac{z+4}{-1};\ x+5y-2z=6,\ 6x-4y+5z=2\ ;$$

show that the lines intersect, and find the equation of the plane containing them. [*B. U.*]

18. Prove that the lines

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \text{ and } \frac{x-l}{\alpha}=\frac{y-m}{\beta}=\frac{z-n}{\gamma}$$

intersect, find the co-ordinates of the point in which they meet and the equation of the plane in which they lie. [*P.U.M.P. 1942*]

19. Show that the lines

$x+2y+3z-4=0=2x+3y+4z-5,\ 2x-3y+3z-5=0=4x-y+5z-7$ **are coplanar, and lie in the plane** $x+y+z=1$.

Find the co-ordinates of their point of intersection.

[See Note in Art. 45, (*b*).]

20. Find the equations of the line drawn through the point $(2, -3, 1)$ **parallel to the plane** $2x+y-z=6$, **so as to meet the line** $\dfrac{x-2}{2}=\dfrac{y}{-3}=\dfrac{z-2}{-1}$.

Find also the co-ordinates of the point of intersection. [*L.U.*]

21. Find the equations of the two straight lines through the origin each of which intersects the straight line $\frac{1}{2}(x-3)=y-3=z$ and is inclined at an angle of 60° to it. [*L. U. 1937*]

22. A square ABCD, of diagonal 2a, is folded along the diagonal AC, so that the planes DAC, BAC are at right angles. Show that the shortest distance between DC and AB is then $2a/\sqrt{3}$. [*P. U. 1951*]

23. Show that (*i*) shortest distance between any two opposite edges of the tetrahedron formed by the planes

$$y+z=0,\ z+x=0,\ x+y=0,\ x+y+z=a$$

is $\dfrac{2a}{\sqrt{6}}$. [*J. & K. U. 1953*]

(*ii*) **the three lines of shortest distance intersect at the point** $x=y=z=-a$. [*P(P). U. 1951*]

**24. Prove that the S. D. between the lines

$$\frac{x-x_1}{\cos\alpha_1}=\frac{y-y_1}{\cos\beta_1}=\frac{z-z_1}{\cos\gamma_1};\ \frac{x-x_2}{\cos\alpha_2}=\frac{y-y_2}{\cos\beta_2}=\frac{z-z_2}{\cos\gamma_2}$$

meets the first line at a point whose distance from (x_1, y_1, z_1) is

$$\frac{\Sigma\,(x_1-x_2)\,(\cos\alpha_1-\cos\theta\cos\alpha_2)}{\sin^2\theta},$$

where θ is the angle between the lines.

25. *Image of a point in a plane.* **Find the co-ordinates of the image of the origin in the plane $2x+3y-4z+1=0$.** [*P.U. 1935 S*]

26. Image of a point in a line. *From a point P, whose co-ordinates are (x, y, z), the perpendicular PM is drawn to the straight line through the origin whose direction-cosines are l, m, n and is produced to P' where $PM=MP'$. If the co-ordinates of P' are (x', y', z'), show that*

$$\frac{x+x'}{l}=\frac{y+y'}{m}=\frac{z+z'}{n}=2(lx+my+nz). \quad [\textit{Bar. U. 1954}]$$

27. Find the locus of a point which moves so that the ratio of its distances from two given lines is constant.

28. A rod of fixed length moves so that its extremities lie on two given skew lines at right angles to each other. Prove that the locus of the middle point of the rod is a circle. [*B. U. 1942*]

29. Lines are drawn to intersect the lines $y-mx=0=z-c$ and $y+mx=0=z+c$ and to make a constant angle with the z-axis. Show that if $-1 < m < 1$, the locus of their mid-points is an ellipse of eccentricity $(1-m^4)^{\frac{1}{2}}$. [*L. U.*]

30. A line moves so as to meet the lines $\dfrac{x}{\cos\alpha}=\dfrac{y}{\pm\sin\alpha}=\dfrac{z\mp c}{0}$ in A and B and passes through the curve $yz=k^2$, $x=0$. Prove that the locus of the mid-point of AB is a curve of the third degree, two of whose asymptotes are parallel to the given lines. [*Ag. U. 1945*]

31. Find the locus of a straight line which meets OX and the circle $x^2+y^2=a^2$, $z=c$ so that the distance between the points of section is $\sqrt{a^2+c^2}$.

[The equations of OX are $\dfrac{x}{1}=\dfrac{y}{0}=\dfrac{z}{0}$ ... (1)

and those of the circle are $x^2+y^2=a^2$, $z=c$ ... (2)

Let the st. line meet the x-axis (1) in $P(r, 0, 0)$ and the circle (2) in $Q(a\cos\theta, a\sin\theta, c)$. Write down the equations of PQ ...(3). Also $PQ=\sqrt{a^2+c^2}$ (Given), $\therefore$? ...(4). Eliminate r and θ from (3) and (4).]

32. *The ends of diameters of the ellipse $z=c$, $x^2/a^2+y^2/b^2=1$, are joined to the corresponding ends of the conjugates of parallel diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=-c$. Find the equation to the surface generated by the joining lines.* [*Ag. U. 1950*]

33. Find the equations to the planes through the point $(1, 0, -1)$ and the lines $4x-y-13=0=3y-4z-1$; $y-2z+2=0=x-5$, and

show that the equations to the line through the given point which intersects the two given lines can be written $x=y+1=z+2$.

[*J. & K. U.*]

****34. Prove that through the point (X, Y, Z) one line can be drawn which intersects the lines**

$$\mathbf{y=x\tan\alpha,\ z=c\ ;\ y=-x\tan\alpha,\ z=-c}$$

and that it meets the plane xy at the point

$$\mathbf{x=(cYZ\cot\alpha-c^2X)/(Z^2-c^2),\ \ y=(cXZ\tan\alpha-c^2Y)/(Z^2-c^2),\ z=0.}$$

[*L. U.*]

35. Find the surface generated by a straight line which intersects the lines $y=0,\ z=c$; $x=0,\ z=-c$ and the hyperbola $z=0,\ xy+c^2=0$.

[*B. U. 1941*]

36. Find the surface generated by a straight line which meets $\mathbf{y=mx,\ z=c\ ;\ y=-mx,\ z=-c}$; and $\mathbf{y^2+z^2=c^2,\ x=0}$.

[The equations of *any* line intersecting the lines $y=mx,\ z=c$; $y=-mx,\ z=-c$, are

$$mx-y+k_1(z-c)=0\ \ldots(1)$$

$$mx+y+k_2(z+c)=0\ \ldots(2)$$

If it meets the circle $y^2+z^2=c^2,\ x=0\ \ldots(3)$

we have to eliminate x, y, z from (1), (2), (3). It will be found that

$$\frac{4c^2k_1^2k_2^2}{(k_1+k_2)^2}+\frac{c^2(k_1-k_2)^2}{(k_1+k_2)^2}=c^2 \qquad [\text{ Cancel } c^2\ ]$$

or $\quad 4k_1^2k_2^2+(k_1-k_2)^2=(k_1+k_2)^2$, or $4k_1^2k_2^2=4k_1k_2 \qquad$ [Cancel $4k_1k_2$]

or $\quad k_1k_2=1\ \ldots(4)$

Eliminate k_1, k_2 from (1), (2), (4).]

37. Show that the locus of lines which meet the lines

$$\mathbf{\frac{x\pm a}{0}=\frac{y}{\sin\alpha}=\frac{z}{\mp\cos\alpha}}\ \textbf{at the same angle is}$$

$$\mathbf{(xy\cos\alpha-az\sin\alpha)(xz\sin\alpha-ay\cos\alpha)=0.}\qquad [\textit{D. U. H.}]$$

**38. Show that the two lines which intersect the four lines,

$$y=1,\ z=-1\ ;\ z=1,\ x=-1\ ;\ x=1,\ y=-1\ ;\ x=0,\ y+z=0,$$

are $\quad z=1,\ y+1=0\ ;\ z+2x+1=0,\ y-z-2=0.$

39. A point P moves on the plane $x/a+y/b+z/c=1$, and the plane through P perpendicular to OP meets the axes in A, B, C. Planes are drawn through A, B, C parallel to YOZ, ZOX, XOY. Show that the locus of their point of intersection is

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}.$$

****40. Show that the planes**

$$x+y-z-2=0,\ 2x-y-z+2=0,\ x-5y+z+4=0$$

form a triangular prism, and calculate the breadth of each face of the prism. [*B. U.*]

41. Show that the planes

$$x=y \sin\psi+z\sin\phi,\ z=x\sin\phi+y\sin\theta,\ y=x\sin\psi+z\sin\theta$$

intersect in the line $\dfrac{x}{\cos\theta}=\dfrac{y}{\cos\phi}=\dfrac{z}{\cos\psi}$ **if** $\theta+\phi+\psi=\dfrac{\pi}{2}$.

[*J. & K. U. 1949*]

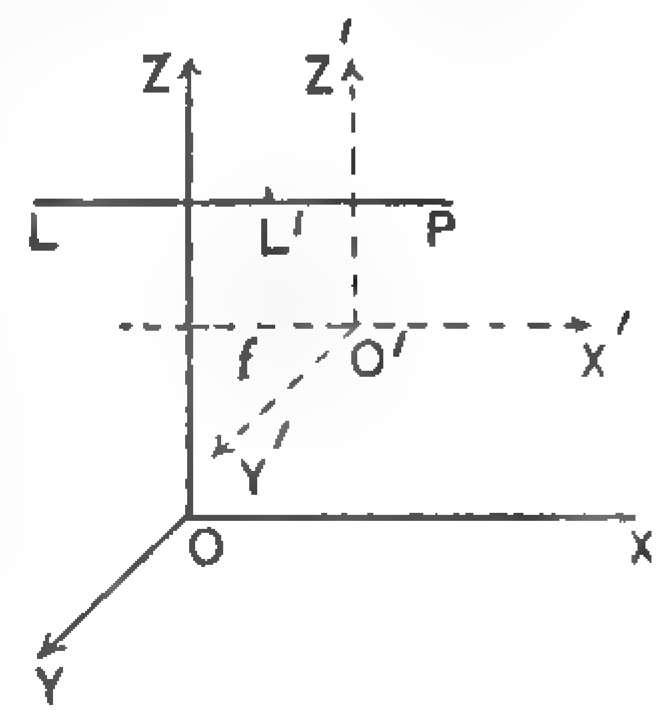
42. The direction-cosines of OA, OB, OC are $l_r, m_r, n_r, r=1, 2, 3$; and OA′, OB′, OC′ bisect the angles BOC, COA, AOB. Show that the planes AOA′, BOB′, COC′ pass through the line

$$\frac{x}{l_1+l_2+l_3}=\frac{y}{m_1+m_2+m_3}=\frac{z}{n_1+n_2+n_3}.$$

CHAPTER VI

CHANGE OF AXES

55. Formulae for change of origin. *To change the origin of co-ordinates without changing the directions of the axes.*



Let OX, OY, OZ be the old axes, O' the new origin, and O'X', O'Y', O'Z' the new axes ∥ to OX, OY, OZ. Let (f, g, h) be the co-ordinates of O' referred to OX, OY, OZ.

Let P be any pt., (x, y, z) its co-ordinates referred to the old axes, and (x', y', z') referred to the new.

(*a*) *To express x, y, z in terms of x', y', z'.*

From P draw PL ⊥ on the plane YOZ to meet the plane Y'O'Z' in L'.

Then $x = LP = LL' + L'P$

[But LL' = distance between ∥ planes YOZ, Y'O'Z'
= ⊥ distance of O' from the plane YOZ
$= f$]

$= f + x'$

or $\mathbf{x = x' + f}$.

Similarly $\mathbf{y = y' + g}$, $\mathbf{z = z' + h}$.

(*b*) *To express x', y', z' in terms of x, y, z.*

∵ $x = x' + f$, $y = y' + g$, $z = z' + h$, [Proved in part (*a*)]

∴ $x' = x - f$, $y' = y - g$, $z' = z - h$.

Cor. 1. Rule to find the transformed equation of a surface when the origin is changed to (f, g, h), the axes remaining parallel to their original directions :

In the given equation change **x** *to* **x+f**, **y** *to* **y+g**, **z** *to* **z+h**. *The resulting equation is the required equation.*

For $x = x' + f$, $y = y' + g$, $z = z' + h$, and in the resulting equation, we have to change x', y', z' to the current co-ordinates x, y, z.

Cor. 2. *If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is transformed by changing the origin, the axes remaining parallel to their original directions, then the coefficients of the second degree terms remain unchanged.*

Proof. The given expression is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \quad \dots (1)$$

Changing the origin to (x_1, y_1, z_1), (1) becomes

$$a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2$$
$$+2f(y+y_1)(z+z_1)+2g(z+z_1)(x+x_1)+2h(x+x_1)(y+y_1),$$

[Rule (Art. 55, Cor. 1)]

in which the second degree terms are

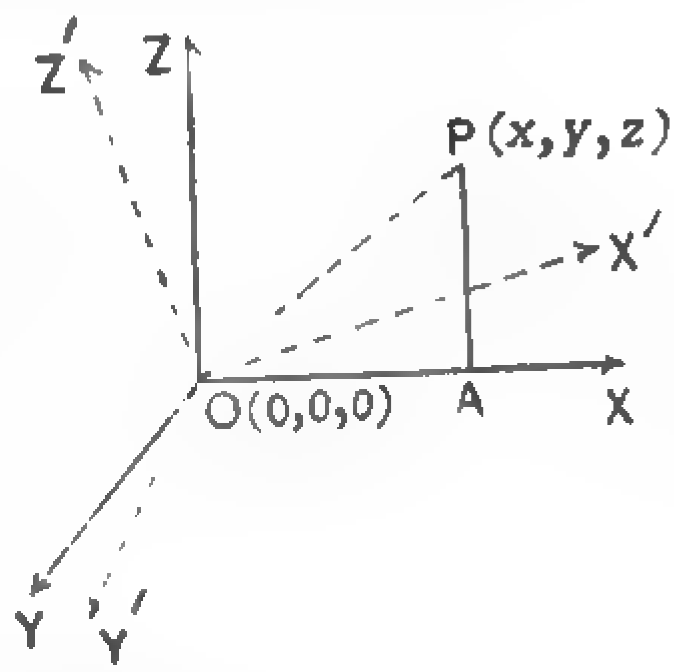
$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$, which are the same as in (1).

EXAMPLES

1. Find the co-ordinates of $(1, 2, 3)$, $(-1, -4, 0)$ referred to parallel axes through $(-2, -3, -4)$.

2. Is the point $(1, -2, 1)$ in the acute or in the obtuse angle between the planes $x+y-z=3$, $x-2y+z=3$?

56. Formulae for change of directions of axes. *To change the directions of the axes without changing the origin, both systems being rectangular.*



Let OX, OY, OZ be the old axes, and OX′, OY′, OZ′ the new. Let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the direction-cosines of OX′, OY′, OZ′ referred to OX, OY, OZ.

Let P be any pt., (x, y, z) its co-ordinates referred to the old axes, and (x', y', z') referred to the new.

(*a*) *To express* x, y, z *in terms of* x', y', z'.

Let A be the foot of the ⊥ from P on OX. Join OP.

Then $x = \text{OA} =$ projection of OP on OX (*direction-cosines* l_1, l_2, l_3 *referred to* OX′, OY′, OZ′)

$$=(x'-0)l_1+(y'-0)l_2+(z'-0)l_3$$

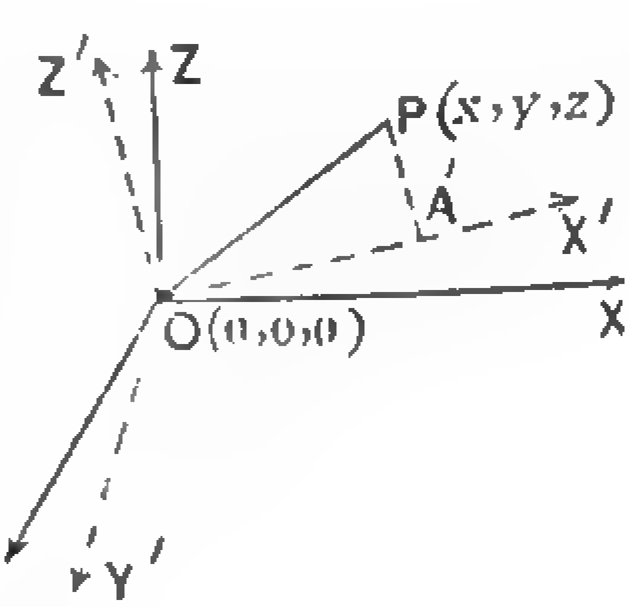
[$(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$ (Art. 14)]

or $\qquad \mathbf{x = l_1x' + l_2y' + l_3z'}$.

Similarly $\qquad \mathbf{y = m_1x' + m_2y' + m_3z'}$,

$\qquad \mathbf{z = n_1x' + n_2y' + n_3z'}$.

(*b*) *To express* x', y', z' *in terms of* x, y, z.



Let A′ be the foot of the ⊥ from P on OX′. Join OP.

Then $x' = \text{OA}' =$ projection of OP on OX′ (direction-cosines l_1, m_1, n_1 referred to OX, OY, OZ)

$$=(x-0)l_1+(y-0)m_1+(z-0)n_1$$

[$(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$ (Art. 14)]

or $\mathbf{x' = l_1x + m_1y + n_1z.}$

Similarly $\mathbf{y' = l_2x + m_2y + n_2z,}$

$\mathbf{z' = l_3x + m_3y + n_3z.}$

[**Rule to write down the formulae for change of axes to lines having direction-cosines l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 the origin remaining the same (both systems being rectangular) :**

Write down in four rows, x, y, z ; l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3. Leave blank the space at the left of the first row, and write down x', y', z' at the left of the second, third and fourth rows.

$$\begin{array}{cccc} & x & y & z \\ x' & l_1 & m_1 & n_1 \\ y' & l_2 & m_2 & n_2 \\ z' & l_3 & m_3 & n_3 \end{array}$$

To get the value of x, multiply the numbers in the x-column by the numbers at the left of their respective rows, and add the products, thus getting $x = l_1x' + l_2y' + l_3z'.$

Similarly $y = m_1x' + m_2y' + m_3z',$

$z = n_1x' + n_2y' + n_3z'.$

To get the value of x', multiply the numbers in the x'-row by the numbers at the heads of their respective columns, and add the products, thus getting $x' = l_1x + m_1y + n_1z.$

Similarly $y' = l_2x + m_2y + n_2z,$

$z' = l_3x + m_3y + n_3z.$]

57. *The degree of an equation is unaltered by any transformation of axes.*

If only the origin is changed to (f, g, h), the axes remaining ‖ to their original directions, x is changed to $x + f$, and so on ...(1)

[Art. 55]

If only the axes are changed to lines having direction-cosines l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3, the origin remaining the same (both systems being rectangular), x is changed to $l_1x + l_2y + l_3z$, and so on ...(2) [Art. 56]

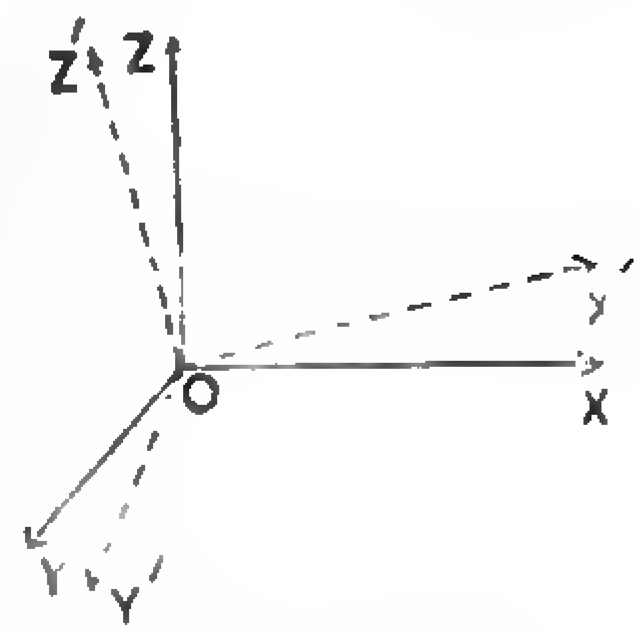
From (1) and (2), however the axes are changed, x is changed to $l_1x + l_2y + l_3z + f$ (*i.e.*, an expression of the first degree in x, y, z), and so on.

$\therefore$ the degree of the equation cannot *increase* ...(3)

Also the degree of the equation cannot *decrease*. For, if possible, suppose that it decreases. Then on changing back to the old axes to get back the given equation, the degree will have to increase, which is impossible. [From (3)]

Hence the degree of the equation is unaltered.

58. *Relations between the direction-cosines of three mutually perpendicular lines.* **To find the relations between the direction-cosines of three mutually perpendicular lines.**



Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction-cosines of three mutually $\perp$ lines OX′, OY′, OZ′. Then

(I) (*i*) $\because$ these are the direction-cosines

$$\therefore \left.\begin{array}{l} \mathbf{l_1^2+m_1^2+n_1^2=1,} \\ \mathbf{l_2^2+m_2^2+n_2^2=1,} \\ \mathbf{l_3^2+m_3^2+n_3^2=1.} \end{array}\right\} \ldots(A) \qquad [\, l^2+m^2+n^2=1 \text{ (Art. 9) }]$$

(*ii*) Again $\because$ OY′ is $\perp$ to OZ′,* OZ′ $\perp$ to OX′, OX′ $\perp$ to OY′

$$\therefore \left.\begin{array}{l} \mathbf{l_2l_3+m_2m_3+n_2n_3=0,} \\ \mathbf{l_3l_1+m_3m_1+n_3n_1=0,} \\ \mathbf{l_1l_2+m_1m_2+n_1n_2=0.} \end{array}\right\} \ldots(B) \qquad [\, ll'+mm'+nn'=0 \text{ (Art. 13,}(a)\text{, Cor. 3)}]$$

(II) (*i*) $\because$ the direction-cosines of OX, OY, OZ referred to OX′, OY′, OZ′ are l_1, l_2, l_3 ; m_1, m_2, m_3 ; n_1, n_2, n_3

$$\therefore \left.\begin{array}{l} \mathbf{l_1^2+l_2^2+l_3^2=1,} \\ \mathbf{m_1^2+m_2^2+m_3^2=1,} \\ \mathbf{n_1^2+n_2^2+n_3^2=1.} \end{array}\right\} \ldots(C) \qquad [\, l^2+m^2+n^2=1 \text{ (Art. 9) }]$$

(*ii*) Again $\because$ OY is $\perp$ to OZ, OZ $\perp$ to OX, OX $\perp$ to OY

$$\therefore \left.\begin{array}{l} \mathbf{m_1n_1+m_2n_2+m_3n_3=0,} \\ \mathbf{n_1l_1+n_2l_2+n_3l_3=0,} \\ \mathbf{l_1m_1+l_2m_2+l_3m_3=0.} \end{array}\right\} \ldots(D) \qquad [\, ll'+mm'+nn'=0 \text{ (Art. 13, }(a)\text{, Cor. 3) }]$$

(A), (B) ; (C), (D) are the required relations.

Note. The two sets (A), (B) ; (C), (D) are not independent. In fact it can be proved that if either of the two sets is given, the other can be deduced algebraically.

[**Rule to write down the relations (C), (D) from (A), (B).** *Write in three rows the direction cosines of the lines thus,*

$$\begin{array}{lll} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3. \end{array}$$

Now change columns into rows thus getting

$$\begin{array}{lll} l_1, & l_2, & l_3 \\ m_1, & m_2, & m_3 \\ n_1, & n_2, & n_3 \end{array}$$

and remember that the numbers in the rows are the direction-cosines of three mutually perpendicular lines ; use (A), (B).]

Cor. If $\mathbf{l_1, m_1, n_1}$; $\mathbf{l_2, m_2, n_2}$; $\mathbf{l_3, m_3, n_3}$ are the direction-

---

***How to write this step.** If, of three things *a, b, c* we have to take two at a time, it is customary to take the second and the third, *i.e.*, *b, c* ; then the third and the first, *i.e.*, *c, a* ; and then the first and the second, *i.e.*, *a, b*.

Here OX′, OY′, OZ′ are three mutually $\perp$ lines,

$\therefore$ we take OY′ $\perp$ to OZ′ ; then OZ′ $\perp$ to OX′ ; and then OX′ $\perp$ to OY′.

cosines of three mutually perpendicular lines, to prove that

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = \pm 1.$$

$$\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} \times \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix}$$

[which, by the Product Rule for determinants]

$$= \begin{vmatrix} l_1^2+m_1^2+n_1^2, & l_1l_2+m_1m_2+n_1n_2, & l_1l_3+m_1m_3+n_1n_3 \\ l_2l_1+m_2m_1+n_2n_1, & l_2^2+m_2^2+n_2^2, & l_2l_3+m_2m_3+n_2n_3 \\ l_3l_1+m_3m_1+n_3n_1, & l_3l_2+m_3m_2+n_3n_2, & l_3^2+m_3^2+n_3^2 \end{vmatrix}$$

$$= \begin{vmatrix} 1, & 0, & 0 \\ 0, & 1, & 0 \\ 0, & 0, & 1 \end{vmatrix}$$ [From the relations (A) and (B) of Art. 58]

$$= 1$$

$$\therefore \quad \begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = \pm 1.$$

EXAMPLES

1. OX, OY, OZ ; $O\xi$, $O\eta$, $O\zeta$ are two sets of rectangular axes through a common origin O. Obtain the relations between the direction-cosines of the lines $O\xi$, $O\eta$, $O\zeta$. [*D.U.H. 1949*]

If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular lines, prove that the line whose direction-cosines are proportional to $l_1+l_2+l_3$, $m_1+m_2+m_3$, $n_1+n_2+n_3$ makes equal angles with them.

2. Three straight lines mutually at right angles meet in a point P, and two of them intersect the axes of x and y respectively, while the third passes through a fixed point (0, 0, c) on the axis of z. Show that the equation of the locus of P is

$$x^2+y^2+z^2 = 2cz.$$ [*D. U. H. 1949*]

3. Two systems of rectangular axes have the same origin ; if a plane cuts them at distances a, b, c, and a′, b′, c′ from the origin, then show that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{a'^2}+\frac{1}{b'^2}+\frac{1}{c'^2}.$$ [*P.U. 1961*]

4. If (x_1, y_1, z_1) is at a distance d from each of the three mutually perpendicular planes $p_r - l_r x - m_r y - n_r z = 0$, $r = 1, 2, 3$; prove that

$$\frac{x_1-(l_1p_1+l_2p_2+l_3p_3)}{l_1+l_2+l_3}=\frac{y_1-(m_1p_1+m_2p_2+m_3p_3)}{m_1+m_2+m_3}$$
$$=\frac{z_1-(n_1p_1+n_2p_2+n_3p_3)}{n_1+n_2+n_3}=d.$$

5. Show that if

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d}$$

be transformed by change of co-ordinates from one set of rectangular axes to another with the same origin, the expressions $\mathbf{a+b+c}$, $\mathbf{u^2+v^2+w^2}$ remain unaltered in value.

[*B. U. 1948*]

**59. *If l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction-cosines of three mutually perpendicular lines, to prove that*

$$l_1=\pm(m_2n_3-m_3n_2),\ m_1=\pm(n_2l_3-n_3l_2),\ n_1=\pm(l_2m_3-l_3m_2).$$

[In words : **In the determinant** $\begin{vmatrix} \mathbf{l_1}, & \mathbf{m_1}, & \mathbf{n_1} \\ \mathbf{l_2}, & \mathbf{m_2}, & \mathbf{n_2} \\ \mathbf{l_3}, & \mathbf{m_3}, & \mathbf{n_3} \end{vmatrix}$,

each constituent = ± (its co-factor).]

The direction-cosines of the lines (1), (2), (3) are

$$l_1, m_1, n_1\ ;\ l_2, m_2, n_2\ ;\ l_3, m_3, n_3.$$

$\because$ the line (1) is $\perp$ to the lines (2), (3)

$\therefore$ $l_1l_2+m_1m_2+n_1n_2=0,$

$l_1l_3+m_1m_3+n_1n_3=0.$ [$ll'+mm'+nn'=0$ (Art. 13, (*a*), Cor. 3)]

Solving for l_1, m_1, n_1 (by cross-multiplication),

$$\frac{l_1}{m_2n_3-m_3n_2}=\frac{m_1}{n_2l_3-n_3l_2}=\frac{n_1}{l_2m_3-l_3m_2}.$$

$\therefore$ l_1, m_1, n_1 are proportional to

$$m_2n_3-m_3n_2,\ n_2l_3-n_3l_2,\ l_2m_3-l_3m_2.$$

Dividing by $\sqrt{\Sigma(m_2n_3-m_3n_2)^2}$

$=\pm\sin\theta$ [Complete formula (Art. 13, (*a*), Cor. 1)]

$=\pm\sin 90°$ [$\because$ line (2) is $\perp$ to line (3)]

$=\pm 1,$

the actual direction-cosines are

$$l_1=\pm(m_2n_3-m_3n_2),\ m_1=\pm(n_2l_3-n_3l_2),\ n_1=\pm(l_2m_3-l_3m_2),$$

the ambiguous signs being taken all +ve or all −ve.

Note. Similarly

$$l_2=\pm(m_3n_1-m_1n_3),\ m_2=\pm(n_3l_1-n_1l_3),\ n_2=\pm(l_3m_1-l_1m_3)\ ;$$
$$l_3=\pm(m_1n_2-m_2n_1),\ m_3=\pm(n_1l_2-n_2l_1),\ n_3=\pm(l_1m_2-l_2m_1).$$

MISCELLANEOUS EXAMPLES ON CHAPTER VI

1. OA, OB, OC are three mutually perpendicular lines through the origin, and their direction-cosines are l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3. If OA=OB=OC=a, prove that the equation to the plane ABC is

$$(l_1+l_2+l_3)x+(m_1+m_2+m_3)y+(n_1+n_2+n_3)\,z = a. \qquad [P.U.\ 1958]$$

2. If the axes of x,y,z are rectangular, prove that the substitutions

$$x=\frac{x'}{\sqrt{6}}+\frac{y'}{\sqrt{2}}+\frac{z'}{\sqrt{3}},\ y=-\frac{2}{\sqrt{6}}x'+\frac{z'}{\sqrt{3}},\ z=\frac{x'}{\sqrt{6}}-\frac{y'}{\sqrt{2}}+\frac{z'}{\sqrt{3}}$$

give a transformation to another set of rectangular axes in which the plane $x+y+z=0$ becomes the plane $z'=0$, and hence prove that the section of the surface $yz+zx+xy+1=0$ by the plane $x+y+z=0$ is a circle of radius $\sqrt{2}$.

CHAPTER VII

THE SPHERE

SECTION I

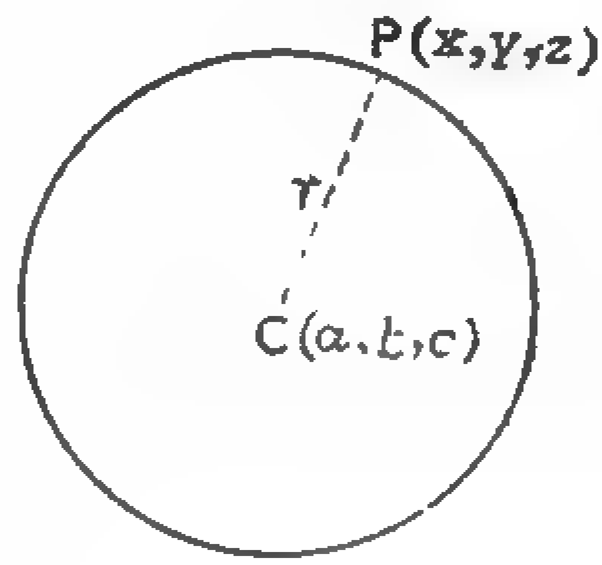
EQUATION OF A SPHERE

60. Sphere. Def. *A* **sphere** *is the locus of a point which moves so that its distance from a fixed point is constant.*

The fixed point is called the **centre,** *and the constant distance the* **radius** *of the sphere.*

61. *Central form.* **To find the equation of a sphere whose centre and radius are given.**

Let C(a, b, c) be the centre, and r the radius of the sphere.



Let P(x, y, z) be *any* pt. on the sphere.

Join CP.

Then CP$=r$. [Def. (Art. 60)]

$\therefore \quad \sqrt{(x-a)^2+(y-b)^2+(z-c)^2}=r$

or, squaring,

$$(\mathbf{x}-\mathbf{a})^2+(\mathbf{y}-\mathbf{b})^2+(\mathbf{z}-\mathbf{c})^2=\mathbf{r}^2,$$

which is the required equation.

Note 1. Central form. Since in the equation

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2,$$

(a, b, c) is the *centre* of the sphere, this form of the equation of a sphere may be called the **central form.**

Note 2. Important. The student must have noticed the close analogy between the equation of a circle

$$(x-h)^2+(y-k)^2=r^2$$

in Analytical Plane Geometry and that of a sphere

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

in Analytical Solid Geometry. He will constantly come across examples of this kind. **He should make use of this analogy as an aid to memory for standard results in sphere.**

Cor. 1. Standard form. *The equation of the sphere, whose centre is the origin and radius a, is* $\mathbf{x}^2+\mathbf{y}^2+\mathbf{z}^2=\mathbf{a}^2$.

Proof. The centre of the sphere is $(0, 0, 0)$, and radius$=a$.

$\therefore$ the equation of the sphere is

$$(x-0)^2+(y-0)^2+(z-0)^2=a^2$$

[$(x-a)^2+(y-b)^2+(z-c)^2=r^2$ (Art. 61)]

or $\qquad x^2+y^2+z^2=a^2.$

Note 3. Standard form. The equation $x^2+y^2+z^2=a^2$ is the *simplest* form of the equation of a sphere, and may be called the **standard form**.

Cor. 2. The equation of a sphere is of the form

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0.}$$

The equation of a sphere, in the central form, is

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \qquad [\text{ Art. } 61 \text{ }]$$

or $\quad x^2-2ax+a^2+y^2-2by+b^2+z^2-2cz+c^2=r^2$

or $\quad x^2+y^2+z^2-2ax-2by-2cz+(a^2+b^2+c^2-r^2)=0,$

which is of the form $x^2+y^2+z^2+2ux+2vy+2wz+d=0,$
where $u=-a,\ v=-b,\ w=-c,\ d=a^2+b^2+c^2-r^2.$

Note 4. The converse of Cor. 2 is *important*, and will be proved in Art. 62.

EXAMPLES

1. Obtain the equation of a sphere in its simplest form.

[*P. U. 1945*]

2. Find the equation to the sphere whose centre is (2, —3, 4) and radius 5. [*P. U. H. 1952*]

The centre of the sphere is $(2, -3, 4)$, and radius$=5$.

$\therefore$ the equation of the sphere is

$$(x-2)^2+[\,y-(-3)\,]^2+(z-4)^2=(5)^2$$

$$[\ (x-a)^2+(y-b)^2+(z-c)^2=r^2 \ \ (\text{Art. } 61)\ ]$$

or $\quad (x-2)^2+(y+3)^2+(z-4)^2=(5)^2$

or $\quad x^2-4x+4+y^2+6y+9+z^2-8z+16=25$

or $\quad x^2+y^2+z^2-4x+6y-8z+4=0.$

3. Find the equation to the sphere whose centre is (–1,0,1) and radius 2.

4. Two systems of rectangular axes have the same origin O ; if a sphere whose centre is O meets them in A, B, C ; A′, B′, C′, then show that Vol. OABC′$=\pm$Vol. OA′B′C.

5. If the axes are rectangular, what locus is represented by

$$x^2+y^2+z^2=a^2,\ y^2=4az\ ?$$

62. *General form.* **To prove that the equation**

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$$

represents a sphere, and to find its centre and radius.

The equation is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(1)$$

Transposing, $(x^2+2ux)+(y^2+2vy)+(z^2+2wz)=-d$

[Constant term on R.H.S.]

Completing the squares in the terms containing x, y, z [by adding $u^2+v^2+w^2$ to both sides],

$$(x^2+2ux+u^2)+(y^2+2vy+v^2)+(z^2+2wz+w^2)=u^2+v^2+w^2-d$$

or $(x+u)^2+(y+v)^2+(z+w)^2=u^2+v^2+w^2-d$

or $[x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2=(\sqrt{u^2+v^2+w^2-d})^2$ ...(2)

Comparing this with the equation of the sphere

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \quad ...(3)$$

(2) is the same as (3),

if $a=-u,\ b=-v,\ c=-w,\ r=\sqrt{u^2+v^2+w^2-d}$.

$\therefore$ (1) represents a sphere,

whose centre is $(\mathbf{-u, -v, -w})$, (a, b, c)

and radius $=\sqrt{\mathbf{u^2+v^2+w^2-d}}$. $[r]$

Note 1. General form. Since the equation

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

can, by a proper choice of u, v, w, d, be made to represent *any* sphere, this form of the equation of a sphere may be called the **general form.** It is also called the **general equation** of a sphere.

Note 2. **Virtual sphere.

The radius of the sphere$=\sqrt{u^2+v^2+w^2-d}$,

$\therefore$ if $u^2+v^2+w^2<d$, the radius is imaginary.

Thus it is a sphere with a real centre and an imaginary radius. It is called a **virtual sphere.**

[**Rule to write down the centre and radius of a sphere (equation in the general form) :**

(i) *Write the equation of the sphere so that the coefficients of x^2, y^2, z^2 are each $=1$ on the L.H.S. (by dividing thro' out by the coefficient of x^2, if necessary) (R.H.S. being zero).*

(ii) *Compare this with $x^2+y^2+z^2+2ux+2vy+2wz+d=0$, and write down the values of*

$u[=\frac{1}{2}$ (coeff. of x)], $v[=\frac{1}{2}$ (coeff. of y)], $w[=\frac{1}{2}$ (coeff. of z)], $d(=$constant term).

(iii) *Then the centre is $(-u, -v, -w)$, and radius$=\sqrt{u^2+v^2+w^2-d}$.*]

EXAMPLES

1. *Most general form.* **Prove that the equation**

$$\mathbf{ax^2+ay^2+az^2+2ux+2vy+2wz+d=0}$$

represents a sphere, and find its centre and radius. [*P.U. 1955S*]

[In words : *If there is an equation of the second degree in x, y, z in which the coefficients of x^2, y^2, z^2 are all equal, and the coefficients of*

yz, zx, xy are all zero, i.e., there are no product terms yz, zx, xy, then the equation represents a sphere.]

The equation is

$$ax^2+ay^2+az^2+2ux+2vy+2wz+d=0 \quad ...(1)$$

Dividing thro' out by a,

$$x^2+y^2+z^2+2\frac{u}{a}x+2\frac{v}{a}y+2\frac{w}{a}z+\frac{d}{a}=0.$$

Transposing,

$$\left(x^2+2\frac{u}{a}x\right)+\left(y^2+2\frac{v}{a}y\right)+\left(z^2+2\frac{w}{a}z\right)=-\frac{d}{a}.$$

[Constant term on R.H.S.]

Completing the squares in the terms containing x, y, z

[by adding $\frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{a^2}$ to both sides],

$$\left(x^2+2\frac{u}{a}x+\frac{u^2}{a^2}\right)+\left(y^2+2\frac{v}{a}y+\frac{v^2}{a^2}\right)+\left(z^2+2\frac{w}{a}z+\frac{w^2}{a^2}\right)$$

$$=\frac{u^2}{a^2}+\frac{v^2}{a^2}+\frac{w^2}{a^2}-\frac{d}{a}$$

or $\left(x+\frac{u}{a}\right)^2+\left(y+\frac{v}{a}\right)^2+\left(z+\frac{w}{a}\right)^2=\frac{u^2+v^2+w^2-ad}{a^2}$

or $\left[x-\left(-\frac{u}{a}\right)\right]^2+\left[y-\left(-\frac{v}{a}\right)\right]^2+\left[z-\left(-\frac{w}{a}\right)\right]^2$

$$=\left[\frac{\sqrt{u^2+v^2+w^2-ad}}{a}\right]^2 \quad ...(2)$$

Comparing this with the equation of the sphere

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \quad ...(3)$$

(2) is the same as (3),

if $\quad$ "a"$=-\frac{u}{a}$, $b=-\frac{v}{a}$, $c=-\frac{w}{a}$, $r=\frac{\sqrt{u^2+v^2+w^2-ad}}{a}$

$\therefore$ (1) represents a sphere,

whose centre is $\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$, $\qquad$ [(a, b, c)]

and radius$=\frac{\sqrt{u^2+v^2+w^2-ad}}{a}$. $\qquad$ [r]

2. Find the centre and radius of the sphere given by

$$\mathbf{x^2+y^2+z^2+2y-4z=4.}$$

The equation of the sphere is

$x^2+y^2+z^2+2y-4z-4=0.$ $\quad$ [Cceffs. of x^2, y^2, z^2 already$=1$]

Comparing this with

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$

here $u=\frac{1}{2}(0)=0,\ v=\frac{1}{2}(2)=1,\ w=\frac{1}{2}(-4)=-2,\ d=-4$

[$u=\frac{1}{2}$ (coeff. of x), $v=\frac{1}{2}$ (coeff. of y), $w=\frac{1}{2}$ (coeff. of z), d=constant term]

$\therefore$ the centre is $(0, -1, 2)$, [$(-u, -v, -w)$]

and radius $=\sqrt{(0)^2+(1)^2+(-2)^2-(-4)}$ [$\sqrt{u^2+v^2+w^2-d}$]

$=\sqrt{1+4+4}=\sqrt{9}=3.$

3. **Find the centre and radius of the sphere given by**

(*i*) $\mathbf{x^2+y^2+z^2+2x-4y+6z=2.}$

(*ii*) $\mathbf{2x^2+2y^2+2z^2-2x+4y-6z=1.}$

4. Prove that $x^2+y^2+z^2+2x-4y+6z-2=0$ represents a sphere whose centre is at $(-1, 2, -3)$ and radius 4.

5. Find the ratios in which the sphere $x^2+y^2+z^2=24$ divides the line joining the points $(-1, 1, 2)$, $(-4, 4, 8)$.

6. The sphere $x^2+y^2+z^2+6y+12z-11=0$ meets the line joining A$(1, -1, -3)$, B$(4, 5, 6)$ in the points P and Q. Prove that

$$\text{AP} : \text{PB} = -\text{AQ} : \text{QB} = 1 : 2.$$

7. A is the point $(1, 3, 4)$ and B the point $(1, -2, -1)$. A point P moves so that 3PA=2PB. Prove that the locus of P is the sphere

$$x^2+y^2+z^2-2x-14y-16z+42=0.$$

Verify that this sphere divides AB internally and externally in the ratio 2 : 3.

8. From the point $(1, -1, 2)$ lines are drawn to meet the sphere $x^2+y^2+z^2=1$, and they are divided in the ratio 2 : 3. Prove that the points of section lie on the sphere

$$5x^2+5y^2+5z^2-6x+6y-12z+10=0.$$

9. **Prove that the locus of a point the sum of the squares of whose distances from any number of given points is constant, is a sphere.**

[Note. For problems relating to any number of given points let the given points be $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots, (x_n, y_n, z_n)$.]

10. **A point moves so that the sum of the squares of its distances from the six faces of a cube is constant ; show that its locus is a sphere.** [*P. U. 1937 S*]

[Note. For problems relating to the faces of a cube, take the centre of the cube as origin, and the three (mutually perpendicular) lines through the centre parallel to the edges as axes, so that the faces are parallel to the co-ordinate planes.

Then if an edge of the cube=2a, the equations of the faces are $x=a,\ x=-a;\ y=a,\ y=-a;\ z=a,\ z=-a.$]

63. A sphere can be found to satisfy four conditions.

The general equation of a sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$

which contains *four* arbitrary constants u, v, w, d. These constants can be determined from the four equations obtained by using the *four* conditions satisfied by the sphere, each condition giving rise to one equation.

$\therefore$ a sphere can be found to satisfy four conditions (*e.g.*, a sphere can be found to pass through any four non-coplanar points).

64. Four-point form. *To find the equation of the sphere through four given points.*

Let $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$ be the four pts.

Let the required equation of the sphere be

$$(x^2+y^2+z^2)+2ux+2vy+2wz+d=0 \quad \ldots (1)$$

$\because$ it passes thro' the four given pts.,

$$\therefore (x_1^2+y_1^2+z_1^2)+2ux_1+2vy_1+2wz_1+d=0 \quad \ldots (2)$$
$$(x_2^2+y_2^2+z_2^2)+2ux_2+2vy_2+2wz_2+d=0 \quad \ldots (3)$$
$$(x_3^2+y_3^2+z_3^2)+2ux_3+2vy_3+2wz_3+d=0 \quad \ldots (4)$$
$$(x_4^2+y_4^2+z_4^2)+2ux_4+2vy_4+2wz_4+d=0 \quad \ldots (5)$$

Eliminating u, v, w, d from (1), (2), (3), (4), (5) [by means of a determinant],

$$\begin{vmatrix} x^2+y^2+z^2, & x, & y, & z, & 1 \\ x_1^2+y_1^2+z_1^2, & x_1, & y_1, & z_1, & 1 \\ x_2^2+y_2^2+z_2^2, & x_2, & y_2, & z_2, & 1 \\ x_3^2+y_3^2+z_3^2, & x_3, & y_3, & z_3, & 1 \\ x_4^2+y_4^2+z_4^2, & x_4, & y_4, & z_4, & 1 \end{vmatrix}=0,$$

which is the required equation.

Note. Short cut for numerical examples. In numerical examples it is better to solve (2), (3), (4), (5) for u, v, w, d, and substitute these values in (1). (See Ex. 1, following.)

EXAMPLES

1. Find the equation to the sphere through the points (0, 0, 0), (0, 1, −1), (−1, 2, 0), (1, 2, 3). [*P(P). U. 1948 Em.*]

Let the required equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \ldots (1)$$

$\because$ it passes thro' $(0, 0, 0)$, $(0, 1, -1)$, $(-1, 2, 0)$, $(1, 2, 3)$,

$\therefore$

$$d=0 \quad \ldots (2)$$

$$1+1+2v-2w+d=0, \quad \text{or} \quad 2v-2w+d+2=0 \quad \ldots (3)$$

$$1+4-2u+4v+d=0, \quad \text{or} \quad -2u+4v+d+5=0 \quad \ldots (4)$$

$$1+4+9+2u+4v+6w+d=0, \quad \text{or} \quad 2u+4v+6w+d+14=0 \quad \ldots (5)$$

Eliminating d [by substituting from (2) in (3), (4), (5)],

$$2v-2w+2=0, \text{ or } v-w+1=0 \quad \ldots (6) \quad | \; 3$$

$$-2u+4v+5=0 \quad \ldots (7)$$

$$2u+4v+6w+14=0, \text{ or } u+2v+3w+7=0 \quad \ldots (8) \quad | \; 1$$

Eliminating w from (6) and (8) [by multiplying (6) by 3, (8) by 1, and adding],

$$u+5v+10=0 \quad \ldots (9)$$

Solving (7) and (9) by cross-multiplication,

$$\frac{u}{40-25}=\frac{v}{5-(-20)}=\frac{1}{-10-4}, \text{ or } \frac{u}{15}=\frac{v}{25}=\frac{1}{-14}, \text{ or } u=-\frac{15}{14},\ v=-\frac{25}{14}.$$

Substituting this value of v in (6),

$$-\frac{25}{14}-w+1=0, \text{ or } w=-\frac{11}{14}.$$

Substituting these values of u, v, w, d in (1),

$$x^2+y^2+z^2+2\left(-\frac{15}{14}\right)x+2\left(-\frac{25}{14}\right)y+2\left(-\frac{11}{14}\right)z+0=0,$$

or
$$x^2+y^2+z^2-\frac{15}{7}x-\frac{25}{7}y-\frac{11}{7}z=0$$

or
$$7(x^2+y^2+z^2)-15x-25y-11z=0,$$

which is the required equation.

[**Check.** This equation of the sphere,

$$7(x^2+y^2+z^2)-15x-25y-11z=0,$$

is satisfied by the co-ordinates of the four given pts.

$$(0, 0, 0), (0, 1, -1), (-1, 2, 0), (1, 2, 3)$$

thus,
$$7(0)-15(0)-25(0)-11(0)=0, \quad \text{or} \quad 0=0,$$
$$7(2)-15(0)-25(1)-11(-1)=0, \quad \text{or} \quad 0=0,$$
$$7(5)-15(-1)-25(2)-11(0)=0, \quad \text{or} \quad 0=0,$$
$$7(14)-15(1)-25(2)-11(3)=0, \quad \text{or} \quad 0=0. \;]$$

[**Rule to find the equation of the sphere through four given points :**

(*i*) *Let the required equation of the sphere be*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \ldots (1)$$

(*ii*) $\because$ *it passes thro' the four given pts.*

$\therefore$ *substitute their co-ordinates in (1), and get four equations (2), (3), (4), (5) in u, v, w, d.*

(iii) Solve (2), (3), (4), (5), and find the values of u, v, w, d.*

(iv) Substitute these values of u, v, w, d in (1). The resulting equation is the required equation.]

2. Find the equation of the sphere which passes through the origin and the points (a, 0, 0), (0, b, 0), (0, 0, c). [*P. U.*]

3. Show that the point $\left(\frac{3}{5}, 0, \frac{4}{5}\right)$ is the centre of the sphere which passes through the four points $(0,-1,-2)$, $(-2,1,2)$, $(2,2,-1)$ and $(0, 2, 3)$.

4. Find the centre of the sphere through the four points (0,0,0), (0, 2, 0), (−1, 0, 0), (0, 0, −4).

5. OA, OB, OC are mutually perpendicular lines through the origin, and their direction-cosines are l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3.

If OA=a, OB=b, OC=c, prove that the equation of the sphere OABC is

$$x^2+y^2+z^2-x(al_1+bl_2+cl_3)-y(am_1+bm_2+cm_3)$$
$$-z(an_1+bn_2+cn_3)=0.$$ [*P. U. 1952*]

6. Prove that the equation to the sphere circumscribing the tetrahedron whose sides are

$$\frac{y}{b}+\frac{z}{c}=0,\ \frac{z}{c}+\frac{x}{a}=0,\ \frac{x}{a}+\frac{y}{b}=0,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

is $$\frac{x^2+y^2+z^2}{a^2+b^2+c^2}-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0.$$ [*P(P). U. 1952 S*]

7. A plane passes through a fixed point (a, b, c) and cuts the axes in A, B, C. Show that the locus of the centre of the sphere OABC is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$. [*J. & K. U. 1956*]

[**Note. Important. For problems relating to a plane meeting the axes in A, B, C, let OA=a, OB=b, OC=c.**

Then the equation of the plane ABC, in the intercept form, is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

[***Rule to solve (2), (3), (4), (5) :**

(*i*) Eliminate *d* from (2) and (3) [by subtracting], and get an equation in *u, v, w*. Again, eliminate *d* from (3) and (4) [by subtracting], and get another equation in *u, v, w*. Again, eliminate *d* from (4) and (5) [by subtracting], and get a third equation in *u, v, w*.

(*ii*) Solve the three equations in *u, v, w* (as in the Rule of Ex. 1, Art. 51).

(*iii*) Substitute these values of *u, v, w* in any one of the equations (2), (3), (4), (5), say (2), and find the value of *d*.]

Here in Ex. 7, in order to avoid confusion with the given a, b, c in the pt. (a, b, c), let $OA=a'$, $OB=b'$, $OC=c'$.

Then the equation of the plane ABC, in the intercept form, is

$$\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}=1.\ ]$$

8. A sphere of constant radius k passes through the origin and meets the axes in A, B, C. Prove that the centroid of the triangle ABC lies on the sphere $9(x^2+y^2+z^2)=4k^2$. [*J. & K. U. 1953*]

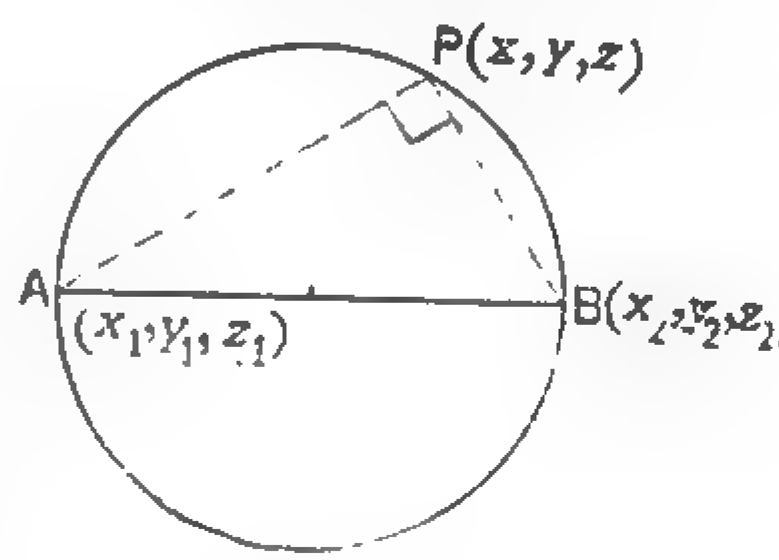
[**Note. Important. For problems relating to a sphere passing through the origin and meeting the axes in A, B, C, let OA=a, OB=b, OC=c. It will be found that the equation of the sphere through O(0, 0, 0), A(a, 0, 0), B(0, b, 0), C(0, 0, c) is**

$$\mathbf{x^2+y^2+z^2-ax-by-cz=0.}\ ]$$

9. A variable sphere passes through the origin O and meets the axes in A, B, C, so that the volume of the tetrahedron OABC is constant. Find the locus of the centre of the sphere.

10. A sphere of constant radius r passes through the origin, O, and cuts the axes (rectangular) in A, B, C. Prove that the locus of the foot of the perpendicular from O to the plane ABC is given by $(x^2+y^2+z^2)^2\,(x^{-2}+y^{-2}+z^{-2})=4r^2$. [*P. U. 1940*]

65] *Diameter form.* **To find the equation of the sphere which has (x_1, y_1, z_1) and (x_2, y_2, z_2) for extremities of a diameter.**



Let A, B be the pts. (x_1, y_1, z_1), (x_2, y_2, z_2).

Let $P(x, y, z)$ be *any* pt. on the sphere on AB as diameter.

Join AP, BP.

Then $\angle APB$, in a semi-circle, is a rt. $\angle$ ...(1)

Now the direction-cosines of AP are proportional to $x-x_1, y-y_1, z-z_1$

[$x_2-x_1, y_2-y_1, z_2-z_1$ (Art. 12)]

and those of BP are proportional to $x-x_2, y-y_2, z-z_2$

$\therefore$ from (1),

$$\mathbf{(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0,}$$

[$aa'+bb'+cc'=0$ (Art. 13, (*b*), Cor. 3)]

which is the required equation.

Note. Diameter form. Since in the equation

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0,$$

(x_1, y_1, z_1), (x_2, y_2, z_2) are the extremities of a *diameter*, this form of the equation of a sphere may be called the **diameter form.**

EXAMPLES

1. Show that the equation

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

represents the sphere on the join of (x_1, y_1, z_1), (x_2, y_2, z_2) as diameter.

2. **Find the equation of the sphere on the join of $(2, -3, 1)$ and $(3, -1, 2)$ as diameter.** [*P. U. 1936*]

The extremities of the diameter of the sphere are $\overset{(1)}{(2,-3,1)}$, $\overset{(2)}{(3,-1,2)}$.

∴ the equation of the sphere is

$$(x-2)(x-3)+[\,y-(-3)\,]\,[y-(-1)]+(z-1)(z-2)=0$$

$$[\,(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0 \text{ (Art. 65) }]$$

or $$(x-2)(x-3)+(y+3)(y+1)+(z-1)(z-2)=0$$

or $$x^2-5x+6+y^2+4y+3+z^2-3z+2=0$$

or $$x^2+y^2+z^2-5x+4y-3z+11=0.$$

3. **Find the equation of the sphere on the join of $(2, -3, 1)$ and $(1, -2, -1)$ as diameter.** [*P. U. 1935 S*]

66. *Equations of a circle.* **Two equations, one of a sphere and the other of a plane, together represent a circle.**

Proof. Let the equations of the sphere and the plane be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,$$

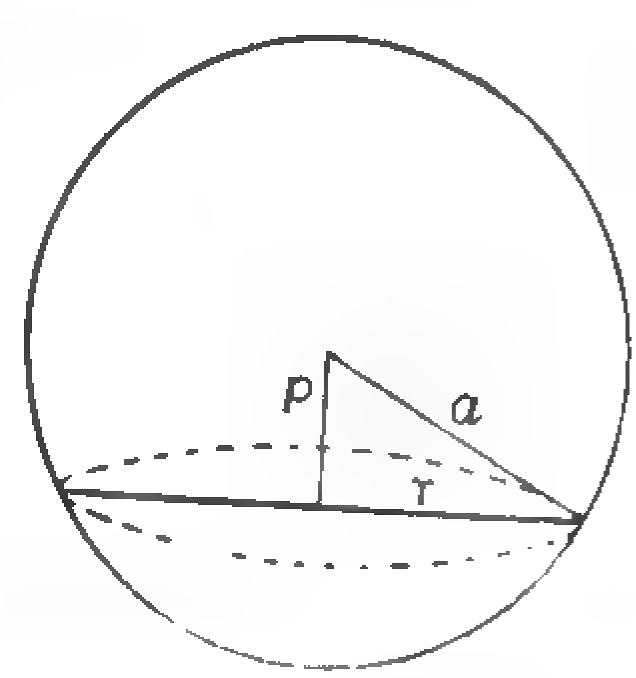
$$Ax+By+Cz+D=0.$$

Then the two equations together represent the curve of intersection of the sphere and the plane, which is a circle. [Art. 19]

Note. Important. *Centre and radius of the circle.* **The centre of the circle is the foot of the perpendicular from the centre of the sphere on the plane of the circle, and radius of the circle**

$=\sqrt{(\textbf{radius of sphere})^2-(\textbf{central} \perp \textbf{on plane})^2}.$

For $\quad r=\sqrt{a^2-p^2}.$



EXAMPLES

1. *Equations of circumcircle of a triangle.* **The plane ABC, whose equation is $x/a+y/b+z/c=1$, meets the axes in A, B, C. Find equations to determine the circumcircle of the triangle ABC, and obtain the co-ordinates of its centre.** [*P(P). U. 1956*]

(*a*) *The circumcircle of the* $\triangle ABC$ *is the circle of intersection of the sphere OABC and the plane ABC.* [*Note this step*]

The plane $\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1$ ...(1) meets the x-axis where, putting $y=0$, $z=0$ in (1), $x=a$,

$\therefore$ A is $(a, 0, 0)$.

Similarly B is $(0, b, 0)$, and C is $(0, 0, c)$.

Let the required equation of the sphere OABC be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(2)$$

$\because$ it passes thro' $O(0, 0, 0)$, $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$,

$\therefore$
$$d=0 \quad ...(3)$$
$$a^2+2ua+d=0 \quad ...(4)$$
$$b^2+2vb+d=0 \quad ...(5)$$
$$c^2+2wc+d=0 \quad ...(6)$$

Eliminating d [by substituting from (3) in (4), (5), (6)],

$$a^2+2ua=0, \quad \therefore u=-\frac{a}{2},$$

$$b^2+2vb=0, \quad \therefore v=-\frac{b}{2},$$

$$c^2+2wc=0, \quad \therefore w=-\frac{c}{2}.$$

Substituting these values of u, v, w, d in (2),

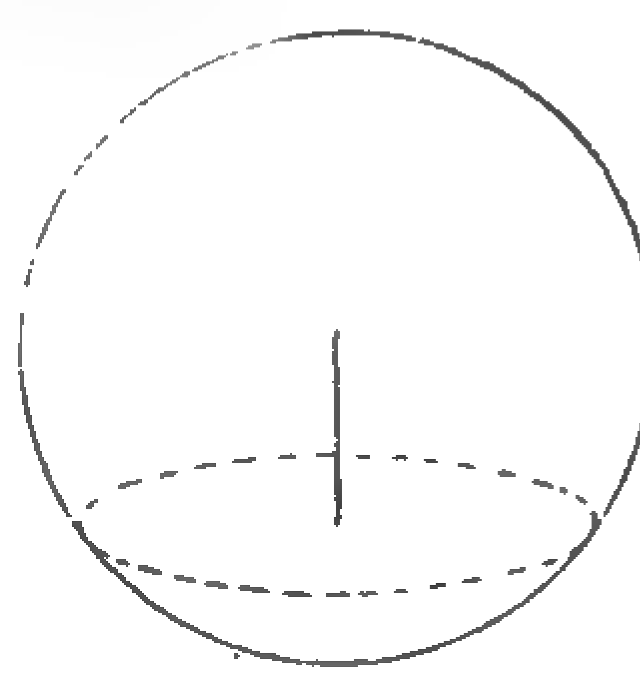
$$x^2+y^2+z^2+2\left(-\frac{a}{2}\right)x+2\left(-\frac{b}{2}\right)y+2\left(-\frac{c}{2}\right)z+0=0,$$

or
$$x^2+y^2+z^2-ax-by-cz=0.$$

$\therefore$ the equations of the circumcircle of the $\triangle ABC$ are

$$x^2+y^2+z^2-ax-by-cz=0, \ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

(*b*) *The centre of the circle is the foot of the* $\perp$ *from the centre of the sphere on the plane of the circle.*



The direction-cosines of the normal to the plane (1) are proportional to

$$\frac{1}{a}, \frac{1}{b}, \frac{1}{c}.$$

$\therefore$ the equations of the $\perp$ from the centre $\left(\dfrac{a}{2}, \dfrac{b}{2}, \dfrac{c}{2}\right)$ of the sphere on the plane are $\dfrac{x-\dfrac{a}{2}}{\dfrac{1}{a}}=\dfrac{y-\dfrac{b}{2}}{\dfrac{1}{b}}=\dfrac{z-\dfrac{c}{2}}{\dfrac{1}{c}}$.

Any pt. on this $\perp$ is

$$\left(\frac{a}{2}+\frac{r}{a},\ \frac{b}{2}+\frac{r}{b},\ \frac{c}{2}+\frac{r}{c}\right)$$

i.e., $$\left(\frac{a}{2}+ra^{-1},\ \frac{b}{2}+rb^{-1},\ \frac{c}{2}+rc^{-1}\right)^* \quad ...(7)$$

If it lies on the plane (1),

then $$\frac{1}{a}\left(\frac{a}{2}+ra^{-1}\right)+\frac{1}{b}\left(\frac{b}{2}+rb^{-1}\right)+\frac{1}{c}\left(\frac{c}{2}+rc^{-1}\right)=1$$

or $$\tfrac{1}{2}+ra^{-2}+\tfrac{1}{2}+rb^{-2}+\tfrac{1}{2}+rc^{-2}=1, \text{ or } r(a^{-2}+b^{-2}+c^{-2})=-\tfrac{1}{2},$$

$$\therefore \qquad r=-\frac{1}{2(a^{-2}+b^{-2}+c^{-2})}.$$

Substituting this value of r in (7), the foot of the $\perp$, *i.e.*, the centre of the circle is

$$\left[\frac{a}{2}-\frac{a^{-1}}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{b}{2}-\frac{b^{-1}}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{c}{2}-\frac{c^{-1}}{2(a^{-2}+b^{-2}+c^{-2})}\right]$$

i.e., $$\left[\frac{a(b^{-2}+c^{-2})}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{b(c^{-2}+a^{-2})}{2(a^{-2}+b^{-2}+c^{-2})},\ \frac{c(a^{-2}+b^{-2})}{2(a^{-2}+b^{-2}+c^{-2})}\right].$$

2. Find the centre and radius of the circle in which the sphere $x^2+y^2+z^2+2x-2y-4z-19=0$ is cut by the plane

$$x+2y+2z+7=0. \qquad [\textit{P.U. 1957 S}]$$

3. If r is the radius of the circle

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0,\ lx+my+nz=0,$$

prove that $(r^2+d)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2.$

4. Find the areas of the projections of the curve

$$x^2+y^2+z^2=25,\ 2x+y+2z=9$$

on the co-ordinate planes, and find the area of the curve. [*L.U.*]

**5. POP′ is a variable diameter of the ellipse $z=0,\ \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, and a circle is described in the plane PP′ZZ′ on PP′ as diameter. Prove that as PP′ varies, the circle generates the surface

$$(x^2+y^2+z^2)(x^2/a^2+y^2/b^2)=x^2+y^2. \qquad [\textit{Ag. U. 1944}]$$

[**Rule to find the surface generated by a curve whose equations involve one variable (parameter) :**

Eliminate the variable from the two equations of the curve.]

67. Any sphere through a given circle. *The equation of* **any** *sphere through the circle of intersection of*

the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0,$

and the plane $Ax+By+Cz+D=0,$

**i.e.*, work in +ve and −ve powers of a, b, c.

is $\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d+k(Ax+By+Cz+D)=0}$, *where k is any constant.*

[In words : **sphere + k (plane) = 0,** *where 'sphere' stands for the 'L.H.S. of the equation of the sphere (R.H.S. being zero)', and so for the 'plane'.*]

Proof. The equations of the sphere and the plane are

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \dots (1)$$

$$Ax+By+Cz+D=0 \quad \dots (2)$$

Consider the equation

$$x^2+y^2+z^2+2ux+2vy+2wz+d+k(Ax+By+Cz+D)=0 \quad \dots (3)$$

where k is any constant.

(*i*) It is an equation of the second degree in x, y, z in which the coeffs. of x^2, y^2, z^2 are all equal, and there are no product terms yz, zx, xy.

$\therefore$ it represents a sphere. [Ex. 1, Art. 62]

(*ii*) The co-ordinates of the pts. which satisfy both (1) and (2) also satisfy (3) [$\because$ substituting from (1) and (2) in (3), we get $0+k(0)=0$, or $0=0$, which is true]

$\therefore$ the pts. of intersection of the sphere (1) and the plane (2) lie on the sphere (3)

$\therefore$ (3) is the equation of any sphere thro' the circle of intersection of the sphere (1) and the plane (2).

Note. The value of k is found from the second condition satisfied by the sphere.

EXAMPLES

1. Find the equation to the sphere through the circle

$$\mathbf{x^2+y^2+z^2=1,\ x+2y+3z=4}$$

and the origin. [*B. U.*]

The equations of the circle are

$$x^2+y^2+z^2-1=0,\ x+2y+3z-4=0 \quad \dots (1)$$

[R. H. S. 's zero (*Note this step*)]

The equation of *any* sphere thro' this circle is

$$x^2+y^2+z^2-1+k(x+2y+3z-4)=0 \quad \dots (2)$$

[sphere $+k$ (plane) $=0$ (Art. 67)]

If it passes thro' (0, 0, 0), then

$$-1+k(-4)=0, \quad \therefore \quad k=-\tfrac{1}{4}.$$

Substituting this value of k in (2),

$$x^2+y^2+z^2-1-\tfrac{1}{4}(x+2y+3z-4)=0$$

or $$4(x^2+y^2+z^2)-4-(x+2y+3z-4)=0$$

or $$4(x^2+y^2+z^2)-x-2y-3z=0,$$

which is the required equation.

2. Find the equation to the sphere through the circle

$$x^2+y^2+z^2=9,\ 2x+3y+4z=5$$

and the point (1, 2, 3). [*P(P). U. 1954*]

3. Find the equation to the sphere which passes through the point (α, β, γ) and the circle $z=0,\ x^2+y^2=a^2$. [*J. & K. U. 1949*]

[The circle of intersection of the plane $z=0$ and the cylinder $x^2+y^2=a^2$ is the same as the circle of intersection of the plane $z=0$ and the sphere $x^2+y^2+z^2=a^2$.

[$\because$ putting $z=0$ in $x^2+y^2+z^2=a^2$, we get $x^2+y^2=a^2$.]

Now find the equation of the sphere thro' the pt. (α, β, γ) and the circle $z=0,\ x^2+y^2+z^2=a^2$.]

****4.** P is a variable point on a given line and A, B, C are its projections on the axes. Show that the sphere OABC passes through a fixed circle. [*Bar. U. 1954*]

5. Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0,\ 5y+6z+1=0\ ;$$
$$x^2+y^2+z^2-3x-4y+5z-6=0,\ x+2y-7z=0\ ;$$

lie on the same sphere and find its equation. [*P(P). U. 1953*]

The equations of the circles are

$$x^2+y^2+z^2-2x+3y+4z-5=0,\ 5y+6z+1=0 \quad \dots (1)$$
$$x^2+y^2+z^2-3x-4y+5z-6=0,\ x+2y-7z=0 \quad \dots (2)$$

The equation of *any* sphere thro' the circle (1) is

$$x^2+y^2+z^2-2x+3y+4z-5+k(5y+6z+1)=0 \quad \dots (3) \text{ [Art. 67]}$$

and the equation of *any* sphere thro' the circle (2) is

$$x^2+y^2+z^2-3x-4y+5z-6+k'(x+2y-7z)=0 \quad \dots (4)$$

If the circles (1) and (2) lie on the same sphere, then for some values of k, k', (3) is the same as (4)

$\therefore$ comparing coeffs. in (3) and (4),

[coeffs. of x], $\quad -2=-3+k' \quad \dots(5)$

[coeffs. of y], $\quad 3+5k=-4+2k' \quad \dots(6)$

[coeffs. of z], $\quad 4+6k=5-7k' \quad \dots(7)$

[constant terms], $\quad -5+k=-6 \quad \dots(8)$

[To solve (5) and (6) for k, k'.]

From (5), $k'=1$, $\therefore$ from (6), $3+5k=-4+2$, $\therefore$ $k=-1$.

Substituting these values of k, k' in (7) and (8), we get

$$4-6=5-7, \text{ and } -5-1=-6$$

or $\quad -2=-2$, and $\quad -6=-6$, each of which is true

$\therefore$ the circles lie on the same sphere.

Substituting the value of $k(=-1)$ in (3),

$$x^2+y^2+z^2-2x+3y+4z-5-(5y+6z+1)=0$$

or $$x^2+y^2+z^2-2x-2y-2z-6=0 \quad \dots(9)$$

which is the required equation of the sphere.

[**Check.** Substituting the value of $k'(=1)$ in (4),

$$x^2+y^2+z^2-3x-4y+5z-6+1(x+2y-7z)=0$$

or $x^2+y^2+z^2-2x-2y-2z-6=0$, which is the same as (9).]

6. Show that the circles

$$\mathbf{x^2+y^2+z^2-2x+3y-z-2=0,\ 2x-3y+z-7=0\ ;}$$

$$\mathbf{x^2+y^2+z^2-2x-4y+6z+1=0,\ x+2y-3z-5=0,}$$

lie on the same sphere and find its equation. [*D. U. H.*]

SECTION II

A SPHERE AND A LINE

68. Intersections of a straight line and a sphere. *To find the points where the line* $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ *meets the sphere*

$$x^2+y^2+z^2=a^2.$$

The equations of the line are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \ldots(1)$$

and the equation of the sphere is

$$x^2+y^2+z^2=a^2 \quad \ldots(2)$$

Any pt. on the line (1) is

$$(x_1+lr,\ y_1+mr,\ z_1+nr) \quad \ldots(3)$$

If it lies on the sphere (2), then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2=a^2$$

or $r^2(l^2+m^2+n^2)+2r(lx_1+my_1+nz_1)+(x_1^2+y_1^2+z_1^2-a^2)=0$

which is a quadratic in r giving two values of r.

Substituting these values of r one by one in (3), we get the two pts. of intersection.

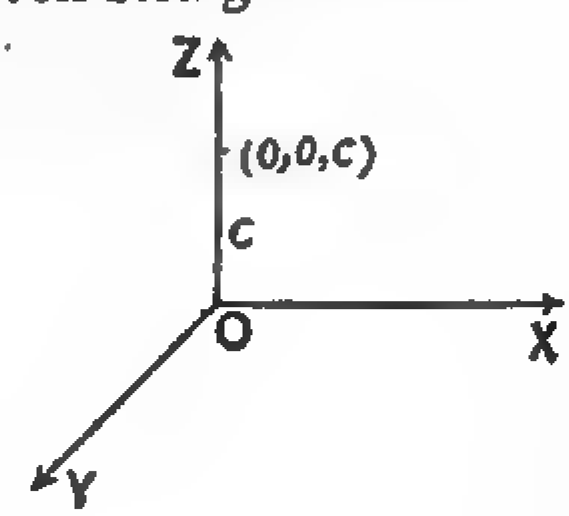
EXAMPLES

1. Find the co-ordinates of the points in which the line

$$\mathbf{\frac{x+2}{4}=\frac{y+9}{3}=\frac{z-8}{-5}}$$

meets the sphere $\mathbf{x^2+y^2+z^2=49}$. [*B. U.*]

2. Find the locus of the centres of spheres which pass through a given point and intercept a fixed length on a given straight line.



[**Note. For problems relating to a given point and a given line, take the foot of the perpendicular from the given point on the given line as origin, the perpendicular as the z-axis, and the given line as the x-axis, so that the line through the origin perpendicular to the zx-plane is the y-axis.**

Then, if the perpendicular distance of the given point from the given line = c, the co-ordinates of the given point are (0, 0, c), and the equations of the given line (x-axis) are y = 0, z = 0.]

3. Prove that the sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular lines through a fixed point is constant. [*J. & K. U. 1954*]

69. Conicoid. Def. A **conicoid** is a surface whose equation is of the second degree in *x, y, z* or, more shortly, a **conicoid** is a surface of the second degree.

Tangent plane. Def. The locus of the tangent lines at a point P of a sphere (or conicoid) is a plane called the **tangent plane** to the sphere (or conicoid) at P.

70. (*a*) *Equation of the tangent plane to a sphere (equation in the standard form).* **To find the equation of the tangent plane at any point (x_1, y_1, z_1) of the sphere $x^2+y^2+z^2=a^2$.**

The equation of the sphere is

$$x^2+y^2+z^2=a^2 \quad ...(1)$$

The equations of *any* line thro' (x_1, y_1, z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(2)$$

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the sphere (1), then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2=a^2$$

or $\quad r^2(l^2+m^2+n^2)+2r(lx_1+my_1+nz_1)+(x_1^2+y_1^2+z_1^2-a^2)=0 \quad ...(3)$

which is a quadratic in *r*.

$\because$ (x_1, y_1, z_1) lies on the sphere (1)

$\therefore$ $$x_1^2+y_1^2+z_1^2=a^2 \quad ...(4)$$

$\therefore$ one root of the quadratic (3) is zero.

If the line *touches* the sphere, the other root is also zero.

$\therefore$ coeff. of $r=0$,

i.e., $\quad lx_1+my_1+nz_1=0 \quad ...(5)$

Eliminating *l, m, n* from (2) and (5) [by substituting their values from (2) in (5)], the locus of the tangent lines is

$$(x-x_1)x_1+(y-y_1)y_1+(z-z_1)z_1=0$$

or $$xx_1+yy_1+zz_1-(x_1^2+y_1^2+z_1^2)=0 \quad ...(6)$$

Adding (4) and (6),

$$\mathbf{xx_1+yy_1+zz_1=a^2},$$

which is the required equation of the tangent plane.

Cor. 1. *A tangent line at any point P of a sphere is perpendicular to the radius through P.*

Let P be the pt. (x_1, y_1, z_1).

The direction-cosines of the line (2) are l, m, n, and those of the radius thro' P, *i.e.*, join of (0, 0, 0) and (x_1, y_1, z_1) are proportional to x_1-0, y_1-0, z_1-0 [$x_2-x_1, y_2-y_1, z_2-z_1$ (Art. 12)]

i.e., proportional to x_1, y_1, z_1

$\therefore$ from (5), the tangent line at P is $\perp$ to the radius thro' P.

[(Art. 13, (*b*), Cor. 3) $\because$ $aa'+bb'+cc'=0$]

Cor. 2. *The tangent plane at any point P of a sphere is perpendicular to the radius through P.*

Let P be the pt. (x_1, y_1, z_1).

The equation of the tangent plane at P is

$$xx_1+yy_1+zz_1 = a^2.$$

$\therefore$ the direction-cosines of the normal to the plane are proportional to x_1, y_1, z_1.

Also the direction-cosines of the radius thro' P, *i.e.*, join of (0, 0, 0) and (x_1, y_1, z_1) are proportional to x_1-0, y_1-0, z_1-0

[$x_2-x_1, y_2-y_1, z_2-z_1$ (Art. 12)]

i.e., proportional to x_1, y_1, z_1

$\therefore$ the normal is $\parallel$ to the radius

$\therefore$ the tangent plane at P is $\perp$ to the radius thro' P.

Note 1. The *geometrical* properties of a sphere (or surface) do not depend on the *form* of its equation, therefore we take the *simplest* form of the equation (as here) to prove these properties. (See Cors. 1 and 2.)

Note 2. Important. *Tangent plane property.* **If a plane touches a sphere,**

the perpendicular distance of the centre from the plane

= the radius. [From Cor. 2]

Tangent line property. **If a line touches a sphere,**

the perpendicular distance of the centre from the line

= the radius. [From Cor. 1]

EXAMPLE

Find the equation of the tangent plane at (α, β, γ) to the sphere

$$\mathbf{x^2+y^2+z^2 = a^2.}$$ [*P(P). U. 1958*]

70. (*b*) *Equation of the tangent plane to a sphere (equation in the general form).* **To find the equation of the tangent plane at any point (x_1, y_1, z_1) of the sphere**

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz + d = 0.}$$

The equation of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \dots (1)$$

The equations of *any* line thro' (x_1, y_1, z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \dots (2)$$

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the sphere (1), then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2+2u(x_1+lr)+2v(y_1+mr)+2w(z_1+nr)+d=0$$

or $\quad r^2(l^2+m^2+n^2)+2r\,[l(x_1+u)+m(y_1+v)+n(z_1+w)]$

$$+(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)=0 \quad \dots (3)$$

which is a quadratic in r.

$\because$ (x_1, y_1, z_1) lies on the sphere (1)

$\therefore$ $$x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0 \quad \dots (4)$$

$\therefore$ one root of the quadratic (3) is zero.

If the line *touches* the sphere, the other root is also zero

$\therefore$ coeff. of $r=0$,

i.e., $\quad l(x_1+u)+m(y_1+v)+n(z_1+w)=0 \quad \dots (5)$

Eliminating l, m, n from (2) and (5) [by substituting their values from (2) in (5)], the locus of the tangent lines is

$$(x-x_1)(x_1+u)+(y-y_1)(y_1+v)+(z-z_1)(z_1+w)=0$$

or $xx_1+yy_1+zz_1+ux+vy+wz-(x_1^2+y_1^2+z_1^2+ux_1+vy_1+wz_1)=0 \dots (6)$

Adding (4) and (6),

$$\mathbf{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0},$$

which is the required equation of the tangent plane.

[**Rule to write down the equation of the tangent plane at the point (x_1, y_1, z_1) of a sphere (or conicoid whose equation contains no* product terms yz, zx, xy) :**

In the equation of the surface, change $\mathbf{x^2}$ *to* $\mathbf{xx_1}$, $\mathbf{y^2}$ *to* $\mathbf{yy_1}$, $\mathbf{z^2}$ *to* $\mathbf{zz_1}$, $\mathbf{x}$ *to* $\frac{1}{2}(\mathbf{x+x_1})$, $\mathbf{y}$ *to* $\frac{1}{2}(\mathbf{y+y_1})$, $\mathbf{z}$ *to* $\frac{1}{2}(\mathbf{z+z_1})$. *The resulting equation is the required equation.*

Note. Important. *If the numerical values of* x_1, y_1, z_1 *are given, substitute them in the above equation.*]

EXAMPLES

1. Find the equation to the tangent plane at (x′, y′, z′) to the sphere $\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$. [*Pesh. U. 1955*]

2. *Find the equation of the tangent plane*

(i) *at* $(-1, 2, 2)$ *to the sphere* $x^2+y^2+z^2=9$,

(ii) *at the origin to the sphere* $x^2+y^2+z^2+2ux+2vy+2wz=0$.

* *If, however, the equation contains the product terms* yz, zx, xy, *change* yz *to* $\frac{1}{2}(yz_1+y_1z)$, zx *to* $\frac{1}{2}(zx_1+z_1x)$, xy *to* $\frac{1}{2}(xy_1+x_1y)$. (See Art. 95.)

(*i*) The equation of the sphere is $x^2+y^2+z^2=9$.

$\therefore$ the equation of the tangent plane at $(-1, 2, 2)$ is

[Rule (Art. 70, (*b*)), here $x_1=-1, y_1=2, z_1=2$]

$$x(-1)+y(2)+z(2)=9^*, \text{ or } x-2y-2z+9=0.$$

(*ii*) The equation of the sphere is $x^2+y^2+z^2+2ux+2vy+2wz=0$.

$\therefore$ the equation of the tangent plane at (0, 0, 0) is

[Rule (Art. 70, (*b*)), here $x_1=0, y_1=0, z_1=0$]

$$x(0)+y(0)+z(0)+2u.\tfrac{1}{2}(x+0)+2v.\tfrac{1}{2}(y+0)+2w.\tfrac{1}{2}(z+0)=0\dagger$$

or $$ux+vy+wz=0.$$

[**Check.** The equation of the tangent plane is satisfied by the co-ordinates of the given pt., thus

in (*i*), $\quad -1-2(2)-2(2)+9=0$, or $0=0$,

and in (*ii*), $\quad u(0)+v(0)+w(0)=0$, or $0=0$.]

3. Find the equation of the tangent plane at

$$(a\cos\theta\sin\phi, a\sin\theta\sin\phi, a\cos\phi)$$

to the sphere $x^2+y^2+z^2=a^2$.

71. (*a*) *Condition of tangency of a plane and a sphere (equation in the standard form).* **To find the condition that the plane**

$$\mathbf{lx+my+nz=p}$$

should touch the sphere $\mathbf{x^2+y^2+z^2=a^2}$.

The equation of the plane is $lx+my+nz=p$ ...(1)

and that of the sphere is $x^2+y^2+z^2=a^2$ ...(2)

If the plane touches the sphere,

the $\perp$ distance of the centre (0, 0, 0) from the plane (1)=the radius

| $lx+my+nz-p=0$

$$\therefore \quad \pm\frac{-p}{\sqrt{l^2+m^2+n^2}}\dagger\dagger=a$$

[Complete $\perp$ distance formula (Art. 29, (*b*))]

or, squaring, $a^2(l^2+m^2+n^2)=p^2$,

which is the required condition.

71. (*b*) *Condition of tangency of a plane and a sphere (equation in the general form).* **To find the condition that the plane**

$$\mathbf{lx+my+nz=p}$$

should touch the sphere

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0.}$$

---

*Explanation. In the equation of the sphere $x^2+y^2+z^2=9$, change x^2 to xx_1 *i.e.*, $x(-1)$, y^2 to yy_1, *i.e.*, $y(2)$, z^2 to zz_1, *i.e.*, $z(2)$.

†Explanation. In the equation of the sphere $x^2+y^2+z^2+2ux+2vy+2wz=0$, change x^2 to xx_1, *i.e.*, $x(0)$, y^2 to yy_1, *i.e.*, $y(0)$, z^2 to zz_1, *i.e.*, $z(0)$, x to $\frac{1}{2}(x+x_1)$, *i.e.*, $\frac{1}{2}(x+0)$, y to $\frac{1}{2}(y+y_1)$, *i.e.*, $\frac{1}{2}(y+0)$, z to $\frac{1}{2}(z+z_1)$, *i.e.*, $\frac{1}{2}(z+0)$.

††See Note in Ex. 15, Art. 43, (*c*).

The equation of the plane is $lx+my+nz=p$ ...(1)
and that of the sphere is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(2)$$

If the plane touches the sphere,
the $\perp$ distance of the centre $(-u,-v,-w)$ from the plane (1) = the radius

$$| \; lx+my+nz-p=0$$

$$\therefore \quad \pm\frac{l(-u)+m(-v)+n(-w)-p}{\sqrt{l^2+m^2+n^2}^*}=\sqrt{u^2+v^2+w^2-d}$$

[Complete $\perp$ distance formula (Art. 29, (b))]

or, squaring, $(lu+mv+nw+p)^2=(l^2+m^2+n^2)(u^2+v^2+w^2-d)$,
which is the required condition.

Note. *For the tangent plane property we use the* **complete** *perpendicular distance formula for the plane.* [See Note 2 in Art. 29, (b)]

EXAMPLES

1. Find the equations to the spheres which pass through the circle $x^2+y^2+z^2=5$, $x+2y+3z=3$, and touch the plane

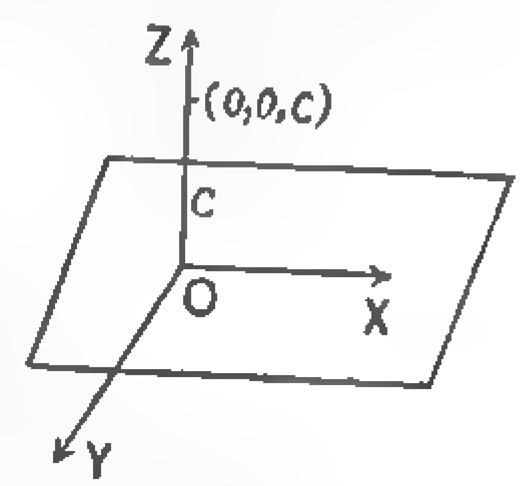
$$4x+3y=15.$$

[*P. U. 1957*]

2. Prove that the locus of the centres of spheres which pass through a given point and touch a given plane is a conicoid.

[*P(P). U. H. 1953*]

[Note. Important. For problems relating to a given point and a given plane, take the foot of the perpendicular from the given point on the given plane as origin, the perpendicular as the z-axis, and two perpendicular lines through the origin in the given plane as the x-, y-axes.



Then, if the perpendicular distance of the given point from the given plane = c, the co-ordinates of the given point are (0, 0, c), and the equation of the given plane (xy-plane) is $z=0$.]

3. *Sphere touching the co-ordinate planes.* **Prove that the equation to a sphere, which lies in the octant OXYZ and touches the co-ordinate planes, is of the form $x^2+y^2+z^2-2\lambda(x+y+z)+2\lambda^2=0$.**

Prove that in general two spheres can be drawn through a given point to touch the co-ordinate planes, and find for what positions of the point the spheres are (i) real, (ii) coincident.

[*P. U. 1944*]

4. Find the locus of the centres of spheres of constant radius which pass through a given point and touch a given line.

[*P(P). U. H. 1952*]

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*See Note in Ex. 15, Art. 43, (c).

5. Prove that the centres of spheres which touch the lines

$$y=mx,\ z=c\ ;\ y=-mx,\ z=-c,$$

lie upon the conicoid $mxy+cz(1+m^2)=0$. [*P. U. 1955*]

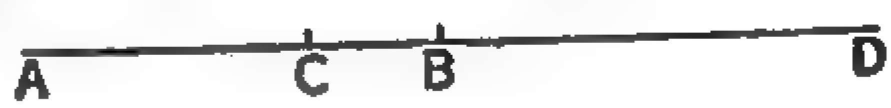
6. Find the equation of a plane which touches each of the circles

$$x=0,\ y^2+z^2=a^2\ ;\ y=0,\ z^2+x^2=b^2\ ;\ z=0,\ x^2+y^2=c^2.$$

How many such planes are there ?

Polar plane.

72. Harmonic division. Harmonic conjugates. Defs.



If a straight line AB is divided internally and externally in the same ratio at C and D, then

(*i*) AB is said to be **divided harmonically** at C, D ; and

(*ii*) C and D are called **harmonic conjugates** with respect to (w.r.t.) A and B ; also D is called the **harmonic conjugate** of C (and C the harmonic conjugate of D) w.r.t. A and B.

Cor. Symmetry of the harmonic relation. *If AB is divided harmonically at C, D, then CD is divided harmonically at A, B.*

Proof. (See Fig. of Art. 72.)

$\because$ AB is divided harmonically at C, D,

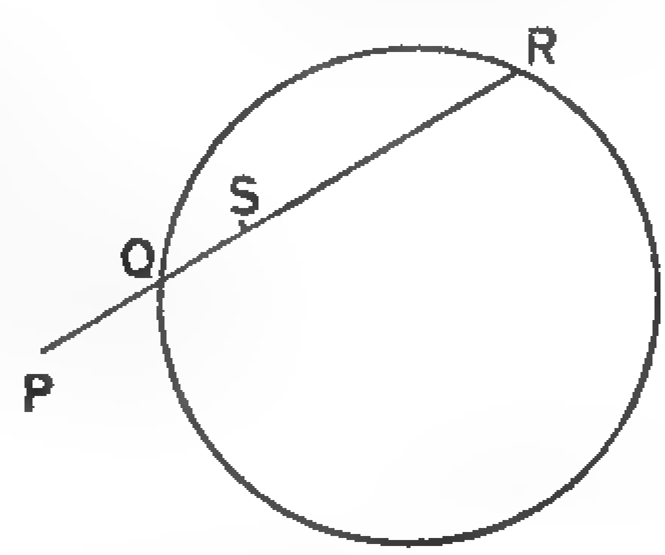
$$\therefore \quad \frac{AC}{CB}=\frac{AD}{BD} \qquad \text{[Def. (Art. 72)]}$$

or $$\frac{AC}{AD}=\frac{CB}{BD},$$

i.e., CD is divided externally and internally in the same ratio at A and B

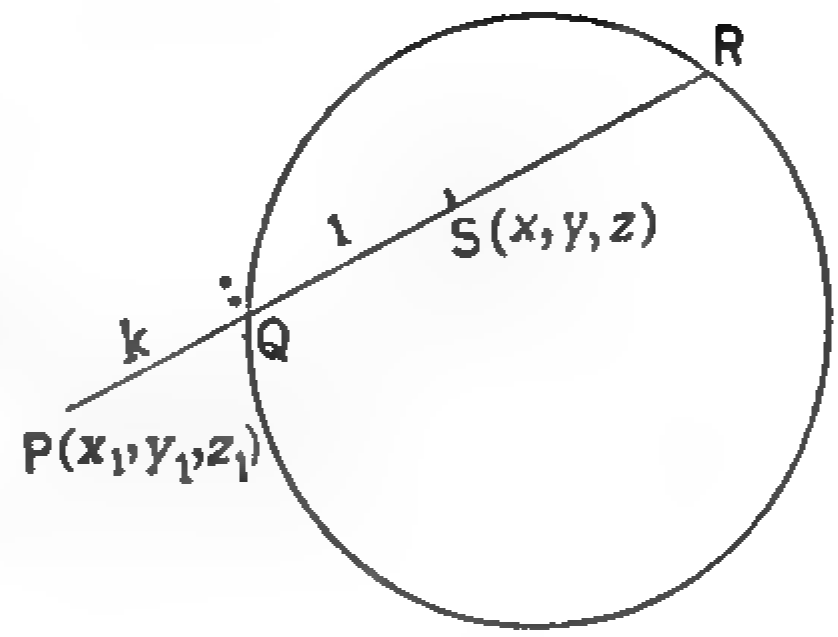
$\therefore$ CD is divided harmonically at A, B. [Def. (Art. 72)]

73. Pole and polar. Def. *If through a point P is drawn* **any** *chord QR of a sphere (or conicoid), and S is the harmonic conjugate of P w.r.t. Q and R, then the locus of S is a plane called the* **polar plane** *of P or, more shortly, the* **polar** *of P w.r.t. the sphere (or conicoid) ;*



P is called the **pole** of the locus of S.

74. Equation of the polar plane w.r.t. a sphere (equation in the standard form). *To find the equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the sphere $x^2+y^2+z^2=a^2$.*



The equation of the sphere is

$$x^2+y^2+z^2=a^2 \quad ...(1)$$

Let P be the pt. (x_1, y_1, z_1), QR *any* chord of the sphere thro' P, and S(x, y, z) the harmonic conjugate of P w.r.t. Q and R.

[To find the locus of S.]

The pt. which divides PS in the ratio $k : 1$ is

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right).$$

If it lies on the sphere (1), then

$$\left(\frac{kx+x_1}{k+1}\right)^2+\left(\frac{ky+y_1}{k+1}\right)^2+\left(\frac{kz+z_1}{k+1}\right)^2=a^2$$

or, multiplying thro' out by $(k+1)^2$,

$$(kx+x_1)^2+(ky+y_1)^2+(kz+z_1)^2=(k+1)^2a^2$$

or $\quad (kx+x_1)^2+(ky+y_1)^2+(kz+z_1)^2-(k+1)^2\,a^2=0$

or $\quad k^2(x^2+y^2+z^2-a^2)+2k(xx_1+yy_1+zz_1-a^2)$

$$+(x_1^2+y_1^2+z_1^2-a^2)=0 \quad ...(2)$$

which is a quadratic in k.

$\because$ PS is divided harmonically, *i.e.*, internally and externally in the same ratio at Q and R [Def. (Art. 72)], the quadratic (2) has equal and opposite roots,

$\therefore$ sum of the roots$=0$, $\quad\therefore$ coeff. of $k=0$,

i.e., $\qquad xx_1+yy_1+zz_1-a^2=0$

or $\qquad \mathbf{xx_1+yy_1+zz_1=a^2}$,

which is the required equation of the polar plane. [Def. (Art. 73)]

Aid to memory. The equation of the polar plane of (x_1, y_1, z_1) w. r. t. a sphere (or conicoid) is of the same *form* as the equation of the tangent plane at (x_1, y_1, z_1).]

Cor. *If a point P is on the sphere, the polar plane of P is the tangent plane at P.*

For the equation of the polar plane of P (Art. 74) is the same as that of the tangent plane at P (Art. 70, (a)).

Note 1. *The locus of points on a sphere the tangent planes at which pass through a point P is the polar plane of P w.r.t. the sphere.*

Proof. Let the equation of the sphere be $x^2+y^2+z^2=a^2$ ...(1) and P be the pt. (x_1, y_1, z_1).

Let (x', y', z') be any pt. on the sphere. The equation of the tangent plane at (x', y', z') to the sphere (1) is $xx'+yy'+zz'=a^2$.

If it passes thro' P, then $x_1x'+y_1y'+z_1z'=a^2$

$\therefore$ the locus of (x', y', z') is [changing (x', y', z') to (x, y, z)]

$$x_1x+y_1y+z_1z=a^2, \text{ or } xx_1+yy_1+zz_1=a^2,$$

which is the polar plane of P. [Art. 74]

Note 2. The property of the polar plane of a point w.r.t. a sphere proved in Note 1 is also true for the polar plane of a point w.r.t. a conicoid. This property is sometimes taken as the *definition* of the polar plane of a point w.r.t. a sphere (or conicoid).

EXAMPLES

1. *Equation of the polar plane w. r. t. a sphere (equation in the general form).* **Find the equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$.** [*P. U. 1954 S*]

[**N.B.** The equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$$\mathbf{xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0.}]$$

Aid to memory. See Aid to memory in Art. 74.]

2. *Reciprocal property.* **If the polar plane of a point P with respect to a sphere passes through a point Q, then the polar plane of Q passes through P.**

[**Note 1.** The reciprocal property of the polar plane w.r.t. a sphere given in Ex. 2 is also true for the polar plane w. r. t. a conicoid.

Note 2. Conjugate points. Def. Two points, which are such that the polar plane of each w. r. t. a sphere (or conicoid) passes through the other, are called **conjugate points** w.r.t. the sphere (or conicoid).

(See Note 1 in Art. 70, (*a*).) Let the equation of the sphere be $x^2+y^2+z^2=a^2$, and P, Q the pts. (x_1, y_1, z_1), (x_2, y_2, z_2).]

3. **Prove that the distances of two points from the centre of a sphere are proportional to the distances of each from the polar of the other.**

SECTION III

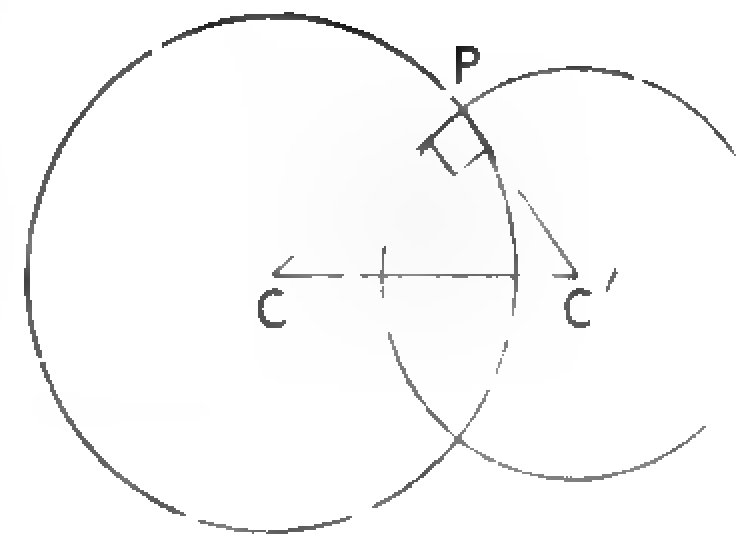
TWO OR MORE SPHERES

Orthogonal spheres.

75. **Orthogonal spheres. Def.** *Two spheres are said to be* **orthogonal** *or to* **cut orthogonally** *if the tangent planes at a point of intersection are at right angles.*

Cor. If two spheres cut orthogonally,
the square of the distance between their centres
= the sum of the squares of their radii.

Proof. Let C, C′ be the centres, and P a pt. of intersection of the spheres which cut orthogonally, so that the tangent planes at P are at rt. ∠s. [Def. (Art. 75)]



Join CC′, CP, C′P.

Then the radii CP, C′P, being ⊥ to the tangent planes at P, are also at rt. ∠s,

∴ $CC'^2 = CP^2 + C'P^2$.

Note. The converse of the above Cor. is also true.

For the order of the steps in the proof of the Cor. can be reversed.

76. *Condition of orthogonality of two spheres.* **To find the condition that the spheres**

$$\mathbf{x^2 + y^2 + z^2 + 2u\,x + 2v\,y + 2w\,z + d = 0,}$$
$$\mathbf{x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0,}$$

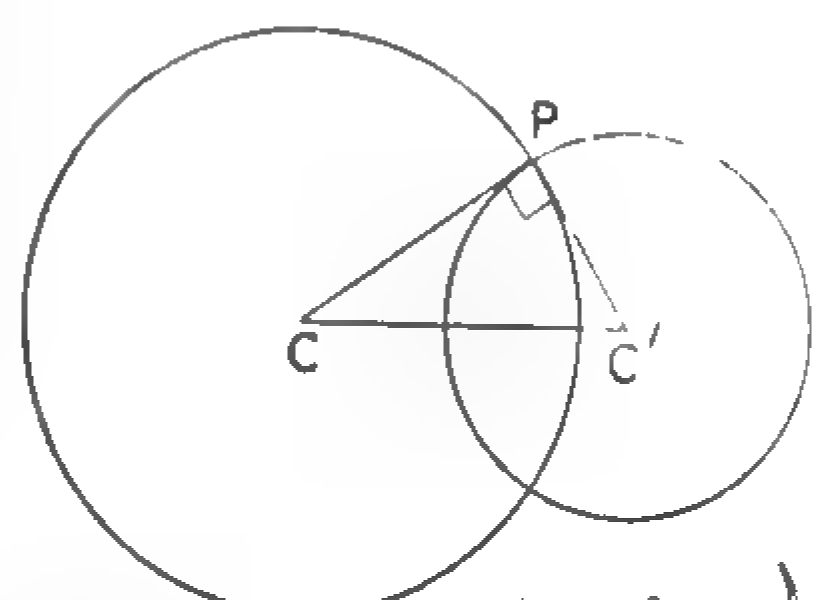
may cut orthogonally.

The equations of the spheres are

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad \dots (1)$$
$$x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0 \quad \dots (2)$$

If the spheres cut orthogonally, the square of the distance between their centres

= the sum of the squares of their radii ... (3)



Now the centres of the spheres (1) and (2) are

$(-u, -v, -w)$, $(-u', -v', -w')$,

and their radii are

$$\sqrt{u^2+v^2+w^2-d}, \quad \sqrt{u'^2+v'^2+w'^2-d'}$$

∴ from (3),

$$(u-u')^2+(v-v')^2+(w-w')^2 = u^2+v^2+w^2-d+u'^2+v'^2+w'^2-d'$$

or $u^2-2uu'+u'^2+v^2-2vv'+v'^2+w^2-2ww'+w'^2$

$$= u^2+v^2+w^2-d+u'^2+v'^2+w'^2-d'$$

[Cancel common terms on both sides]

or $$-2uu'-2vv'-2ww' = -d-d'$$

or $$\mathbf{2uu'+2vv'+2ww' = d+d',}$$

which is the required condition.

EXAMPLES

1. (*a*) **Prove that the tangent planes to the spheres**

$$\mathbf{x^2+y^2+z^2+2u\,x+2v\,y+2w\,z+d=0,}$$
$$\mathbf{x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0}$$

at any common point are at right angles if

$$\mathbf{2uu_1+2vv_1+2ww_1=d+d_1.} \qquad [\textit{P. U. 1958}]$$

(*b*) Show that the two spheres $x^2+y^2+z^2+6y+2z+8=0$ and $x^2+y^2+z^2+6x+8y+4z+20=0$ are orthogonal. [*P. U. 1957*]

2. Show that the spheres whose equations are

$$x^2+y^2+z^2+2Ax+2By+2Cz+D=0,$$

and
$$x^2+y^2+z^2+2ax+2by+2cz+d=0,$$

cut one another at right angles, if

$$2Aa+2Bb+2Cc-D-d=0.$$

Find the equation of a sphere which cuts four given spheres orthogonally. [*P. U. H.*]

3. **Find the equation of the sphere which touches the plane

$$\mathbf{3x+2y-z+2=0}$$

at the point P(1, −2, 1) and also cuts orthogonally the sphere

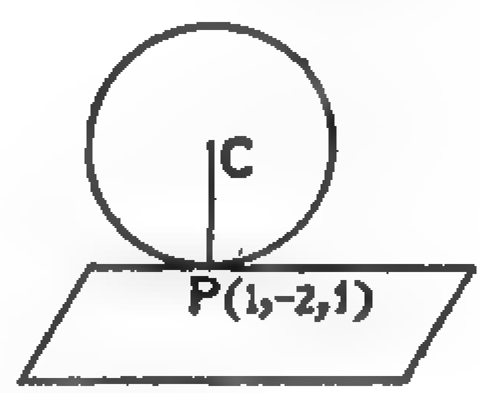
$$\mathbf{x^2+y^2+z^2-4x+6y+4=0.} \qquad [\textit{P. U. 1959}]$$

[The equation of the plane is

$$3x+2y-z+2=0 \quad ...(1)$$

and that of the sphere is

$$x^2+y^2+z^2-4x+6y+4=0 \quad ...(2)$$



∵ the sphere (whose equation is required) touches the plane at P(1, −2, 1),

∴ its centre lies on the normal to the plane thro' P.

Now the equations of the normal to the plane (1), thro' P are

$$\frac{x-1}{3}=\frac{y+2}{2}=\frac{z-1}{-1}.$$

Let the centre C of the sphere on this normal be

$$(1+3r,\ -2+2r,\ 1-r).$$

Then radius of the sphere = PC (which will be found) $=\sqrt{14}\,r$.

∵ this sphere cuts the sphere (2) orthogonally,

∴ the square of the distance between their centres

= the sum of the squares of their radii.]

4. **Prove that a sphere which cuts the two spheres S = 0 and S′ = 0 at right angles, will cut $l\mathrm{S}+m\mathrm{S}'=0$ at right angles.**

[*P. U. 1946*]

[**Abridged notation.** The equation of the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

is, for shortness, denoted by S = 0.

Similarly the equation of the sphere

$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0$$

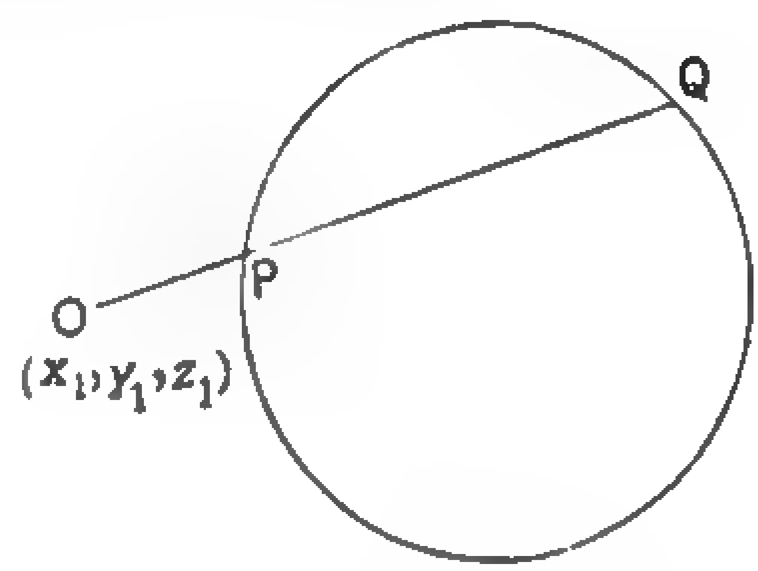
is denoted by $S'=0$.]

Radical plane of two spheres.

77. *Power of a point w.r.t. a sphere.* **If any secant through a given point O meets a given sphere in P and Q, OP. OQ is constant.**

Let O be the pt. (x_1, y_1, z_1), and the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \ldots(1)$$



Let the equations of any secant thro' O be $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$,

where l, m, n are the actual direction-cosines.

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the sphere (1), then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2$$
$$+2u(x_1+lr)+2v(y_1+mr)+2w(z_1+nr)+d=0$$

or $\quad r^2(l^2+m^2+n^2)+2r\,[\,l(x_1+u)+m(y_1+v)+n(z_1+w)\,]$

$$+(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)=0,$$

or $\quad r^2+2r\,[\,l\,(x_1+u)+m(y_1+v)+n(z_1+w)\,]$

$$+(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)=0$$

[$\because l^2+m^2+n^2=1$]

which is a quadratic in r, whose roots are OP, OQ.

$\therefore$ OP.OQ$=x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d \quad \ldots(2)$

[Product of the roots]

which is independent of l, m, n, and $\therefore$ constant.

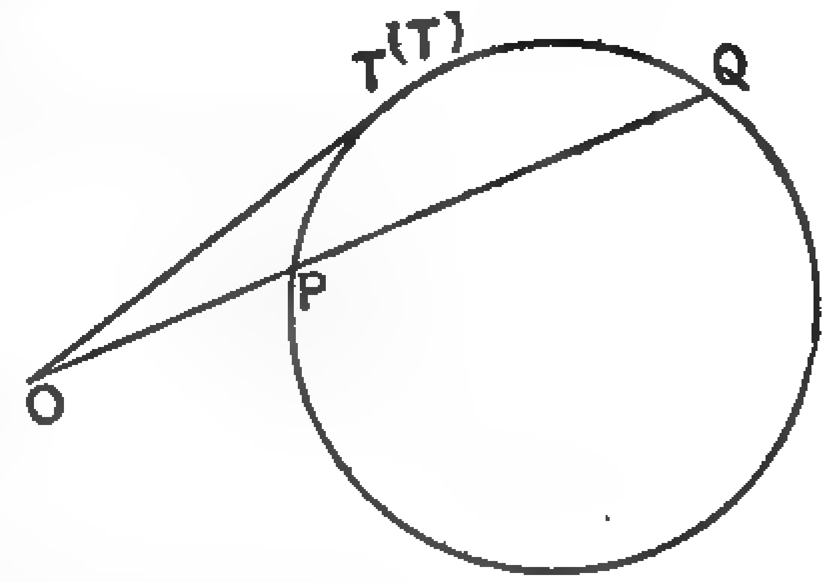
Note 1. Power of a point w. r. t. a sphere. Def. This constant, OP.OQ is called the **power** of O w.r.t. the sphere.

Note 2. Formula for the power. The power of the point (x_1, y_1, z_1) w.r.t. the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is

$$\mathbf{x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d}.$$

Cor. Length of the tangent. *To find the length of the tangent from the point* (x_1, y_1, z_1) *to the sphere*

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$



Let O be the pt. (x_1, y_1, z_1), and OT a tangent from O to the sphere.

Thro' O draw a chord PQ of the sphere.

Then OP . OQ

$=x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d.$ [Art. 77]

$\therefore$ when P, Q coincide at T,

OT.OT, *i.e.*, $OT^2=x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d$

or $\quad OT=\sqrt{\mathbf{x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d}}.$

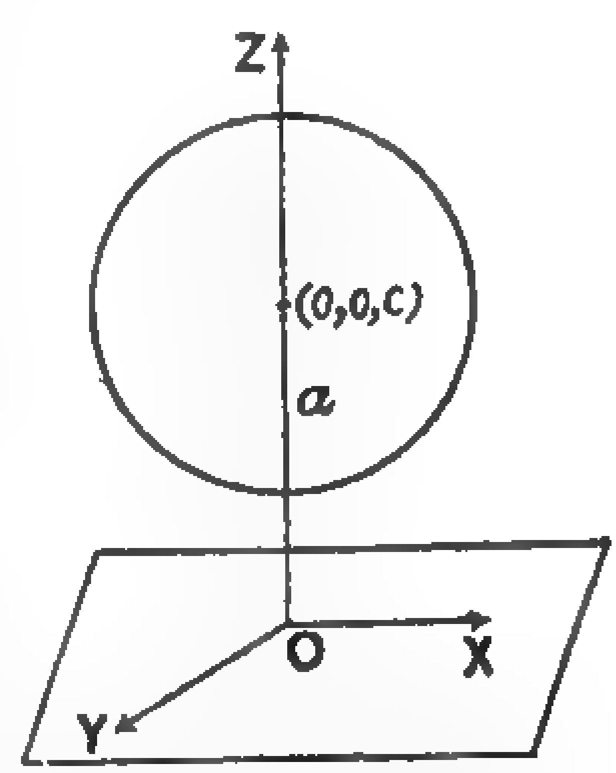
[**Rule to find the power of a point w. r. t. a sphere or the square of the tangent from a point to a sphere :**

(*i*) *Write the equation of the sphere so that the coefficients of* x^2, y^2, z^2 *are each=1 on the L.H.S. (by dividing thro' out by the coefficient of* x^2*, if necessary), R.H.S. being zero.*

(*ii*) *In the L.H.S. substitute the co-ordinates of the point. The result is the power of the point or the square of the tangent from the point.*]

EXAMPLE

**With any point P of a given plane as centre and the tangent from P to a given sphere as radius, a sphere is described. Prove that all such spheres pass through two fixed points.



[**Note.** For problems relating to a given plane and a given sphere, take the foot of the perpendicular from the centre of the given sphere on the given plane as origin, the perpendicular as the z-axis, and two perpendicular lines through the origin in the given plane as the x-, and y-axes.

Then, if the perpendicular distance of the centre of the given sphere from the given plane is c, the co-ordinates of the centre of the sphere are $(0, 0, c)$, let the radius be $=a$, so that the equation of the given sphere is $x^2+y^2+(z-c)^2=a^2$ ...(1) and the equation of the given plane (xy-plane) is $z=0$.

Let *any* pt. P of the given plane $(z=0)$ be $(x_1, y_1, 0)$.

It will be found that the equation of the sphere whose centre is P, and

radius=the tangent from P to the sphere (1),

is $\quad x^2+y^2+z^2-2xx_1-2yy_1+a^2-c^2=0,$

which, for *all* values of x_1 and y_1, passes thro' the pts. given by [*equating to zero the coeffs. of* x_1 *and* y_1] [*Note this step*]

$$x=0, y=0,$$
$$z^2+a^2-c^2=0, \text{ or } z=\pm\sqrt{c^2-a^2},$$

i.e., thro' the two fixed pts. $(0, 0, \pm\sqrt{c^2-a^2})$.]

78. Radical plane. Def. *The locus of a point, whose powers w.r.t. two spheres are equal, is a plane called the* **radical plane** *of the two spheres.*

Note. Since the power of an (external) point w.r.t. a sphere is equal to the square of the tangent from the point to the sphere [Art. 77, Cor.]

we have another definition.

Def. Radical plane. *The locus of a point, the tangents from which to two spheres are equal, is a plane called the* **radical plane** *of the two spheres.*

79. *Equation of the radical plane of two spheres.* **To find the equation of the radical plane of the spheres**

$$\mathbf{x^2+y^2+z^2+2ux+2vy+2wz+d=0,}$$
$$\mathbf{x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0.}$$

The equations of the spheres are

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \ldots(1)$$
$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0 \quad \ldots(2)$$

Let (x, y, z) be *any* pt. on the radical plane.

Then the power of (x, y, z) w.r.t. the sphere (1)

$=$ the power of (x, y, z) w.r.t. the sphere (2) [Def. (Art. 78)]

$$\therefore\ x^2+y^2+z^2+2ux+2vy+2wz+d$$
$$=x^2+y^2+z^2+2u'x+2v'y+2w'z+d' \quad \ldots(3) \text{ [Art. 77]}$$

or $\mathbf{2x(u-u')+2y(v-v')+2z(w-w')+d-d'=0}$ ...(4)

which is the required equation.

Abridged notation.

If $S=x^2+y^2+z^2+2ux+2vy+2wz+d,$

$S'=x^2+y^2+z^2+2u'x+2v'y+2w'z+d',$

the equation of the radical plane of the spheres $S=0$, $S'=0$, *is* $\mathbf{S-S'=0}$.

[**Rule to find the equation of the radical plane of two spheres :**

(*i*) *Write the equation of each sphere so that the coefficients of* x^2, y^2, z^2 *are each=1 on the L.H.S. (by dividing thro' out by the coefficient of* x^2, *if necessary), R.H.S. being zero.*

(*ii*) *Subtract one equation from the other. The resulting equation is the required equation.*]

Cor. 1. The radical plane of two spheres is at right angles to the line joining the centres.

Proof. The direction-cosines of the normal to the radical plane (4), (Art. 79), are proportional to $2(u-u')$, $2(v-v')$, $2(w-w')$.

[Coeffs. of x, y, z (Art. 24, Cor.)]

i.e., proportional to $u-u'$, $v-v'$, $w-w'$.

Also the direction-cosines of the join of the centres $(-u, -v, -w)$ and $(-u', -v', -w')$ are proportional to $u-u'$, $v-v'$, $w-w'$.

[x_2-x_1, y_2-y_1, z_2-z_1 (Art. 12)]

$\therefore$ the normal is $\parallel$ to the join of the centres

$\therefore$ the radical plane is $\perp$ to the join of the centres.

Cor. 2. The radical plane of two spheres passes through their points of intersection.

Proof. Let the equations of the spheres be

$$S=0 \quad ...(1)$$
$$S'=0 \quad ...(2)$$

where $S=x^2+y^2+z^2+2ux+2vy+2wz+d$,

$S'=x^2+y^2+z^2+2u'x+2v'y+2w'z+d'$.

Then the equation of their radical plane is

$$S-S'=0 \quad ...(3)$$ [Rule (Art. 79)]

The co-ordinates of the pts. which satisfy both (1) and (2) also satisfy (3) [$\because$ substituting from (1) and (2) in (3), we get $0-0=0$, or $0=0$, which is true]

$\therefore$ the pts. of intersection of the spheres (1) and (2) lie on the radical plane (3),

i.e., the radical plane of the two spheres passes thro' their pts. of intersection.

Note. The property of the radical plane of two spheres proved in the above Cor. is *important*. It is sometimes taken as the definition.

Def. Radical plane. The plane through the points of intersection of two spheres is called the **radical plane** of the two spheres.

80. *Radical line of three spheres.* **The radical planes of three spheres taken two by two pass through one line.**

Proof. Let the equations of the spheres be

$$S_1=0, \; S_2=0, \; S_3=0,$$

where $S_1=x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1$, and so for S_2, S_3.

Then the equations of the radical planes of the spheres taken two by two are $S_1-S_2=0$, $S_2-S_3=0$, $S_3-S_1=0$, [Rule (Art. 79)]

which pass thro' the line $S_1=S_2=S_3$.

Note. Radical line. Def. The common line of intersection of the radical planes of three spheres taken two by two is called the

radical line of the three spheres.

81. *Radical centre of four spheres.* **The radical planes of four spheres taken two by two pass through one point.**

Proof. Let the equations of the spheres be

$$S_1=0,\ S_2=0,\ S_3=0,\ S_4=0,$$

where $S_1=x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1$, and so for S_2, S_3, S_4.

Then the equations of the radical planes of the spheres taken two by two are

$S_1-S_2=0,\ S_1-S_3=0,\ S_1-S_4=0\ ;\ S_2-S_3=0,\ S_2-S_4=0\ ;\ S_3-S_4=0,$*
which pass thro' the pt. $S_1=S_2=S_3=S_4$.

Note 1. Radical centre. Def. The common point of intersection of the radical planes of four spheres taken two by two is called the **radical centre** of the four spheres.

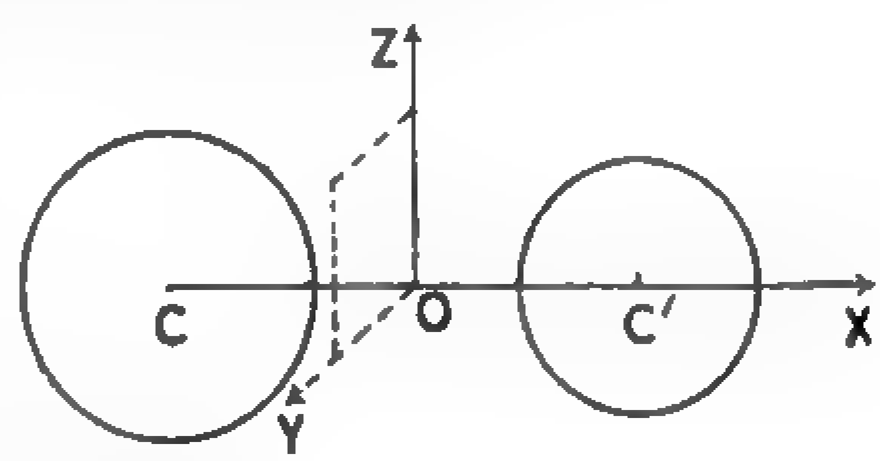
[**Note 2.** Thus we have the radical *plane* of *two* spheres, the radical *line* of *three* spheres, and the radical *centre* of *four* spheres.]

82. *Equations of two spheres in the simplest form.* **To prove that the equations of any two spheres can be put in the form**

$$\mathbf{x^2+y^2+z^2+2\lambda_1 x+d=0,\ x^2+y^2+z^2+2\lambda_2 x+d=0.}$$

Take (i) the join of the centres of the two spheres as the x-axis, and

(ii) their radical plane as the yz-plane.



Let the equations of the spheres be

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \quad ...(1)$$

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \quad ...(2)$$

(*i*) $\because$ their centres $(-u_1, -v_1, -w_1)$, $(-u_2, -v_2, -w_2)$ lie on the x-axis, *i.e.*, $y=0$, $z=0$

$\therefore\ -v_1=0,\ -w_1=0\ ;\ -v_2=0,\ -w_2=0$

or $\quad v_1=0,\quad w_1=0\ ;\quad v_2=0,\quad w_2=0.$

$\therefore$ from (1) and (2), the equations of the spheres become

$$x^2+y^2+z^2+2u_1x+d_1=0 \quad ...(3)$$

$$x^2+y^2+z^2+2u_2x+d_2=0 \quad ...(4)$$

* **How to write this step.** Take the sphere (1) with the *following*, *i.e.*, with the spheres (2), (3), (4), thus getting $S_1-S_2=0$, $S_1-S_3=0$, $S_1-S_4=0$; now take the sphere (2) with the *following*, *i.e.*, with the spheres (3), (4), thus getting $S_2-S_3=0$, $S_2-S_4=0$; now take the sphere (3) with the *following*, *i.e.*, with the sphere (4), thus getting $S_3-S_4=0$.

(*ii*) Now the equation of their radical plane is [From (3) and (4)]

$$2(u_1-u_2)x+d_1-d_2=0 \quad \ldots (5) \quad \text{[Rule (Art. 79)]}$$

$\because$ this is the same as the yz-plane, *i.e.*, $x=0$ ... (6)

$\therefore$ comparing coeffs. in (5) and (6),

$$\frac{2(u_1-u_2)}{1}=\frac{d_1-d_2}{0}$$

or, cross-multiplying, $d_1-d_2=0$, $\therefore$ $d_1=d_2=d$ (say)

$\therefore$ from (3) and (4), the equations of the spheres become

$$x^2+y^2+z^2+2u_1x+d=0,\ x^2+y^2+z^2+2u_2x+d=0,$$

or $$x^2+y^2+z^2+2\lambda_1x+d=0,\ x^2+y^2+z^2+2\lambda_2x+d=0,$$

where $\lambda_1=u_1$, $\lambda_2=u_2$.

Note. Important. For problems relating to two given spheres, let the equations of the spheres be

$$\mathbf{x^2+y^2+z^2+2u_1x+d=0,\ x^2+y^2+z^2+2u_2x+d=0.}$$

EXAMPLE

**Show that the spheres which cut two given spheres along great circles all pass through two fixed points. [*P. U. 1954*]

[**Note. Great circle. Def.** The section of a sphere by a plane passing through its centre is called a **great circle.**]

Coaxal spheres.

83. **Coaxal spheres. Def.** *A system of spheres every two of which have the same radical plane is said to be* **coaxal.**

84. **Equation of a coaxal system.** *The equation*

$$\mathbf{x^2+y^2+z^2+2ux+d=0,}$$

where u is a parameter, represents a coaxal system of spheres.*

Proof. The equation is $x^2+y^2+z^2+2ux+d=0$ ... (1)

where u is a parameter.

It clearly represents a system of spheres. [$\because$ u varies]

Let the equations of two spheres of the system be

$$x^2+y^2+z^2+2u_1x+d=0,$$

$$x^2+y^2+z^2+2u_2x+d=0.$$

Then the equation of their radical plane is

$$2(u_1-u_2)x=0 \quad \text{[Rule (Art. 79)]}$$

or $x=0$, which is independent of u_1, u_2,

and $\therefore$ the same for every two spheres of the system

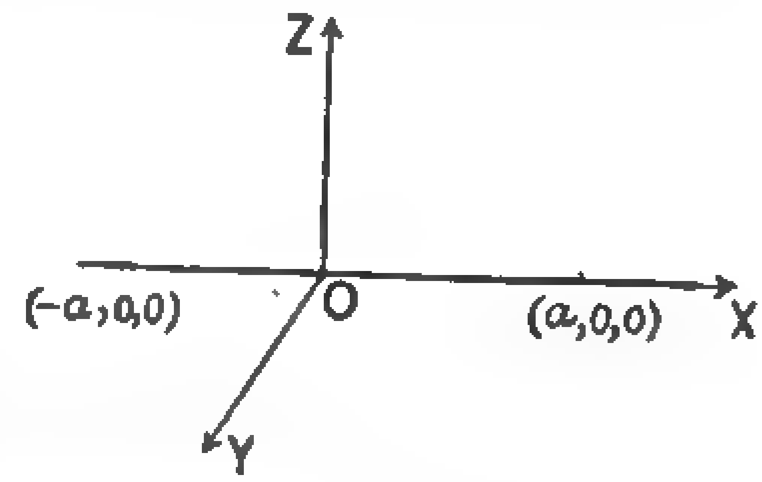
$\therefore$ (1) represents a coaxal system of spheres. [Def. (Art. 83)]

* **Parameter. Def.** A symbol which is constant for the same member of the system but different for different members is called a parameter.

EXAMPLES

1. A, B are two fixed points, and P moves so that PA = n PB ; show that the locus of P is a sphere. Show also that all such spheres, for different values of n, have a common radical plane. [*P. U.*]

[Note. Important. For problems relating to two given points, take the mid-point of their join as origin, the join as the x-axis, and two perpendicular lines through the origin in the plane through the origin and perpendicular to the x-axis as the y-, and z-axes.



Then, if the join of the two given points = 2a, the co-ordinates of the two given points are (−a, 0, 0) and (a, 0, 0).]

2. Prove that the locus of points whose powers with respect to two given spheres are in a constant ratio is a sphere coaxal with the two given spheres. [*P. U.*]

3. *Three types of coaxal systems. Intersecting, tangential and non-intersecting.* **Prove that the members of the coaxal system, $x^2+y^2+z^2+2ux+d=0$, intersect one another, touch one another, or do not intersect one another according as $d \lesseqgtr 0$.**

The equation of the coaxal system is $x^2+y^2+z^2+2ux+d=0$.
Let the equations of two members be

$$x^2+y^2+z^2+2u_1x+d=0 \quad \dots (1)$$

$$x^2+y^2+z^2+2u_2x+d=0 \quad \dots (2)$$

They meet where, subtracting (2) from (1),

$$2(u_1-u_2)x=0, \text{ or } x=0.$$

Substituting this value of x in (1),

$$y^2+z^2+d=0,$$

or

$$y^2+z^2=-d \quad \dots (3)$$

which is a circle *in the yz-plane* ;

$$\text{its radius} = \sqrt{-d} \quad \dots (4)$$

The spheres (1) and (2) intersect one another, touch one another, or do not intersect one another according as the circle (3) is a real circle, a pt. circle, or an imaginary circle,
i.e., according as its radius is real, 0, or imaginary,
i.e., from (4), according as d is −ve, 0 or +ve.

4. *Limiting points of a coaxal system.* **Prove that the centres of the two spheres of the coaxal system, $x^2+y^2+z^2+2ux+d=0$, which have zero radius are at the points $(\pm\sqrt{d}, 0, 0)$.**

The equation of *any* sphere of the system is

$$x^2+y^2+z^2+2ux+d=0 \quad \ldots (1)$$

If its radius $=0$, then $\sqrt{u^2-d}=0$, or $u^2-d=0$, or $u^2=d$

$$\therefore \quad u=\pm\sqrt{d} \quad \ldots (2)$$

Now the centre of the sphere (1) is $(-u, 0, 0)$,

i.e., from (2), $(\mp\sqrt{d}, 0, 0)$, or $(\pm\sqrt{d}, 0, 0)$.

Note. Limiting points of a coaxal system. Def. The centres of the two spheres of a coaxal system, which have zero radius, are called the **limiting points** of the system.

Thus the limiting points of the coaxal system

$$x^2+y^2+z^2+2ux+d=0$$

are $(\pm\sqrt{d}, 0, 0)$.

5. Show that the equation $x^2+y^2+z^2+2\mu y+2\nu z-d=0$, where μ and ν are parameters, represents a system of spheres passing through the limiting points of the system $x^2+y^2+z^2+2\lambda x+d=0$, and cutting every member of that system at right angles. [*P(P). U. 1952*]

MISCELLANEOUS EXAMPLES ON CHAPTER VII

1. **Touching spheres.** Show that the spheres

$$x^2+y^2+z^2=25, \quad x^2+y^2+z^2-18x-24y-40z+225=0$$

touch and find the co-ordinates of their common point. [*P.U. 1959S*]

2. **Touching spheres.** Show that the spheres

$$x^2+y^2+z^2=64, \text{ and } x^2+y^2+z^2-12x+4y-6z+48=0$$

touch internally and find the point of contact. [*P.U. 1953*]

3. **A sphere of radius R passes through the origin. Show that the ends of the diameter which is parallel to the x-axis lie on each of the spheres $x^2+y^2+z^2\pm 2Rx=0$.** [*L. U. 1937*]

[It will be found that the equation of the sphere thro' the origin is $x^2+y^2+z^2+2ux+2vy+2wz=0$.

Its radius$=$R, $\therefore$ $u^2+v^2+w^2=R^2 \ldots (1)$.

The equations of the diameter, *i.e.*, line thro' the centre $(-u, -v, -w)$ of the sphere, $\parallel$ to the x-axis are $\dfrac{x+u}{1}=\dfrac{y+v}{0}=\dfrac{z+w}{0}$. Putting each member$=\pm$R, the co-ordinates of the extremities of the diameter are $\quad x=-u\pm R, \; y=-v, \; z=-w \ldots (2)$

Eliminate u, v, w from (1) and (2) [by substituting their values from (2) in (1)].]

4. Spheres are described to contain the circle $z=0, x^2+y^2=a^2$. Prove that the locus of the ends of their diameters which are parallel

to the x-axis is the rectangular hyperbola $y=0, x^2-z^2=a^2$. [*L. U.*]

**5. Find the equation of the sphere which passes through the points $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$ and has its radius as small as possible. [*L. U.*]

6. Find the centre of the circle which passes through the points $(-1, 0, 0)$, $(0, 2, 0)$, $(0, 0, 3)$. [*L. U.*]

**7. Find the conditions that the circles

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0, \quad lx+my+nz=p;$$

$$x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0, \quad l'x+m'y+n'z=p';$$

should lie on the same sphere. [*P. U.*]

8. Find the equation of a sphere described on the line joining $(2, -1, 4)$ and $(-2, 2, -2)$ as diameter, and find also the equation of the tangent plane at $(2, -1, 4)$ to the sphere. [*B. H. U. 1940*]

Find also the equations of the normal to the sphere at $(2, -1, 4)$. [*Poona U.*]

9. Find the equation of a sphere which touches the sphere

$$x^2+y^2+z^2+2x-6y+1=0$$

at the point $(1, 2, -2)$ and passes through the origin. [*P.U. 1950*]

[The required sphere passes thro' the (pt.) circle of intersection of the given sphere and its tangent plane at $(1, 2, -2)$, and also thro' the origin.]

10. A sphere is inscribed in the tetrahedron whose faces are

$$x=0, \; y=0, \; z=0, \; 2x+6y+3z=14.$$

Find its centre and radius and write down its equation. [*D. U.*]

11. Find the equations of the spheres which pass through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$ and touch the plane $z=0$. [*P. U. 1958*]

12. Find the equations of the spheres which pass through the circle $x^2+z^2-2x+2z=2$, $y=0$ and touch the plane $y-z=7$. [*L. U.*]

13. Find the equations of the two spheres whose centres lie in the positive octant and which touch the planes

$$x=0, \; y=0, \; z=0, \; 2x+2y+z=8a. \qquad [L. U.]$$

[Proceed as in Ex. 3, Art. 71, (*b*). It will be found that the centre of the sphere is (r, r, r), and radius$=r$. If it touches the plane $2x+2y+z=8a$, then ? (Use the tangent plane property.)]

14. *Sphere touching the co-ordinate planes.* **Find the equation of a sphere touching the three co-ordinate planes. How many such spheres can be drawn ?** [*P. U. 1946 S*]

15. Find the equations of the tangent planes to the sphere $x^2+y^2+z^2+2x-4y+6z-7=0$ which intersect in the line

$$6x-3y-23=0=3z+2. \qquad [\textit{D. U. H. 1954}]$$

16. Obtain the equations of the tangent planes to the sphere $x^2+y^2+z^2=9$ which can be drawn through the line

$$\frac{x-5}{2}=-\frac{y-1}{2}=\frac{z-1}{1}. \qquad [\textit{P. U. 1959}]$$

17. *Sphere touching the axes.* **Find the equation of the sphere which touches the three co-ordinate axes. How many spheres can be so drawn ?** [*P. U.*]

18. Find the equation of the sphere which has its centre at (a, b, c) and which touches the line $(x-f)/l=(y-g)/m=(z-h)/n$. [*L. U.*]

19. Spheres are drawn to pass through the points (2, 0, 0), (8, 0, 0), and to touch the axes of y and z. Find how many such spheres can be drawn and give their equations.

Show that the polar planes of the origin with respect to these spheres all pass through the same point and find the co-ordinates of this point. [*L. U.*]

20. *Find the locus of the points of contact of the tangent planes to $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ which pass through a given point (α, β, γ).* [*P. U. 1961*]

21. Find the equation of the sphere which cuts orthogonally each of the spheres $x^2+y^2+z^2=a^2+b^2+c^2$, $x^2+y^2+z^2+2ax=a^2$, $x^2+y^2+z^2+2by=b^2$, $x^2+y^2+z^2+2cz=c^2$. [*P.U.M.P. 1942*]

22. Show that all the spheres that can be drawn through the origin and each set of points where the planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere

$$x^2+y^2+z^2+2fx+2gy+2hz=0,$$

if $af+bg+ch=0$. [*Bar. U. 1954*]

23. Show that the locus of a point from which equal tangents may be drawn to the spheres

$$x^2+y^2+z^2=1,\ x^2+y^2+z^2+2x-2y+2z-1=0,$$
$$x^2+y^2+z^2-x+4y-6z-2=0$$

is the straight line $\frac{x-1}{2}=\frac{y-2}{5}=\frac{z-1}{3}$. [*L. U.*]

24. Show that the locus of the points from which the tangents to the three spheres

$(x-2)^2+y^2+z^2=1$, $x^2+(y-3)^2+z^2=6$, $(x+2)^2+(y+1)^2+(z-2)^2=6$

are all equal is the line $\dfrac{x}{3}=\dfrac{y}{2}=\dfrac{z}{7}$.

Find the co-ordinates of the point on this line from which the length of tangents to the three spheres is also equal to that of the tangent to the sphere $(2x+1)^2+4y^2+(2z-1)^2=6$. [*L. U.*]

25. Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $\dfrac{r_1 r_2}{\sqrt{r_1^2+r_2^2}}$. [*P. U. 1956*]

26. **Angle of intersection of two spheres.** At what angle does the sphere $x^2+y^2+z^2-2x-4y-6z+10=0$ intersect the sphere which has the points $(1, 2, -3)$ and $(5, 0, 1)$ as the extremities of a diameter? Find the equation to the sphere through the point $(0,0,0)$ coaxal with the above two spheres. [*P. U. 1948*]

[**Note 1. Angle of intersection of two spheres. Def.** *The* **angle of intersection of two spheres** *is the angle between their tangent planes at a point of intersection.*

Note 2. Equation of a sphere coaxal with two given spheres.

The equation of *any* sphere coaxal with the spheres $S=0$, $S'=0$, is $\mathbf{S+kS'=0}$, where k is any constant.

Note 3. The value of k is found from the second condition satisfied by the sphere.]

27. **Find the equation of the sphere having the circle**

$$x^2+y^2+z^2=9,\ x-2y+2z=5$$

for a great circle ; determine its centre and radius. [*B. U.*]

CHAPTER VIII
THE CONE
SECTION I
EQUATION OF A CONE

85. Cone. Def. *A* **cone** *is a surface generated by a straight line which passes through a fixed point, and satisfies some other condition, e.g., it may intersect a fixed curve.*

The straight line in any position is called a **generator,** *the fixed point the* **vertex,** *and the fixed curve is called the* **guiding curve** *of the cone.*

86. Homogeneous equation. Def. The equation $f(x, y, z)=0$ is said to be **homogeneous** if $f(rx, ry, rz)=0$, for *all* values of r.

Example. *The equation* $\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$ *is homogeneous.*

Proof. The equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \ldots(1)$$

Here $f(x, y, z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy$.

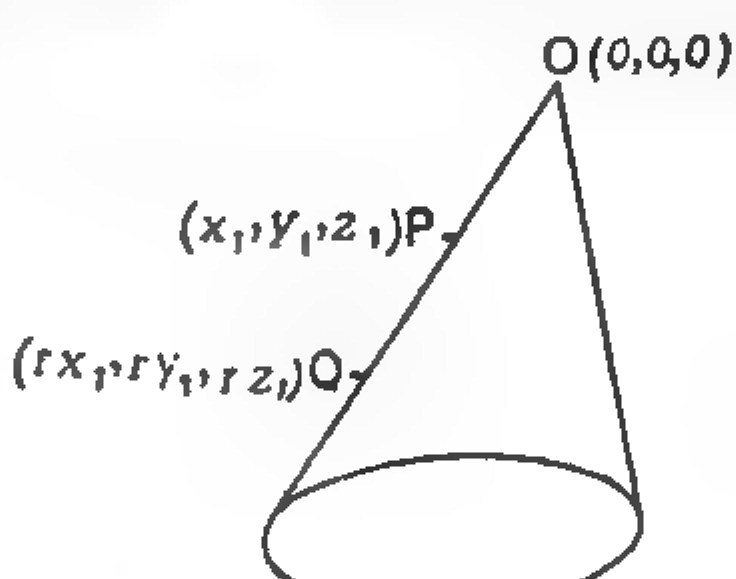
$\therefore$ [changing x, y, z to rx, ry, rz]

$$\begin{aligned} f(rx, ry, rz) &= a(rx)^2+b(ry)^2+c(rz)^2+2f(ry)(rz)+2g(rz)(rx)+2h(rx)(ry) \\ &= r^2(ax^2+by^2+cz^2+2fyz+2gzx+2hxy) \\ &= r^2(0) \quad [\text{ From (1) }] \\ &= 0, \text{ for } all \text{ values of } r. \end{aligned}$$

$\therefore$ the equation (1) is homogeneous. [Def. (Art. 86)]

Note. *An equation in x, y, z in which every term is of the same degree is homogeneous.*

87. *Equation of a cone whose vertex is the origin.* **The equation of a cone, whose vertex is the origin, is homogeneous in x, y, z.**



Let the equation of the cone, whose vertex is the origin, be

$$f(x, y, z)=0 \quad \ldots(1)$$

Let $P(x_1, y_1, z_1)$ be *any* pt. on the cone.

Then $f(x_1, y_1, z_1)=0 \quad \ldots(2)$

The equations of OP are

(1)	(2)
(0, 0, 0)	(x_1, y_1, z_1)

$$\frac{x-0}{x_1-0}=\frac{y-0}{y_1-0}=\frac{z-0}{z_1-0} \qquad \left[\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \text{ (Art. 40)}\right]$$

or $\dfrac{x}{x_1}=\dfrac{y}{y_1}=\dfrac{z}{z_1}$.

Any pt. on OP is $Q(rx_1, ry_1, rz_1)$.

$\because$ Q lies on the cone (1) [Def. (Art. 85)]

$\therefore$ $f(rx_1, ry_1, rz_1)=0$, for *all* values of r ...(3)

From (2) and (3), the equation $f(x_1, y_1, z_1)=0$ is homogeneous in x_1, y_1, z_1 [Def. (Art. 86)]

$\therefore$ $f(x, y, z)=0$ is homogeneous in x, y, z.

Cor. General equation of a quadric cone whose vertex is the origin. The general equation of a cone of the second degree, called a *quadric cone,* whose vertex is the origin, is

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.} \qquad \text{[Ex., Art. 86]}$$

Note. If in the equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, $abc+2fgh-af^2-bg^2-ch^2=0$, the cone *degenerates* into a pair of planes [Art. 35].

Converse of Art. 87. *Any homogeneous equation in* x, y, z *represents a cone whose vertex is the origin.*

Let the homogeneous equation be $f(x, y, z)=0$ ...(1)

(See Fig. of Art. 87.)

Let $P(x_1, y_1, z_1)$ be *any* pt. on the locus of (1).

Then $f(x_1, y_1, z_1)=0$ ...(2)

$\because$ the equation is homogeneous

$\therefore$ $f(rx_1, ry_1, rz_1)=0$...(3), for *all* values of r. [Def. (Art. 86)]

The equations of OP are　　(1) $(0, 0, 0)$　(2) (x_1, y_1, z_1)

$$\frac{x-0}{x_1-0}=\frac{y-0}{y_1-0}=\frac{z-0}{z_1-0} \qquad \left[\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \text{ (Art. 40)}\right]$$

or $\dfrac{x}{x_1}=\dfrac{y}{y_1}=\dfrac{z}{z_1}$.

Any pt. on OP is $Q(rx_1, ry_1, rz_1)$.

$\therefore$ from (3), *any* pt. Q on OP, and $\therefore$ OP itself lies on the locus

$\therefore$ the locus of (1) is a cone whose vertex is O. [Def. (Art. 85)]

EXAMPLES

1. *Equation of a cone whose vertex is the origin, and which passes through the curve of intersection of a plane and a surface.*

Find the equation to the cone whose vertex is the origin and which passes through the curve of intersection of the plane $\mathbf{lx+my+nz=p}$ and the surface $\mathbf{ax^2+by^2+cz^2=1}$. [*P.U. 1958 S*]

The equation of the plane is $lx+my+nz=p$,

or $\dfrac{lx+my+nz}{p}=1$ ...(1) [First degree terms in x, y, z on L. H. S. and 1 on R.H.S. (*Note this step*)]

and that of the surface is

$$ax^2+by^2+cz^2=(1)^2 \quad ...(2)$$

Making (2) homogeneous by means of (1), [by substituting from (1) in (2)],

$$ax^2+by^2+cz^2=\left(\frac{lx+my+nz}{p}\right)^2$$

or $p^2(ax^2+by^2+cz^2)=(lx+my+nz)^2$,

which is the required equation.

2. Find the equations to the cones with vertex at the origin which pass through the curves given by

(*i*) $ax^2+by^2=2z,\ lx+my+nz=p$;

(*ii*) $x^2+y^2+z^2+2ux+d=0,\ lx+my+nz=p.$ [*P(P). U.*]

3. **Find the equation to the cone whose vertex is the origin and base the circle $x=a,\ y^2+z^2=b^2$, and show that the section of the cone by a plane parallel to the plane XOY is a hyperbola.**

[*P. U. 1959 S*]

4. **Equation of a cone whose vertex is the origin and which passes through the curve of intersection of any two surfaces.** Find the equation of the cone whose vertex is the origin and which passes through the curve given by $ax^2+by^2+cz^2=1,\ \alpha x^2+\beta y^2=2z$.

[Rule to make one equation in x, y, z homogeneous by means of another :

(*i*) *Make each equation homogeneous in x, y, z, t by introducing proper powers of another variable t.*

(*ii*) *Eliminate t from the two resulting equations.*

The resulting equation is the required equation.

Note. The above Rule is always applicable. But in the particular case when one of the two surfaces is a plane, the method of Ex. 1 is easier.]

5. **The plane $x/a+y/b+z/c=1$ meets the co-ordinate axes in A, B, C. Prove that the equation to the cone generated by lines drawn from O to meet the circle ABC is**

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0. \qquad [P.\ U.\ 1957]$$

6. **A variable plane is parallel to the given plane**

$$x/a+y/b+z/c=0,$$

and meets the axes in A, B, C. Prove that the circle ABC lies

on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0. \quad [P.\,U.\ 1960]$$

7. A is a point on OX and B on OY so that the angle OAB is constant ($=\alpha$). On AB as diameter a circle is described whose plane is parallel to OZ. Prove that as AB varies the circle generates the cone $2xy-z^2 \sin 2\alpha=0$. [*Ag. U. 1946*]

8. Planes through OX and OY include an angle α. Show that their line of intersection lies on the cone

$$z^2(x^2+y^2+z^2)=x^2y^2 \tan^2 \alpha. \quad [\,P.\,U.\ 1955\,]$$

88. *If* $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ *is a generator of the cone represented by the homogeneous equation* $f(x, y, z)=0$, *then* $f(l, m, n)=0$.

[In words : **The direction-cosines of a generator of a cone satisfy the (homogeneous) equation of the cone.**]

The equations of the generator are $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ ...(1)

and the (homogeneous) equation of the cone is $f(x, y, z)=0$ ...(2)

Any pt. on the generator (1) is (lr, mr, nr).

$\because$ it lies on the cone (2),

$\therefore$ $f(lr, mr, nr)=0$, for *all* values of r.

But the equation is homogeneous,

$\therefore$ $f(l, m, n)=0$. [Def. (Art. 86)]

Converse of Art. 88. If the direction-cosines of a straight line, which passes through a fixed point, satisfy a homogeneous equation, the line is a generator of a cone whose vertex is the fixed point.

[****Proof.** Take the fixed pt. as origin.

Let l, m, n be the direction-cosines of the line, and let them satisfy the homogeneous equation $f(l, m, n)=0$ ...(1)

The equations of the line [thro' the origin and having direction-cosines l, m, n] are $\dfrac{x-0}{l}=\dfrac{y-0}{m}=\dfrac{z-0}{n}$, or $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ ...(2)

Eliminating l, m, n from (1) and (2) [by substituting their values from (2) in (1)], the locus of the line is

$$f(x, y, z)=0^*,$$

---

*Explanation. Putting each member of the equations (2), *viz.*,

$\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}=\dfrac{1}{r}$ (say), $l=rx,\ m=ry,\ n=rz.$

Substituting these values of l, m, n in (1),

$$f(rx, ry, rz)=0$$

or $f(x, y, z)=0.$ [$\because$ the equation is homogeneous]

which, being homogeneous, represents a cone whose vertex is the origin (Converse of Art. 87), *i.e.*, the fixed pt.]

Henceforth we shall deal with quadric cones only.

EXAMPLES

1. Show that a cone of the second degree can be found to pass through any five concurrent lines.

2. Show that the line $x/l=y/m=z/n$, where $l^2+2m^2-3n^2=0$ is a generator of the cone $x^2+2y^2-3z^2=0$. [*Sind U.*]

3. **Show that the lines drawn through the point (α, β, γ) whose direction-ratios satisfy $al^2+bm^2+cn^2=0$, generate the cone**

$$\mathbf{a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2 = 0.} \qquad [\textit{ B. U. 1953 }]$$

89. *Equation of a quadric cone through the axes.* **To show that the general equation to the cone of the second degree, which passes through the axes, is**

$$\mathbf{fyz+gzx+hxy = 0.}$$

The general equation of a cone of the second degree, whose vertex is the origin, is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(1) \qquad [\text{ Art. 87, Cor. }]$$

If it passes thro' the x-axis, then (1) is satisfied by the direction-cosines 1, 0, 0 [Art. 88], $\therefore\ a=0$.

Similarly if the cone passes thro' the y-, z-axes,

then $$b = 0, \quad c = 0.$$

Substituting these values of a, b, c in (1),

$$2fyz + 2gzx + 2hxy = 0$$

or $$\mathbf{fyz + gzx + hxy = 0,}$$

which is the required equation.

EXAMPLES

1. **Find the general equation of a cone of the second degree referred to three of its generators as axes of co-ordinates.**

[*P. U. 1945*]

2. **Show that a cone of the second degree can be found to pass through any two sets of rectangular axes through the same origin.** [*P. U. B. Sc. 1958 S*]

Take one set OX, OY, OZ as the axes of co-ordinates, and let the direction-cosines of the other set OX′, OY′, OZ′ referred to them be l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3.

The equation of *any* cone thro' OX, OY, OZ is

$$fyz+gzx+hxy = 0 \; ... (1) \qquad [\text{ Art. 89 }]$$

If it passes thro' OX', then (1) is satisfied by the direction-cosines l_1, m_1, n_1

$\therefore \quad fm_1n_1 + gn_1l_1 + hl_1m_1 = 0 \; ... (2)$ [Art. 88]

Similarly if the cone passes thro' OY',

then $\quad fm_2n_2 + gn_2l_2 + hl_2m_2 = 0 \; ... (3)$

Adding (2) and (3) vertically,

$$f(m_1n_1+m_2n_2)+g(n_1l_1+n_2l_2)+h(l_1m_1+l_2m_2) = 0 \; ... (4)$$

But $\quad m_1n_1 + m_2n_2 + m_3n_3 = 0$

[(Art. 58, (D)) $\because$ l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the direction-cosines of three mutually $\perp$ lines]

$\therefore \quad m_1n_1 + m_2n_2 = -m_3n_3.$

Similarly $\quad n_1l_1+n_2l_2 = -n_3l_3,\; l_1m_1+l_2m_2 = -l_3m_3.$

Substituting in (4),

$$-fm_3n_3-gn_3l_3-hl_3m_3 = 0$$

or $$fm_3n_3+gn_3l_3+hl_3m_3 = 0$$

$\therefore$ from (1), the cone also passes thro' OZ', [Conv. of Art. 88]
i. e., the cone passes thro' OX, OY, OZ ; OX', OY', OZ'.

3. Prove that the equation to the cone through the co-ordinate axes and the lines in which the plane $lx+my+nz = 0$ cuts the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0$ is

$$\mathbf{l(bn^2+cm^2-2fmn)yz+m(cl^2+an^2-2gnl)zx}$$
$$\mathbf{+n(am^2+bl^2-2hlm)xy=0.} \qquad [\textit{B.U. 1942}]$$

[The equation of the plane is $\quad lx + my + nz = 0 \; ... (1)$
and that of the cone is $\quad ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0 \; ... (2)$

The equation of *any* cone thro' the lines of intersection of the plane (1) and the cone (2), whose vertex is the origin, is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$$
$$+(lx+my+nz)(l'x+m'y+n'z) = 0 \; ...(3) \qquad [\text{ Art. 87 }]$$

If it passes thro' the x-axis, then (3) is satisfied by the direction-cosines 1, 0, 0

$\therefore \quad a+ll'=0 \; ...(4)$

Similarly if the cone passes thro' the y-axis,

then $\quad b+mm'=0 \; ...(5)$

and if it passes thro' the z-axis,

then $\quad c+nn'=0 \; ...(6)$

Find the values of l', m', n' from (4), (5), (6), and substitute them in (3).]

Right circular cone.

90. Right circular cone. Def. *A* **right circular cone** *is a surface generated by a straight line which passes through a fixed point, and makes a constant angle with a fixed line through the fixed point.*

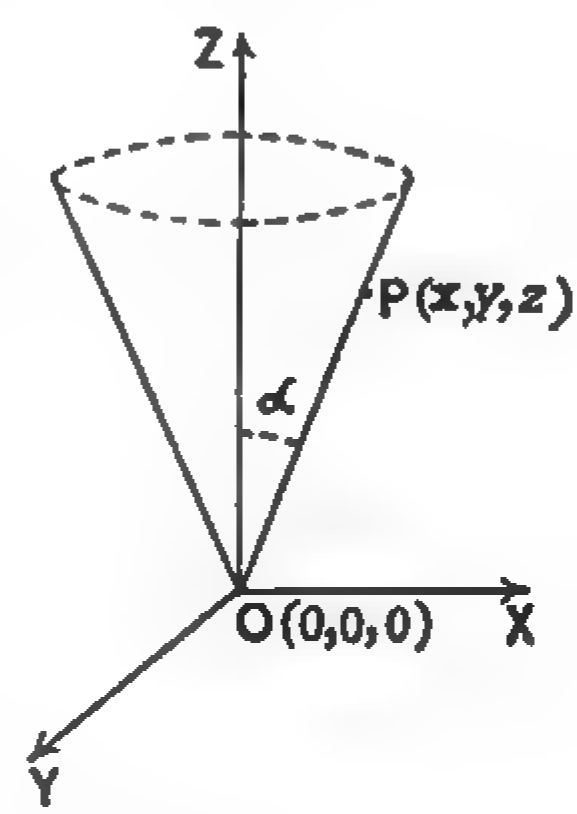
The fixed point is called the **vertex,** *the constant angle the* **semi-vertical angle,** *and the fixed line is called the* **axis** *of the cone.*

EXAMPLES

1. Find the direction-cosines of the axis of the right circular cone which passes through the lines drawn from O having direction-cosines proportional to $(-1, 2, 2)$, $(2, -1, 2)$, $(-2, 3, 6)$, and prove that the cone also passes through the co-ordinate axes.

2. Lines are drawn from O with direction-cosines proportional to $(1, 2, 2)$, $(2, 3, 6)$, $(3, 4, 12)$; find the direction-cosines of the axis of the right circular cone through them, and prove that the semi-vertical angle of the cone is $\cos^{-1} 1/\sqrt{3}$.

91. *Standard form.* **To show that the equation of the right circular cone whose vertex is the origin, axis the z-axis and semivertical angle α is $\mathbf{x^2+y^2=z^2 \tan^2 \alpha}$.**



Let $P(x, y, z)$ be *any* pt. on the cone, so that $\angle POZ=\alpha$ ...(1)

Now the direction-cosines of OP are proportional to $x-0$, $y-0$, $z-0$ (Art. 12), *i.e.*, proportional to x, y, z,

and those of OZ are proportional to $0, 0, 1$,

$\therefore$ from (1),

$$\tan \alpha^{*}=\frac{\sqrt{(y-0)^2+(0-x)^2+(0-0)^2}}{x(0)+y(0)+z(1)} \quad \begin{array}{|l} x,\ y,\ z \\ 0,\ 0,\ 1 \end{array}$$

[Art. 13, (*b*), Cor. 2]

$$=\frac{\sqrt{y^2+x^2}}{z}$$

or, squaring, $\tan^2 \alpha=\dfrac{y^2+x^2}{z^2}$,

or $\mathbf{x^2+y^2=z^2 \tan^2 \alpha}$,

which is the required equation.

Note. Standard form. The equation $x^2+y^2=z^2 \tan^2 \alpha$ is the *simplest* form of the equation of a right circular cone, and may be called the **standard form.**

---

*Why this step. As suggested by the result required to be proved, *viz.*, $x^2+y^2=z^2 \tan^2 \alpha$, we have used the angle formula (tangent form) (Art. 13, (*b*), Cor. 2).

EXAMPLES

1. Find the equation to the right circular cone whose vertex is P(2, −3, 5), axis PQ which makes equal angles with the axes, and semivertical angle is 30°. [*B. U. 1952*]

[Let $R(x, y, z)$ be *any* pt. on the cone, so that

$$\angle QPR=30° \quad ...(1)$$

Now the direction-cosines of PR are proportional to $x-2, y+3, z-5$, and those of PQ are proportional to 1, 1, 1 [Ex. 2, (*b*), Art. 10]

$\therefore$ from (1),

$$\cos 30°=\frac{(x-2)\,1+(y+3)\,1+(z-5)\,1}{\sqrt{(x-2)^2+(y+3)^2+(z-5)^2}\,.\,\sqrt{3}}$$

or
$$\frac{\sqrt{3}}{2}=\frac{x+y+z-4}{\sqrt{x^2+y^2+z^2-4x+6y-10z+38}\,.\,\sqrt{3}}.$$

Square and simplify.

Note. To expand $(a+b+c+d)^2$.

$(\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d})^2=\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2+\mathbf{d}^2+2(\mathbf{ab}+\mathbf{ac}+\mathbf{ad}+\mathbf{bc}+\mathbf{bd}+\mathbf{cd})$*.]

2. Prove that $x^2-y^2+z^2-2x+4y+6z+6=0$ represents a right circular cone whose vertex is the point $(1, 2, -3)$ whose axis is parallel to OY and whose semivertical angle is 45°. [*P(P). U.*]

3. The axis of a right cone, vertex O, makes equal angles with the co-ordinate axes, and the cone passes through the line drawn from O with direction-cosines proportional to (1, −2, 2). Find the equation to the cone. [*P. U. 1960 S*]

[**Note. Important.** *The semivertical angle, α, of the cone is the angle between any generator of the cone and the axis.*

Here the direction-cosines of the generator are proportional to $1, -2, 2$; and those of the axis are proportional to 1, 1, 1.

$$\therefore \cos\alpha=\frac{1(1)+(-2)(1)+2(1)}{\sqrt{(1)^2+(-2)^2+(2)^2}\sqrt{(1)^2+(1)^2+(1)^2}}=\frac{1}{3\sqrt{3}}.\ ]$$

4. Find the equation to the right circular cone whose vertex is P(2, −3, 5), axis PQ which makes equal angles with the axes, and which passes through A(1, −2, 3). [*P. U. 1955 S*]

5. Show that the equation to the right circular cone whose vertex is at the origin, whose axis has direction-cosines, cos α,

---

*How to write this step. In order to find the sum of the products of the four quantities *a, b, c, d,* taken two at a time, take *a* with the *following, i.e.,* with *b, c, d,* thus getting $ab+ac+ad$;

now take *b* with the *following, i.e.,* with *c, d,* thus getting $bc+bd$;

now take *c* with the *following, i.e.,* with *d,* thus getting *cd*.

(Compare the footnote* on page 159.)

cos β, cos γ, and whose semivertical angle is θ, is

$$(y\cos\gamma - z\cos\beta)^2 + (z\cos\alpha - x\cos\gamma)^2 + (x\cos\beta - y\cos\alpha)^2 = \sin^2\theta\,(x^2+y^2+z^2). \quad [B.\ U.\ 1953]$$

[(As suggested by the result required to be proved) Use the angle formula (sine form) (Art. 13, (b), Cor. 1).]

**6. The sum of the squares of the distances of a point from the planes $x+y+z=0$, $x-2y+z=0$ is equal to the square of its distance from the plane $z=x$. Prove that the equation of the locus of the point is $y^2+2zx=0$. By turning the axes of x and z in their plane through angles of 45°, show that the locus is a right circular cone whose semivertical angle is 45°.

92. *Equation to a cone with given vertex and given conic for base.* **To find the equation to the cone whose vertex is the point (α, β, γ) and base the conic**

$$f(x, y) \equiv ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0.$$

The equations of the conic are

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0 \quad \ldots(1)$$

The equations of *any* line thro' (α, β, γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \ldots(2)$$

It meets $z=0$, where $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=-\dfrac{\gamma}{n}$

or $$x-\alpha=-\frac{l}{n}\gamma, \quad y-\beta=-\frac{m}{n}\gamma,$$

or $$x=\alpha-\frac{l}{n}\gamma, \quad y=\beta-\frac{m}{n}\gamma.$$

Substituting these values of x, y in (1),

$$a\left(\alpha-\frac{l}{n}\gamma\right)^2+2h\left(\alpha-\frac{l}{n}\gamma\right)\left(\beta-\frac{m}{n}\gamma\right)+b\left(\beta-\frac{m}{n}\gamma\right)^2$$
$$+2g\left(\alpha-\frac{l}{n}\gamma\right)+2f\left(\beta-\frac{m}{n}\gamma\right)+c=0 \quad \ldots(3)$$

Eliminating l, m, n from (2) and (3) [by substituting their values from (2) in (3)], the locus of the line is

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)^2+2h\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)$$
$$+b\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2+2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)+2f\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)+c=0$$

or $$a\frac{(\alpha z-\gamma x)^2}{(z-\gamma)^2}+2h\frac{(\alpha z-\gamma x)(\beta z-\gamma y)}{(z-\gamma)^2}+b\frac{(\beta z-\gamma y)^2}{(z-\gamma)^2}$$
$$+2g\frac{\alpha z-\gamma x}{z-\gamma}+2f\frac{\beta z-\gamma y}{z-\gamma}+c=0$$

or, multiplying thro'out by $(z-\gamma)^2$,

$$a(\alpha z-\gamma x)^2+2h(\alpha z-\gamma x)(\beta z-\gamma y)+b(\beta z-\gamma y)^2$$
$$+2g(\alpha z-\gamma x)(z-\gamma)+2f(\beta z-\gamma y)(z-\gamma)+c(z-\gamma)^2=0,$$

which is the required equation of the cone.

[**Check.** The equation of the cone is satisfied by the co-ordinates of the vertex (α, β, γ), and also by the equations of the base conic $ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0$ thus,

$$a(\alpha\gamma-\gamma\alpha)^2+2h(\alpha\gamma-\gamma\alpha)(\beta\gamma-\gamma\beta)+b(\beta\gamma-\gamma\beta)^2$$
$$+2g(\alpha\gamma-\gamma\alpha)(\gamma-\gamma)+2f(\beta\gamma-\gamma\beta)(\gamma-\gamma)+c(\gamma-\gamma)^2=0$$

or $0=0$;

also putting $z=0$ in the equation of the cone,

$$a\gamma^2x^2+2h\gamma^2xy+b\gamma^2y^2+2g\gamma^2x+2f\gamma^2y+c\gamma^2=0$$

or, dividing thro' out by γ^2,

$$ax^2+2hxy+by^2+2gx+2fy+c=0.\ ]$$

EXAMPLES

1. Find the equation to the cone whose vertex is (α, β, γ) and base

(i) $\mathbf{ax^2+by^2=1,\ \ z=0}$; [*P(P). U. 1950*]

(ii) $\mathbf{y^2=4ax,\ \ z=0.}$ [*J. & K. U. 1954*]

2. (a) **Find the equation of the cone whose vertex is the point (α, β, γ) and whose generating lines pass through the conic**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\quad z=0.$$ [*Bar. U. 1954*]

(b) The section of a cone whose vertex is P and guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ by the plane $x=0$ is a rectangular hyperbola. Show that the locus of P is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$. [*P.U. 1954 S*]

3. Prove that a line which passes through the point (α, β, γ) and meets the parabola $z^2=4ax,\ y=0$, lies on the surface

$$(\beta z-\gamma y)^2=4a(\beta-y)(\beta x-\alpha y).$$

4. Show that the equation to the cone whose vertex is the origin and base the curve $z=c,\ f(x, y)=0$ is $f\left(\frac{xc}{z}, \frac{yc}{z}\right)=0.$ [*P. U.*]

[Proceed as in Art. 92.]

5. A cone has as base the circle $z=0,\ x^2+y^2+2ax+2by=0$, and passes through the fixed point $(0, 0, c)$ on the z-axis. If the section of the cone by the plane ZOX is a rectangular hyperbola, prove that the vertex lies on a fixed circle. [*B. H. U. 1954*]

[It will be found that the locus of the vertex of the cone is the curve of intersection of the surface

$$c(x^2+y^2+2ax+2by)-2z(ax+by)=0 \quad ...(A)$$

and the sphere $\quad x^2+y^2+z^2+2ax+2by=0 \quad ...(B)$

Substituting from (B) in (A),

$$c(-z^2)-2z(ax+by)=0, \text{ or } -z(2ax+2by+cz)=0$$

$\therefore$ $2ax+2by+cz=0$.

$\therefore$ the locus of the vertex is the fixed circle

$$x^2+y^2+z^2+2ax+2by=0,\ 2ax+2by+cz=0.]$$

93. Tangent cone or **enveloping cone.** The locus of the tangents from a given point to a sphere (or conicoid) is a cone called the **tangent cone** or the **enveloping cone** from the point to the sphere (or conicoid).

94. *Equation of the (tangent cone or) enveloping cone from a point to a sphere.* **To find the equation of the (tangent cone or) enveloping cone from the point (x_1, y_1, z_1) to the sphere**

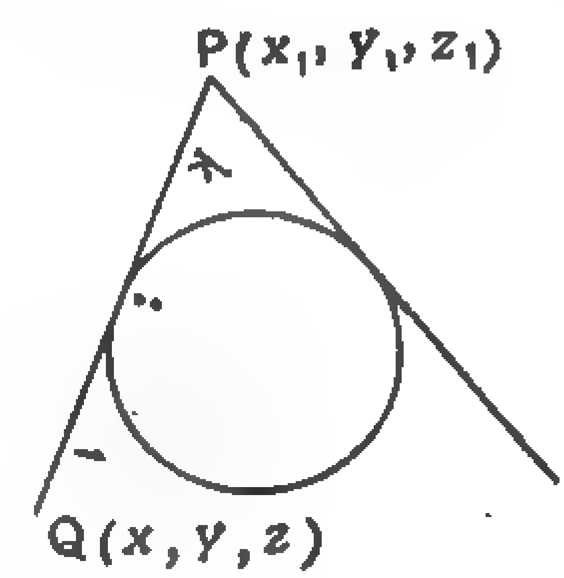
$$x^2+y^2+z^2=a^2.$$

The equation of the sphere is

$$x^2+y^2+z^2=a^2 \quad ...(1)$$

Let P be the pt. (x_1, y_1, z_1).

Let $Q(x, y, z)$ be *any* pt. on a tangent from P to the sphere.



The pt. which divides PQ in the ratio $k:1$ is $\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right)$.

If it lies on the sphere (1), then

$$\left(\frac{kx+x_1}{k+1}\right)^2+\left(\frac{ky+y_1}{k+1}\right)^2+\left(\frac{kz+z_1}{k+1}\right)^2=a^2$$

or, multiplying thro' out by $(k+1)^2$,

$$(kx+x_1)^2+(ky+y_1)^2+(kz+z_1)^2=(k+1)^2a^2$$

or $\quad (kx+x_1)^2+(ky+y_1)^2+(kz+z_1)^2-(k+1)^2a^2=0$

or $\quad k^2(x^2+y^2+z^2-a^2)+2k(xx_1+yy_1+zz_1-a^2)$

$$+(x_1^2+y_1^2+z_1^2-a^2)=0 \quad ...(2)$$

which is a quadratic in k.

$\because$ PQ *touches* the sphere, the quadratic (2) has equal roots

$\therefore$ $4(xx_1+yy_1+zz_1-a^2)^2=4(x^2+y^2+z^2-a^2)(x_1^2+y_1^2+z_1^2-a^2)$

or $\quad (x^2+y^2+z^2-a^2)(x_1^2+y_1^2+z_1^2-a^2)=(xx_1+yy_1+zz_1-a^2)^2 \quad ...(3)$

which is the required equation of the enveloping cone.

Abridged notation. If $S=x^2+y^2+z^2-a^2$, so that $S=0$ is the equation of the sphere,

$S_1=x_1^2+y_1^2+z_1^2-a^2$, so that S_1 is the result of substituting the co-ordinates of the pt. (x_1, y_1, z_1) in S,

$T=xx_1+yy_1+zz_1-a^2$, so that $T=0$ is the equation of the tangent plane at (x_1, y_1, z_1), then from (3), the equation of the enveloping cone is $$SS_1=T^2.$$

[**Check.** The equation of the enveloping cone (3) is satisfied by the co-ordinates of the given pt. (x_1, y_1, z_1) thus,

$$(x_1^2+y_1^2+z_1^2-a^2)(x_1^2+y_1^2+z_1^2-a^2)=(x_1^2+y_1^2+z_1^2-a^2)^2$$

or $$(x_1^2+y_1^2+z_1^2-a^2)^2=(x_1^2+y_1^2+z_1^2-a^2)^2.]$$

Note. Compare, in Analytical Plane Geometry, the equation of the pair of tangents $(SS_1=T^2)$ from the point (x_1, y_1) to the circle $S=0$.

EXAMPLES

1. *Equation of the enveloping cone of a sphere (equation in the general form).* **Find the locus of the tangents drawn from the point (x_1, y_1, z_1) to the sphere**

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0.$$

[**N.B.** *The equation of the enveloping cone from the point (x_1, y_1, z_1) to a sphere (or conicoid) is*

$$SS_1=T^2,$$

where $S=0$ is the equation of the sphere (or conicoid),

S_1 is the result of substituting the co-ordinates of the point (x_1, y_1, z_1) in S,

$T=0$ is the equation of the tangent plane at (x_1, y_1, z_1).]

2. Find the enveloping cone of the sphere

$$x^2+y^2+z^2+2x-2y=2$$

with its vertex at (1, 1, 1). [*P. U. 1952*]

The equation of the sphere is $x^2+y^2+z^2+2x-2y-2=0$.

[Form $S=0$ (*Note this step*)]

and the pt. is (1, 1, 1).

Here $S=x^2+y^2+z^2+2x-2y-2$; $x_1=1$, $y_1=1$, $z_1=1$

$$\therefore \quad S_1=(1)^2+(1)^2+(1)^2+2(1)-2(1)-2=1+1+1+2-2-2=1$$
$$T=x(1)+y(1)+z(1)+2.\tfrac{1}{2}(x+1)-2.\tfrac{1}{2}(y+1)-2$$
$$=x+y+z+x+1-(y+1)-2=2x+z-2$$

$\therefore$ the equation of the enveloping cone is

$$(x^2+y^2+z^2+2x-2y-2)\,1=(2x+z-2)^2 \quad [SS_1=T^2 \text{ (Art. 94)}]$$

or $x^2+y^2+z^2+2x-2y-2=4x^2+z^2+4+4zx-4z-8x$

or $3x^2-y^2+4zx-10x+2y-4z+6=0$ ...(1)

[**Check.** The equation of the enveloping cone (1) is satisfied by the co-ordinates of the given pt. (1, 1, 1) thus,

$$3(1)^2-(1)^2+4(1)(1)-10(1)+2(1)-4(1)+6=0, \text{ or } 0=0.]$$

3. Prove that the lines drawn from the origin so as to touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ lie on the cone

$$d(x^2+y^2+z^2)=(ux+vy+wz)^2. \qquad [P. U. 1957 S]$$

SECTION II

TANGENT PLANE

95. Equation of the tangent plane. *To find the equation of the tangent plane at any point (x_1, y_1, z_1) of the cone*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

The equation of the cone is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(1)$$

The equations of *any* line thro' (x_1, y_1, z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(2)$$

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the cone (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2$$
$$+2f(y_1+mr)(z_1+nr)+2g(z_1+nr)(x_1+lr)+2h(x_1+lr)(y_1+mr)=0$$

or
$$r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)$$
$$+2r[l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)]$$
$$+(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1)=0 \quad ... (3)$$

which is a quadratic in r.

$\because$ (x_1, y_1, z_1) lies on the cone (1)

$\therefore$ $ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0$ ... (4)

$\therefore$ one root of the quadratic (3) is zero.

If the line *touches* the cone, the other root is also zero.

$\therefore$ coeff. of $r=0$,

i.e., $l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)=0$ ... (5)

Eliminating l, m, n from (2) and (5) [by substituting their values from (2) in (5)], the locus of the tangents is

$$(x-x_1)(ax_1+hy_1+gz_1)+(y-y_1)(hx_1+by_1+fz_1)$$
$$+(z-z_1)(gx_1+fy_1+cz_1)=0$$

or
$$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)$$
$$-(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1)=0 \quad ... (6)$$

Adding (4) and (6),

$$\mathbf{x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0} \quad ...(7)$$

which is the required equation of the tangent plane.

[**Rule to write down the equation of the tangent plane at the point (x_1, y_1, z_1) of a cone :** See Rule of Art. 70, (*b*) and its footnote.]

Cor. 1. The tangent plane at any point of a cone passes through its vertex.

[$\because$ the tangent plane (7) passes thro' (0, 0, 0)]

Cor. 2. Generator of contact. *The tangent plane at any point P of a cone touches the cone along the generator through P.*

Let P be the pt. (x_1, y_1, z_1).

The equation of the tangent plane at P to the cone is

$$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0 \quad ...(1)$$

The equations of OP, the generator thro' P, are

$$\frac{x-0}{x_1-0}=\frac{y-0}{y_1-0}=\frac{z-0}{z_1-0} \qquad O(0, 0, 0) \qquad P(x_1, y_1, z_1)$$

or

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}.$$

Any pt. on OP is $Q(x_1r, y_1r, z_1r)$.

The equation of the tangent plane at Q is

$$x(ax_1r+hy_1r+gz_1r)+y(hx_1r+by_1r+fz_1r)+z(gx_1r+fy_1r+cz_1r)=0$$

or, dividing thro' out by r,

$$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0,$$

which is the same as the equation of the tangent plane (1) at P.

$\therefore$ the tangent plane at P touches the cone also at *any* pt. of OP, *i.e.*, it touches the cone along OP.

Note. Generator of contact. OP is called the **generator of contact.**

EXAMPLE

Points of intersection of a line and a cone. Find the points in which the line $\frac{x+1}{-1}=\frac{y-12}{5}=\frac{z-7}{2}$ cuts the surface $11x^2-5y^2+z^2=0$.

[*P(P). U. 1957*]

96. Notation. Important. It is convenient to use the following notation :

(1) $\mathbf{D}=\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=\mathbf{abc+2fgh-af^2-bg^2-ch^2}$*.

---

*Explanation. $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}=a(bc-f^2)-h(ch-fg)+g(hf-bg)$

$=abc-af^2-ch^2+fgh+fgh-bg^2$

$=abc+2fgh-af^2-bg^2-ch^2.$

(2) A, B, C, F, G, H are the co-factors of a, b, c, f, g, h respectively in the determinant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$,

i.e., $\mathbf{A = bc - f^2}$*, $B = ca - g^2$, $C = ab - h^2$;
$\mathbf{F = gh - af}$†, $G = hf - bg$, $H = fg - ch$.
[$\because$ $F = -(af - gh) = gh - af$. Similarly for G, H.]

Cor. 1. $\mathbf{BC - F^2 = aD}$.

Proof.
$$\begin{aligned} BC - F^2 &= (ca - g^2)(ab - h^2) - (gh - af)^2 \\ &= a^2bc - cah^2 - abg^2 + g^2h^2 - (g^2h^2 - 2afgh + a^2f^2) \\ &= a^2bc - cah^2 - abg^2 + 2afgh - a^2f^2 \\ &= a(abc + 2fgh - af^2 - bg^2 - ch^2) = aD. \quad \text{[From (1)]} \end{aligned}$$

Similarly $CA - G^2 = bD$, $AB - H^2 = cD$.

Cor. 2. $\mathbf{GH - AF = fD}$.

Proof.
$$\begin{aligned} GH - AF &= (hf - bg)(fg - ch) - (bc - f^2)(gh - af) \\ &= f^2gh - cfh^2 - bfg^2 + bcgh - (bcgh - abcf - f^2gh + af^3) \\ &= 2f^2gh - cfh^2 - bfg^2 + abcf - af^3 \\ &= f\,(abc + 2fgh - af^2 - bg^2 - ch^2) = fD. \quad \text{[From (1)]} \end{aligned}$$

Similarly $HF - BG = gD$, $FG - CH = hD$.

(3) $$\mathbf{P}^2 = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & 0 \end{vmatrix}$$
$$= -(\mathbf{Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv}).$$
[From Higher Algebra]

SECTION III

A CONE AND A PLANE THROUGH THE VERTEX

97. *Angle between two lines in which a plane cuts a cone.* **The axes being rectangular, to find the angle between the lines in which the plane $\mathbf{ux + vy + wz = 0}$ cuts the cone

$$\mathbf{f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.}$$

The equation of the plane is $ux + vy + wz = 0$ ...(1)

and that of the cone is $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ ...(2)

Let the equations of a line of section be $\dfrac{x}{l} = \dfrac{y}{m} = \dfrac{z}{n}$.

---

*Remember : $\mathbf{A = bc - f^2}$, and put down B, C *cyclically*, *i.e.*, changing a to b, b to c, c to a; f to g, g to h, h to f, thus getting
$$B = ca - g^2, \ C = ab - h^2.$$

†Remember : $\mathbf{F = gh - af}$, and put down G, H *cyclically*, *i.e.*, changing a to b, b to c, c to a; f to g, g to h, h to f, thus getting
$$G = hf - bg, \ H = fg - ch.$$

Then $\because$ it lies in the plane (1),

$\therefore$ it is $\perp$ to the normal to the plane

$\therefore$ $ul + vm + wn = 0$ ... (3)

Again $\because$ it lies on the cone (2),

$\therefore$ $al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0$ ...(4) [Art. 88]

Eliminating n from (3) and (4) [by substituting its value $\left(n = -\dfrac{ul+vm}{w}\right)$ from (3) in (4)],

$$al^2 + bm^2 + c\left(\frac{ul+vm}{w}\right)^2 + 2fm\left(-\frac{ul+vm}{w}\right) + 2g\left(-\frac{ul+vm}{w}\right)l + 2hlm = 0$$

or, multiplying thro' out by w^2,

$$al^2w^2 + bm^2w^2 + c(ul+vm)^2 - 2fmw(ul+vm) - 2glw(ul+vm) + 2hlmw^2 = 0$$

or $\quad l^2(aw^2 + cu^2 - 2gwu) + 2lm(cuv - fwu - gvw + hw^2)$

$$+ m^2(bw^2 + cv^2 - 2fvw) = 0$$

or, dividing thro' out by m^2,

$$\frac{l^2}{m^2}(aw^2 + cu^2 - 2gwu) + 2\frac{l}{m}(cuv - fwu - gvw + hw^2)$$
$$+ (bw^2 + cv^2 - 2fvw) = 0 \ \dots (5)$$

which is a quadratic in $\dfrac{l}{m}$.

If l_1, m_1, n_1 ; l_2, m_2, n_2 are the direction-cosines of the lines of section, then $\dfrac{l_1}{m_1}$, $\dfrac{l_2}{m_2}$ are the roots of the quadratic (5).

$$\therefore \quad \frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \frac{bw^2 + cv^2 - 2fvw}{aw^2 + cu^2 - 2gwu}, \qquad \text{[Product of roots]}$$

$$\frac{l_1}{m_1} + \frac{l_2}{m_2} = -\frac{2(cuv - fwu - gvw + hw^2)}{aw^2 + cu^2 - 2gwu} \qquad \text{[Sum of roots]}$$

$$\therefore \quad \frac{l_1 l_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1 m_2}{cu^2 + aw^2 - 2gwu}$$

$$= \frac{l_1 m_2 + l_2 m_1}{-2(cuv - fwu - gvw + hw^2)}$$

[To find $l_1m_2 - l_2m_1$]

$$= \frac{[\,(l_1m_2 + l_2m_1)^2 - 4l_1l_2m_1m_2\,]^{\frac{1}{2}}}{[4(cuv - fwu - gvw + hw^2)^2 - 4(bw^2 + cv^2 - 2fvw)(cu^2 + aw^2 - 2gwu)]^{\frac{1}{2}}}$$

$$= \frac{l_1m_2 - l_2m_1}{2w[-(Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv)]^{\frac{1}{2}}}{}^{*} \quad \text{[Art. 96, (2)]}$$

***Explanation.**

The expression whose square root occurs in the denominator

(*Continued on page 182*)

$$=\frac{l_1m_2-l_2m_1}{2wP} \quad \ldots (6) \qquad [\text{ Art. 96, (3) }]$$

or $$\frac{l_1l_2}{bw^2+cv^2-2fvw}=\frac{m_1m_2}{cu^2+aw^2-2gwu}$$

(which from symmetry is further)

$$=\frac{n_1n_2}{av^2+bu^2-2huv}$$

(which from symmetry of (6) is further)

$$=\frac{m_1n_2-m_2n_1}{2uP}=\frac{n_1l_2-n_2l_1}{2vP}=\frac{l_1m_2-l_2m_1}{2wP}$$

$$=\frac{\Sigma l_1l_2}{\Sigma(b+c)u^2-2\Sigma fvw}=\frac{[\Sigma(m_1n_2-m_2n_1)^2]^{\frac{1}{2}}}{[4(u^2+v^2+w^2)P^2]^{\frac{1}{2}}}$$

$$=\frac{[\Sigma(m_1n_2-m_2n_1)^2]^{\frac{1}{2}}}{2(u^2+v^2+w^2)^{\frac{1}{2}}\ P} \quad \ldots (7)$$

$\therefore$ if θ is the angle between the lines of section,

[adding and subtracting Σau^2 from the denom. of L. H. S.]

$$\frac{\cos\theta}{\Sigma(a+b+c)u^2-\Sigma(au^2+2fvw)}=\frac{\sin\theta}{2(u^2+v^2+w^2)^{\frac{1}{2}}\ P}$$

or $$\tan\theta=\frac{2(u^2+v^2+w^2)^{\frac{1}{2}}\ P}{(a+b+c)(u^2+v^2+w^2)-f(u,v,w)}.$$

[$\because f(x, y, z)=ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ (Given)
$=\Sigma(ax^2+2fyz)$
$\therefore f(u, v, w)=\Sigma(au^2+2fvw)$]

(Continued from page 181)

$$=4[c^2u^2v^2+f^2w^2u^2+g^2v^2w^2+h^2w^4-2cfu^2vw-2cguv^2w+2chuvw^2$$
$$+2fguvw^2-2fhuw^3-2ghvw^3$$
$$-(bcu^2w^2+abw^4-2bguw^3+c^2u^2v^2+cav^2w^2-2cguv^2w$$
$$-2cfu^2vw-2afvw^3+4fguvw^2)]$$

[Cancel $c^2u^2v^2$ and other such terms, and take the common factor w^2 outside the brackets]

$$=4w^2[f^2u^2+g^2v^2+h^2w^2+2chuv+2fguv-2fhuw$$
$$-2ghvw-(bcu^2+abw^2-2bguw+cav^2-2afvw+4fguv)]$$
$$=4w^2[u^2(f^2-bc)+v^2(g^2-ca)+w^2(h^2-ab)$$
$$+2uv(ch-fg)+2vw(af-gh)+2wu(bg-hf)]$$
$$=-4w^2[u^2(bc-f^2)+v^2(ca-g^2)+w^2(ab-h^2)$$
$$+2vw(gh-af)+2wu(hf-bg)+2uv(fg-ch)]$$
$$=-4w^2[Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv].$$

****Complete result.** If θ is the angle between the lines in which the plane $ux+vy+wz=0$ cuts the cone

$$f(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

then $\tan\theta = \pm \dfrac{2(u^2+v^2+w^2)^{\frac{1}{2}}\ P}{(a+b+c)\ (u^2+v^2+w^2)-f\ (u, v, w)}$.

For in (7), $\sin\theta = \pm [\ \Sigma(m_1n_2-m_2n_1)^2\ ]^{\frac{1}{2}}$

[Complete angle formula (sine form) (Art. 13, (*a*), Cor. 1)]

or $[\ \Sigma\ (m_1n_2-m_2n_1)^2\ ]^{\frac{1}{2}} = \pm \sin\theta.$

$\therefore$ from (7),

$$\frac{\cos\theta}{\Sigma(a+b+c)u^2-\Sigma(au^2+2fvw)} = \frac{\pm\sin\theta}{2(u^2+v^2+w^2)^{\frac{1}{2}}\ P}$$

or $$\tan\theta = \pm \frac{2(u^2+v^2+w^2)^{\frac{1}{2}}\ P}{(a+b+c)(u^2+v^2+w^2)-f(u, v, w)}.$$

[$\because$ $f(x, y, z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ (Given)

$= \Sigma(ax^2+2fyz)$

$\therefore$ $f(u, v, w) = \Sigma(au^2+2fvw)$]

Cor. 1. Condition of perpendicularity of lines of section. *The condition, that the lines in which the plane $ux+vy+wz=0$ cuts the cone*

$$f(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

may be perpendicular, is

$$\mathbf{(a+b+c)(u^2+v^2+w^2)=f\ (u, v, w).}$$

If the lines of section are $\perp$,

θ, the angle between them, $= 90^\circ$

$\therefore$ $\tan\theta = \tan 90^\circ = \infty$,

i.e., $\dfrac{2(u^2+v^2+w^2)^{\frac{1}{2}}\ P}{(a+b+c)(u^2+v^2+w^2)-f(u, v, w)} = \infty$ [Art. 97]

$\therefore$ the *denom.* $(a+b+c)(u^2+v^2+w^2)-f\ (u, v, w)=0$

or $\mathbf{(a+b+c)(u^2+v^2+w^2)=f\ (u, v, w)}$,

which is the required condition.

Cor. 2. Condition of perpendicularity of a pair of lines. *The condition, that the lines whose direction-cosines are given by*

$$ul+vm+wn=0,\ f(l, m, n)=al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$$

may be perpendicular, is

$$\mathbf{(a+b+c)(u^2+v^2+w^2)=f\ (u, v, w).}$$

[From Art. 97, (3), (4), and Cor. 1]

EXAMPLES

1. Find the angle between the lines of section of the plane $6x-y-2z=0$ and the cone $103x^2-7y^2-20z^2=0$.

The equation of the plane is $6x-y-2z=0$ ...(1)

and that of the cone is $108x^2-7y^2-20z^2=0$ ...(2)

Let the equations of a line of section be $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$.

Then $\because$ it lies in the plane (1),

$\therefore$ it is $\perp$ to the normal to the plane

$\therefore$ $6l-m-2n=0$ ...(3)

Again $\because$ it lies on the cone (2),

$\therefore$ $108l^2-7m^2-20n^2=0$ ...(4) [Art. 88]

[Compare Ex. 11, Art. 13, (*b*)]

Eliminating n from (3) and (4) [by substituting its value $\left(n=\dfrac{6l-m}{2}\right)$ from (3) in (4)],

$$108l^2-7m^2-20\left(\frac{6l-m}{2}\right)^2=0$$

or $$108l^2-7m^2-5\,(36l^2-12lm+m^2)=0$$

or $$-72l^2+60lm-12m^2=0 \qquad [\text{ Cancel } (-12)\ ]$$

or $$6l^2-5lm+m^2=0$$

$$\therefore\quad l=\frac{5m\pm\sqrt{25m^2-24m^2}}{12}=\frac{5m\pm m}{12}=\frac{m}{2} \text{ or } \frac{m}{3}$$

$\therefore$ *either* $2l-m=0$

i.e., $2l-m+0.n=0$

also $6l-m-2n=0$

$$\therefore\ \frac{l}{2-(-0)}=\frac{m}{0-(-4)}=\frac{n}{-2-(-6)}$$

or $\dfrac{l}{2}=\dfrac{m}{4}=\dfrac{n}{4}$

or $\dfrac{l}{1}=\dfrac{m}{2}=\dfrac{n}{2}$

or $3l-m=0$

i.e., $3l-m+0.n=0$

also $6l-m-2n=0$ [From (3)]

$$\therefore\ \frac{l}{2-(-0)}=\frac{m}{0-(-6)}=\frac{n}{-3-(-6)}$$

or $\dfrac{l}{2}=\dfrac{m}{6}=\dfrac{n}{3}$

$\therefore$ the direction-cosines of the lines are proportional to 1, 2, 2 ; 2, 6, 3

$\therefore$ if θ is the angle between the lines, then

$$\cos\theta=\frac{1(2)+2(6)+2(3)}{\sqrt{(1)^2+(2)^2+(2)^2}\ \sqrt{(2)^2+(6)^2+(3)^2}}$$

$$\left[\frac{aa'+bb'+cc'}{\sqrt{a^2+b^2+c^2}\ \sqrt{a'^2+b'^2+c'^2}}\ (\text{Art. 13, }(b))\right]$$

$$=\frac{20}{3(7)}=\frac{20}{21}$$

$\therefore \qquad \theta=\cos^{-1}\dfrac{20}{21}$.

2. Find the angles between the lines of section of the following planes and cones :

(i) $3x+y+5z=0,\ 6yz-2zx+5xy=0$; [*P(P). U. 1957*]

(ii) $3x-2y+z=0,\ 5x^2-3y^2+7z^2-20yz+36zx+2xy=0$.

3. Show that the angle between the lines given by

$$x+y+z=0,\ ayz+bzx+cxy=0,$$

is $\pi/2$ if $a+b+c=0$, but $\pi/3$ if $1/a+1/b+1/c=0$. [*Ag. U. 1941*]

4. Find the angle between the lines given by

$$x+y-z=0,\ \frac{yz}{q-r}+\frac{zx}{r-p}+\frac{xy}{p-q}=0.$$

5. Find the equations to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$. [*P(P). U. 1958 S*]

98. *Condition of tangency of a plane and a cone.* **To find the condition that the plane $lx+my+nz=0$ may touch the cone**

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

[Method of point of contact.]

The equation of the plane is $lx+my+nz=0$ ...(1)

and that of the cone is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ ...(2)

Let (x_1, y_1, z_1) be the pt. of contact.

The equation of the tangent plane at (x_1, y_1, z_1) to the cone (2) is

$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0$ ...(3)

[Art. 95]

$\because$ it is the same as the equation of the given plane (1)

$\therefore$ comparing coeffs. in (3) and (1),

$$\frac{ax_1+hy_1+gz_1}{l}=\frac{hx_1+by_1+fz_1}{m}=\frac{gx_1+fy_1+cz_1}{n}=-k^*\text{ (say)}$$

$\therefore \quad ax_1+hy_1+gz_1+lk=0$ ...(4)

$hx_1+by_1+fz_1+mk=0$ ...(5)

$gx_1+fy_1+cz_1+nk=0$ ...(6)

Also $\because$ *(x_1, y_1, z_1) lies on the given plane (1)*

$\therefore \qquad lx_1+my_1+nz_1=0$ [*Note this step*]

or $\qquad lx_1+my_1+nz_1+0.k=0$ ...(7)

Eliminating x_1, y_1, z_1, k from (4), (5), (6), (7),

$$\begin{vmatrix} a, & h, & g, & l \\ h, & b, & f, & m \\ g, & f, & c, & n \\ l, & m, & n, & 0 \end{vmatrix}=0$$

*and *not* k to avoid negative signs in the equations (4), (5), (6) following. (See the footnote on page 110.)

or $\quad -(Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm)=0$ [Art. 96, (3)]

or $\quad \mathbf{Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0,}$

which is the required condition,

where A, B, C, F, G, H are the co-factors of a, b, c, f, g, h in the determinant $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$,

i.e., $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$;
$F=gh-af$, $G=hf-bg$, $H=fg-ch$.

Reciprocal cone.

99. Reciprocal cone. Def. *The locus of the normals through the vertex of a cone to the tangent planes is another cone called the* **reciprocal cone.**

100. Equation of the reciprocal cone. *To find the equation of the cone reciprocal to the cone*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

The equation of the cone is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(1)$$

The equation of *any* tangent plane to the cone (1) is

$$lx+my+nz=0 \quad ...(2)$$

where $\quad Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0 \quad ...(3)$ [Art. 98]

The direction-cosines of the normal to the tangent plane (2) are proportional to l, m, n

$\therefore$ the equations of the normal thro' the vertex (0, 0, 0) are

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n}$$

$$\left[\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c} \text{ (Art. 37, Cor. 1)}\right]$$

or $\quad \dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n} \quad ...(4)$

Eliminating l, m, n from (3) and (4) [by substituting their values from (4) in (3)], the locus of the normals is

$$\mathbf{Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0,}$$

which is the required equation of the reciprocal cone.

Note. Why the name "reciprocal" cone. The locus of the normals thro' the vertex to the tangent planes to the cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(1)$$

is the cone $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ ...(2) [Art. 100]

Also the locus of the normals thro' the vertex to the tangent planes to the cone (2) is

$$A'x^2+B'y^2+C'z^2+2F'yz+2G'zx+2H'xy=0 \quad ...(3)$$

where $A' = BC - F^2 = aD$, and so on ;

$F' = GH - AF = fD$, and so on. [Art. 96, Cors. 1 and 2]

$\therefore$ (3) becomes $aDx^2 + bDy^2 + cDz^2 + 2fDyz + 2gDzx + 2hDxy = 0$ or, dividing thro' out by D, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, which is the cone (1).

Thus the two cones (1) and (2) are such that each is the locus of the normals thro' the vertex to the tangent planes to the other, hence they are called *reciprocal cones.*

Reciprocal cones. Def. Two cones, which are such that each is the locus of the normals through the vertex to the tangent planes to the other, are called **reciprocal cones.**

EXAMPLES

1. (*a*) Define reciprocal cones. [*P. U. 1955*]

(*b*) **Prove that the cones**

$$ax^2 + by^2 + cz^2 = 0 \text{ and } \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$$

are reciprocal. [*Bar. U. 1953*]

2. Prove that tangent planes to the cone $fyz + gzx + hxy = 0$ are perpendicular to generators of the cone

$$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0. \quad [D.U.H. \ 1952]$$

3. Prove that perpendiculars drawn from the origin to tangent planes to the cone $2x^2 + 3y^2 + 4z^2 + 2yz + 4zx + 6xy = 0$ lie on the cone

$$11x^2 + 4y^2 - 3z^2 + 8yz - 6zx - 20xy = 0.$$

4. Show that the general equation to a cone which touches the co-ordinate planes is

$$a^2x^2 + b^2y^2 + c^2z^2 - 2bcyz - 2cazx - 2abxy = 0. \quad [\ J. \ \& \ K. \ U. \ 1956\ ]$$

5. Prove that the equation $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ represents a cone which touches the co-ordinate planes, [*P. U. 1956*] **and that the equation of the reciprocal cone is $fyz + gzx + hxy = 0$.** [*Ag. U.*]

6. Show that any two sets of perpendicular planes which meet in a point all touch a cone of the second degree.

SECTION IV

INVARIANTS

101. Invariants. *If $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ is transformed by any change of rectangular axes, the expressions $a + b + c$, $A + B + C$, D remain unaltered in value.*

(*i*) If only the origin is changed, the axes remaining to their original directions, the coeffs. of the second degree terms, *i.e.*, *a, b, c, f, g, h* remain unchanged, [Art. 55, Cor. 2]

$\therefore$ $a+b+c$, $A+B+C(=bc-f^2+ca-g^2+ab-h^2)$,
$D(=abc+2fgh-af^2-bg^2-ch^2)$ remain unchanged.

(*ii*) If only the axes are changed (remaining rectangular), the origin remaining the same, let
$ax^2+by^2+cz^2+2fyz+2gzx+2hxy$ be changed into
$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy.$$

$\because$ $x^2+y^2+z^2$ remains unchanged
[being = square of the distance of the same pt. from the same origin]

$$\therefore \quad ax^2+by^2+cz^2+2fyz+2gzx+2hxy-\lambda(x^2+y^2+z^2) \quad \dots (1)$$
is changed into
$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy-\lambda(x^2+y^2+z^2) \quad \dots (2)$$

If (1) is the product of two linear factors for some value of λ, then (2) is also the product of two linear factors for the same value of λ.

Now (1) is the product of two linear factors if
$$(a-\lambda)(b-\lambda)(c-\lambda)+2fgh-(a-\lambda)f^2-(b-\lambda)g^2-(c-\lambda)h^2=0$$
$$[\,abc+2fgh-af^2-bg^2-ch^2=0 \quad (\text{Art. 35, } (a))\,]$$
or, multiplying thro' out by -1,
$$(\lambda-a)(\lambda-b)(\lambda-c)-2fgh+(a-\lambda)f^2+(b-\lambda)g^2+(c-\lambda)h^2=0$$
or $$\lambda^3-\lambda^2(a+b+c)+\lambda(bc+ca+ab-f^2-g^2-h^2)$$
$$-(abc+2fgh-af^2-bg^2-ch^2)=0$$
or $$\lambda^3-\lambda^2(a+b+c)+\lambda(A+B+C)-D=0 \quad \dots (3)$$

Similarly (2) is the product of two linear factors if
$$\lambda^3-\lambda^2(a'+b'+c')+\lambda(A'+B'+C')-D'=0 \quad \dots (4)$$

The cubics (3) and (4) in λ have the same roots

$\therefore$ equating coeffs., [$\because$ coeff. of λ^3 in each, $=1$]
$$a+b+c=a'+b'+c', \quad A+B+C=A'+B'+C', \quad D=D',$$
i.e., $a+b+c$, $A+B+C$, D remain unchanged.

(*iii*) $\therefore$ from (*i*) and (*ii*), for *any* change of rectangular axes, $a+b+c$, $A+B+C$, D remain unchanged.

Cor. Practical form. If, by a change of rectangular axes, origin remaining the same, $a\,x^2+b\,y^2+c\,z^2+2f\,yz+2g\,zx+2h\,xy$
is changed into $a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy$
then
$$\mathbf{a+b+c=a'+b'+c'},$$
$$\mathbf{A+B+C=A'+B'+C'},$$
i.e., $$bc-f^2+ca-g^2+ab-h^2=b'c'-f'^2+c'a'-g'^2+a'b'-h'^2,$$
$$\mathbf{D=D'},$$
i.e., $$abc+2fgh-af^2-bg^2-ch^2=a'b'c'+2f'g'h'-a'f'^2-b'g'^2-c'h'^2.$$

Note. Invariants. Since, the expressions $a+b+c$, $A+B+C$, D remain *unchanged in value* by any change of rectangular axes, they are called **invariants.**

102. *Condition that a cone may have three mutually perpendicular generators.* **To find the necessary and sufficient condition that a cone may have three mutually perpendicular generators.**

Let the equation of the cone be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy = 0 \quad ... \ (1)$$

(*i*) [*To find the necessary condition.*] Let the cone have three mutually $\perp$ generators.

(To find the condition.)

Take the three mutually $\perp$ generators as the axes. Then the equation of the cone becomes $f'yz+g'zx+h'xy = 0$ ... (2) [Art. 89]

$\therefore$ from (1) and (2), by invariants,

$$a+b+c = a'+b'+c' \qquad [\text{ Art. 101, Cor. }]$$
$$= 0+0+0$$

or $$\mathbf{a+b+c=0,}$$

which is the required *necessary* condition.

(*ii*) [*To prove that the condition is sufficient.*] Let $a+b+c=0$.

(To prove that the cone has an infinite number of sets of three mutually $\perp$ generators.)

Take *any* generator as the x-axis, and let the equation of the cone become

$$a'x^2 + b'y^2 + c'z^2 + 2f'yz + 2g'zx + 2h'xy = 0 \quad ... \ (3)$$

$\because$ it passes thro' the x-axis, (3) is satisfied by the direction-cosines 1, 0, 0 [Art. 88]

$$\therefore \qquad a' = 0.$$

$\therefore$ the equation (3) becomes

$$b'y^2+c'z^2+2f'yz+2g'zx+2h'xy = 0 \quad ... \ (4)$$

Now from (1) and (4), by invariants, $a+b+c = b'+c'$.

But $a+b+c = 0$ (Given), $\therefore$ $b'+c' = 0$ ... (5)

The plane thro' the vertex $\perp$ to the x-axis, *viz.*, the yz-plan *i.e.*, $x=0$ meets the cone (4), where

$$b'y^2+c'z^2+2f'yz = 0,$$

which is the equation of the lines of section *in that plane.*

But $b'+c' = 0$ (Proved in (5)),

$\therefore$ by Analytical Plane Geometry these lines are $\perp$.

$\therefore$ the cone has three mutually $\perp$ generators.

Further $\because$ *any* generator has been taken as the x-axis

$\therefore$ the cone has an infinite number of sets of three mutually $\perp$ generators

****Cor.** *If the general equation of the second degree, viz.,*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d = 0$$

represents a cone, then the necessary and sufficient condition, that it may

have three mutually perpendicular generators, is

$$\mathbf{a+b+c=0.}$$

Proof. The equation of the cone is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad ...(1)$$

Changing the origin to the vertex, the equation becomes

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(2)$$

[$\because$ the coeffs. of the second degree terms remain unchanged (Art. 55, Cor. 2), and the equation becomes homogeneous (Art. 87)]

Now the necessary and sufficient condition that the cone (2) may have three mutually $\perp$ generators is

$$a+b+c=0,$$

$\therefore$ the necessary and sufficient condition, that the cone (1) may have three mutually $\perp$ generators, is

$$a+b+c=0.$$

EXAMPLES

1. If a right circular cone has three mutually perpendicular generators, the semi-vertical angle is $\tan^{-1}\sqrt{2}$. [*P. U. 1961*]

Let the equation of the rt. circular cone be

$$x^2+y^2=z^2\tan^2\alpha \qquad [\text{ Art. 91 }]$$

or $$x^2+y^2-z^2\tan^2\alpha=0 \quad ...(1)$$

If it has three mutually $\perp$ generators, then

$$1+1-\tan^2\alpha=0 \qquad [\ a+b+c=0 \text{ (Art. 102) }]$$

or $2-\tan^2\alpha=0$, or $\tan^2\alpha=2$, or $\tan\alpha=\sqrt{2}$

$\therefore\quad \alpha=\tan^{-1}\sqrt{2}$.

2. Prove that the plane $\mathbf{ax+by+cz=0}$ cuts the cone $\mathbf{yz+zx+xy=0}$ in perpendicular lines if $\mathbf{\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=0}$.

[*P(P). U. 1955 S*]

The equation of the plane is

$$ax+by+cz=0 \quad ... (1)$$

and that of the cone is $yz+zx+xy=0 \quad ... (2)$

[Compare it with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.]

Here $\quad$ "$a+b+c$" $=0+0+0=0$,

$\therefore$ the cone has three mutually $\perp$ generators [Art. 102]

$\therefore$ the plane (1) cuts the cone (2) in $\perp$ lines if the normal to the plane (1) thro' the vertex, (direction-cosines proportional to a, b, c), lies on the cone (2)

i.e., if $\quad bc+ca+ab=0 \qquad$ [Art. 88]

or, dividing thro' out by abc, if $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}=0$.

3. Show that the two straight lines represented by the equations $ax+by+cz=0$, $yz+zx+xy=0$ will be perpendicular if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$. [*P. U. 1942*]

4. Prove that the plane $lx+my+nz=0$ cuts the cone

$$(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$$

in perpendicular lines if

$$(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gnl+2hlm=0.$$

[*P. U. 1940 S*]

5. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations to the other two. [*P. U. 1954 S*]

6. Show that the cone whose vertex is the origin and which passes through the curve of intersection of the surface $2x^2-y^2+2z^2=3a^2$, and any plane at a distance a from the origin, has three mutually perpendicular generators.

103. *Condition that a cone may have three mutually perpendicular tangent planes.* **To prove that the necessary and sufficient condition, that the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ may have three mutually perpendicular tangent planes, is**

$$A+B+C=0,$$

where $A=bc-f^2$, $B=ca-g^2$, $C=ab-h^2$.

The equation of the cone is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad ...(1)$$

$\therefore$ that of the reciprocal cone is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \quad ...(2)$$

The necessary and sufficient condition, that the cone (1) may have three mutually $\perp$ tangent planes, is that the reciprocal cone (2) may have three mutually $\perp$ generators

i.e., $A+B+C=0$. [$a+b+c=0$ (Art. 102)]

EXAMPLE

Prove that the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ has three mutually perpendicular tangent planes if

$$bc+ca+ab=f^2+g^2+h^2.$$

[*P(P). U. H. 1953*]

MISCELLANEOUS EXAMPLES ON CHAPTER VIII

1. Find the equation of the cone whose vertex is the origin and which passes through

(*i*) the circle $z=2$ and $x^2+y^2=4$. [*P. U.*]

(*ii*) the curve of intersection of the plane $ax+by+cz+d=0$ and the surface $\alpha x^2+\beta y^2+\gamma z^2=1$. [*P. U. 1936 S*]

(*iii*) the intersections of the two spheres

$$x^2+y^2+z^2-x-1=0,\ x^2+y^2+z^2+y-2=0. \qquad [\textit{P. U.}]$$

[(*iii*) The equations of the spheres are

$$x^2+y^2+z^2-x-1=0 \quad ...(1)$$
$$x^2+y^2+z^2+y-2=0 \quad ...(2)$$

(To obtain an equation of the first degree in x, y, z from (1) and (2).)

Subtracting (2) from (1),

$$-x-y+1=0, \text{ or } x+y-1=0 \quad ...(3)$$

The intersections of the spheres (1) and (2) are the same as those of the sphere (1) and the plane (3). Now make (1) homogeneous by means of (3).]

2. Cone represented by the general equation of the second degree.

Prove that if

$$F(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

represents a cone, the co-ordinates of the vertex satisfy the equations $F_x=0$, $F_y=0$, $F_z=0$, $F_t=0$, *where t is used to make* $F(x, y, z)$ *homogeneous and is equated to unity after differentiation.* [*P. U. 1944 S*]

Let $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \ ...(1)$ represent a cone, and (x_1, y_1, z_1) be the vertex.

Changing the origin to (x_1, y_1, z_1), (1) becomes

$$a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2$$
$$+2f(y+y_1)(z+z_1)+2g(z+z_1)(x+x_1)+2h(x+x_1)(y+y_1)$$
$$+2u(x+x_1)+2v(y+y_1)+2w(z+z_1)+d=0$$

or $\quad ax^2+by^2+cz^2+2fyz+2gzx+2hxy$

$$+2[x(ax_1+hy_1+gz_1+u)+y(hx_1+by_1+fz_1+v)+z(gx_1+fy_1+cz_1+w)]$$
$$+(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d)=0 \quad ...(2)$$

$\because$ the new origin is the vertex, (2) is homogeneous. [Art. 87]

$\therefore$ coeff. of $x=0$, coeff. of $y=0$, coeff. of $z=0$, constant term$=0$,

i.e.,
$$ax_1+hy_1+gz_1+u=0 \quad ...(3)$$
$$hx_1+by_1+fz_1+v=0 \quad ...(4)$$
$$gx_1+fy_1+cz_1+w=0 \quad ...(5)$$
$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d=0 \quad ...(6)$$

From (6),

$$x_1(ax_1+hy_1+gz_1+u)+y_1(hx_1+by_1+fz_1+v)+z_1(gx_1+fy_1+cz_1+w)$$
$$+ux_1+vy_1+wz_1+d=0$$

or, substituting from (3), (4), (5),

$$x_1(0)+y_1(0)+z_1(0)+ux_1+vy_1+wz_1+d=0$$

or $$ux_1+vy_1+wz_1+d=0 \quad ...(7)$$
Now $F(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$.

Making it homogeneous in x, y, z, t by introducing proper powers of t,

$F(x,y,z,t) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2uxt+2vyt+2wzt+dt^2$.

Differentiating *partially* w.r.t. x, y, z, t,

$$F_x=2(ax+gz+hy+ut),\ F_y=2(by+fz+hx+vt),$$
$$F_z=2(cz+fy+gx+wt),\ F_t=2(ux+vy+wz+dt).$$

Putting $t=1$ in these, and equating to zero, we get

$$ax+hy+gz+u=0,\ hx+by+fz+v=0,$$
$$gx+fy+cz+w=0,\ ux+vy+wz+d=0.$$

$\therefore$ from (3), (4), (5), (7), the co-ordinates of the vertex (x_1, y_1, z_1) satisfy $F_x=0, F_y=0, F_z=0, F_t=0$, where t is put $=1$ after differentiation.

Cor. 1. To find the condition that the general equation of the second degree in x, y, z may represent a cone.

Eliminating x_1, y_1, z_1 from (3), (4), (5), (7),

$$\begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix}=0,$$

which is the required condition.

Cor. 2. To find the co-ordinates of the vertex.

From (3), (4), (5),

$$\frac{x_1}{\begin{vmatrix} h, & g, & u \\ b, & f, & v \\ f, & c, & w \end{vmatrix}}=\frac{-y_1}{\begin{vmatrix} a, & g, & u \\ h, & f, & v \\ g, & c, & w \end{vmatrix}}=\frac{z_1}{\begin{vmatrix} a, & h, & u \\ h, & b, & v \\ g, & f, & w \end{vmatrix}}=\frac{-1}{\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}},$$

which give the required co-ordinates (x_1, y_1, z_1) of the vertex.

[Rule to prove that a given numerical equation F(x, y, z)=0 represents a cone, and to find the vertex of the cone :

(*a*) (To prove that the equation represents a cone.)

(*i*) *Make $F(x, y, z)$ homogeneous in x, y, z, t by introducing proper powers of t, and get $F(x, y, z, t)$.*

(*ii*) *Find* $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}, \dfrac{\partial F}{\partial t}$.

(*iii*) *Put $t=1$, and equate each of* $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial F}{\partial z}, \dfrac{\partial F}{\partial t}$ *to zero.*

(*iv*) *Solve for x, y, z any three of the four equations, say, the first three equations obtained in step (iii).*

(v) *Substitute these values of x, y, z in the fourth equation of step (iii), and show that it is satisfied.*

(b) (To find the vertex.)

The values of x, y, z found in step (iv) are the co-ordinates of the vertex.]

3. Prove that $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if $u^2/a+v^2/b+w^2/c=d$. [*P(P). U. H. 1951*]

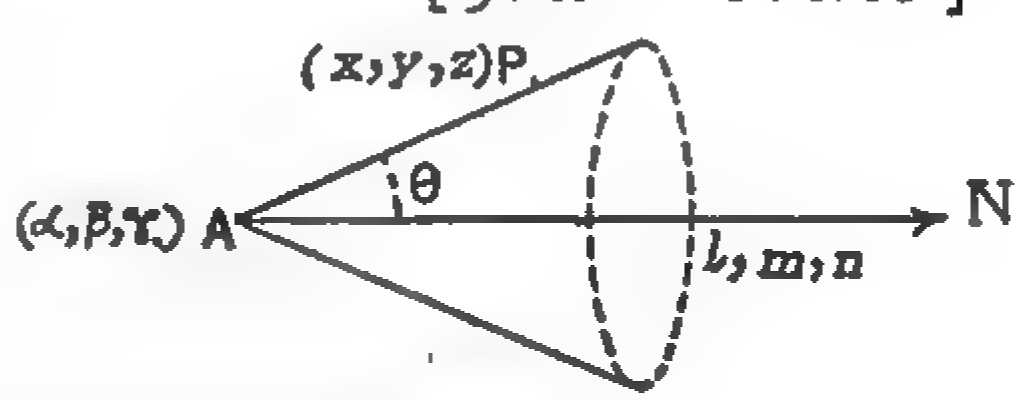
4. **Prove that the equation**

(i) $\mathbf{2y^2-8yz-4zx-8xy+6x-4y-2z+5=0}$ **represents a cone whose vertex is** $(-7/6, 1/3, 5/6)$. [*D. U. H. 1950*]

(ii) $\mathbf{7x^2+2y^2+2z^2-10zx+10xy+26x-2y+2z-17=0}$ **represents a cone whose vertex is** $(1, -2, 2)$. [*P.U. B. Sc. 1961*]

5. OP and OQ are two straight lines that remain at right angles and move so that the plane OPQ always passes through the z-axis. If OP describes the cone $F(y/x, z/x)=0$, prove that OQ describes the cone $F\left\{\dfrac{y}{x}, \left(-\dfrac{x}{z}-\dfrac{y^2}{zx}\right)\right\}=0.$ [*Ag. U. 1947*]

6. *Most general form.* **Find the equation of a right circular cone whose vertex is (α, β, γ), semivertical angle θ, and axis has the direction-cosines l, m, n.** [*J. & K. U. 1953*]



Let A be the vertex (α,β,γ), and AN the axis (direction-cosines l, m, n).

Let $P(x, y, z)$ be *any* pt. on the cone, so that $\angle\, NAP=\theta$...(1)

Now the direction-cosines of AP are proportional to $x-\alpha, y-\beta, z-\gamma$, and those of AN are proportional to l, m, n.

$\therefore$ from (1),

$$\cos\theta=\frac{(x-\alpha)l+(y-\beta)m+(z-\gamma)n}{\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}\ \sqrt{l^2+m^2+n^2}}$$

[But $l^2+m^2+n^2=1$ ($\because$ l, m, n are *actual* direction-cosines (Given))]

$$=\frac{(x-\alpha)l+(y-\beta)m+(z-\gamma)n}{\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}}$$

or, squaring,

$$[(x-\alpha)l+(y-\beta)m+(z-\gamma)n]^2=[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]\cos^2\theta,$$

which is the required equation.

7. *General form.* **Show that the equation of a right circular cone whose vertex is the origin, semivertical angle α, and axis the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ (l, m, n actual direction-cosines) is**

$$(\mathbf{lx+my+nz})^2=(\mathbf{x^2+y^2+z^2})\cos^2\alpha. \qquad [\textit{ P.U. 1956 S}]$$

8. Find the equation of the right circular cone whose vertex is at the origin, whose axis is the line $x/1=y/2=z/3$, and which has a vertical angle of 60°. [*Gujarat U. 1950*]

Show that the sections by planes parallel to $x=0$ are hyperbolas and by planes parallel to $y=0$ and $z=0$ are ellipses. [*L.U.*]

**9. Show that $33x^2+13y^2-95z^2-144yz-96zx-48xy=0$ represents a right circular cone whose axis is the line $3x=2y=z$. Find its vertical angle. [*P. U. 1950 S*]

****10. If $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ represents a right circular cone of semi-vertical angle α,**

$$\frac{gh}{f}-a=\frac{hf}{g}-b=\frac{fg}{h}-c=\left(\frac{gh}{f}+\frac{hf}{g}+\frac{fg}{h}\right)\cos^2\alpha.$$

Cor. Conditions for a right circular cone. *The conditions, that the equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ may represent a right circular cone, are $\frac{F}{f}=\frac{G}{g}=\frac{H}{h}$, where F, G, H are the co-factors of f, g, h in the determinant* $\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$,

i. e., $F=gh-af$, $G=hf-bg$, $H=fg-ch$.

11. If α is the semivertical angle of the right circular cone which passes through the lines OX, OY, $x=y=z$, show that

$$\cos\alpha=(9-4\sqrt{3})^{-\frac{1}{2}}.$$

12. The axis of the right circular cone, vertex at the origin, which passes through the three lines, whose direction-cosines are (l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3) is normal to the plane

$$\begin{vmatrix} 0, & 1, & 1, & 1 \\ x, & l_1, & l_2, & l_3 \\ y, & m_1, & m_2, & m_3 \\ z, & n_1, & n_2, & n_3 \end{vmatrix}=0.$$

[*P. U. B. Sc. 1961*]

13. Prove that the line $x=pz+q$, $y=rz+s$, intersects the conic $z=0$, $ax^2+by^2=1$, if $aq^2+bs^2=1$. [*P. U. 1947*]

**Hence show that the co-ordinates of any point on a line which intersects the conic and passes through the point (α, β, γ) satisfy the equation $a(\gamma x-\alpha z)^2+b(\gamma y-\beta z)^2=(z-\gamma)^2$.

14. The vertex of a cone is (a, b, c), and the yz-plane meets it in the curve $f(y, z)=0$, $x=0$. Show that the zx-plane meets it in the

curve $y=0, f\left(\dfrac{bx}{x-a}, \dfrac{cx-az}{x-a}\right)=0.$

**15. Two cones pass through the curves

$$y=0,\ z^2=4ax\ ;\ x=0,\ z^2=4by,$$

and they have a common vertex. Show that if the plane $z=0$ cuts them in two conics which meet in four concyclic points, the vertex lies on the surface $z^2(x/a+y/b)=4(x^2+y^2)$.

16. Find the equation of the cone with vertex at the origin and generators touching the sphere $x^2+y^2+z^2-2x+4z=1$.

[*Sind U. 1950*]

**17. Find the equation of the planes through the z-axis and the lines of section of the plane $ux+vy+wz=0$ and the cone

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

and prove that the plane touches the cone if $P=0$.

[**Note.** $P^2=-(Au^2+Bv^2+Cw^2+2Fvw+2Gwu+2Huv)$ (Art. 96,(3)). For simplification see the footnote on page 181.]

18. **Find the equation of the cone generated by straight lines drawn from the origin to cut the circle through the three points (1, 0, 0), (0, 2, 0), (2, 1, 1) and prove that the acute angle between the two lines in which the plane x=2y cuts the cone is**

$$\cos^{-1}\sqrt{\frac{5}{14}}.$$

[*Ag. U. 1937*]

**19. Show that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in two lines inclined at an angle

$$\tan^{-1}\left[\frac{\{(a^2+b^2+c^2)(a^2+b^2+c^2-2bc-2ca-2ab)\}^{\frac{1}{2}}}{bc+ca+ab}\right],$$

[*P.U.H. 1961*]

and by considering the value of this expression when $a+b+c=0$, show that the cone is of revolution, and that its axis is $x=y=z$ and vertical angle $\tan^{-1} 2\sqrt{2}$.

[*P. U. H. 1958*]

[When $a+b+c=0$, $(a+b+c)^2=0$,

i.e., $a^2+b^2+c^2+2bc+2ca+2ab=0$, or $2bc+2ca+2ab=-(a^2+b^2+c^2)$.

It will be found that the acute angle between the two lines of section$=\tan^{-1} 2\sqrt{2}$, which is independent of a, b, c.

$\therefore$ *any* plane thro' the line $x=y=z$ cuts the cone in two lines the angle between which $=\tan^{-1} 2\sqrt{2}$

$\therefore$ the cone is a right circular cone whose axis is $x=y=z$ and vertical angle $=\tan^{-1} 2\sqrt{2}$.]

20. **Prove that the equation of the cone through the co-ordinate axes and the lines of section of the cone $3x^2-7y^2+z^2=0$ and the plane $2x-7y+z=0$ is $12yz-7zx+17xy=0$, and that the**

other two common generators of the two cones lie in the plane $3x+2y+2z=0$. [*B. U.*]

21. Find the conditions that the lines of section of the plane $lx+my+nz=0$ and the cones $ax^2+by^2+cz^2=0$, $fyz+gzx+hxy=0$ may be coincident.

[Let the equations of a line of section of the plane and the first cone be $\dfrac{x}{L}=\dfrac{y}{M}=\dfrac{z}{N}$. Proceed as in Art. 97. It will be found

that $$\frac{L_1 L_2}{bn^2+cm^2}=\frac{M_1 M_2}{cl^2+an^2}=\frac{N_1 N_2}{am^2+bl^2} \quad \dots \text{(A)}$$

Again from the plane and the second cone, it will be found that
$$\frac{L_1 L_2}{fm}=\frac{M_1 M_2}{gl}.$$

[*Multiply the denominators of both sides by n** (*Note this step*)]

or $\dfrac{L_1 L_2}{fmn}=\dfrac{M_1 M_2}{gnl}$ (which, from symmetry, is further) $=\dfrac{N_1 N_2}{hlm}$ ...(B)

Divide (B) by (A).]

22. Prove that the common generators of the cones
$$(m^2n^2-l^4)x^2+(n^2l^2-m^4)y^2+(l^2m^2-n^4)z^2=0,$$
$$\frac{mn-l^2}{lx}+\frac{nl-m^2}{my}+\frac{lm-n^2}{nz}=0$$
lie in the planes $(mn\pm l^2)x+(nl\pm m^2)y+(lm\pm n^2)z=0$.

23. Prove that the equation to the planes through the origin perpendicular to the lines of section of the plane $lx+my+nz=0$ and the cone $ax^2+by^2+cz^2=0$, is
$$x^2(bn^2+cm^2)+y^2(cl^2+an^2)+z^2(am^2+bl^2)-2amnyz-2bnlzx-2clmxy=0.$$

[The equation of the plane is
$$lx+my+nz=0 \quad \dots (1)$$
and that of the cone is $ax^2+by^2+cz^2=0$ ... (2)

Let the equations of a line of section be $\dfrac{x}{L}=\dfrac{y}{M}=\dfrac{z}{N}$ ... (3)

It will be found that
$$Ll+Mm+Nn=0 \quad \dots (4)$$
$$aL^2+bM^2+cN^2=0 \quad \dots (5)$$

The equation of the plane thro' the origin $\perp$ to the line of section (3) is $Lx+My+Nz=0$ ... (6)

Eliminate L, M, N from (4), (5), (6) [by finding their values from (4) and (6) by cross-multiplication and substituting them in (5).]

24. Find the semivertical angle of a right circular cone when the cone has three tangent planes perpendicular to one another.

***Why this step.** To get symmetrical (*i.e., cyclic*) results in the denominators.

CHAPTER IX
THE CYLINDER
EQUATION OF A CYLINDER

104. Cylinder. Def. *A* **cylinder** *is a surface generated by a straight line which is parallel to a fixed line, and satisfies some other condition, e.g., it may intersect a fixed curve.*

The straight line in any position is called a **generator,** *and the fixed curve the* **guiding curve** *of the cylinder.*

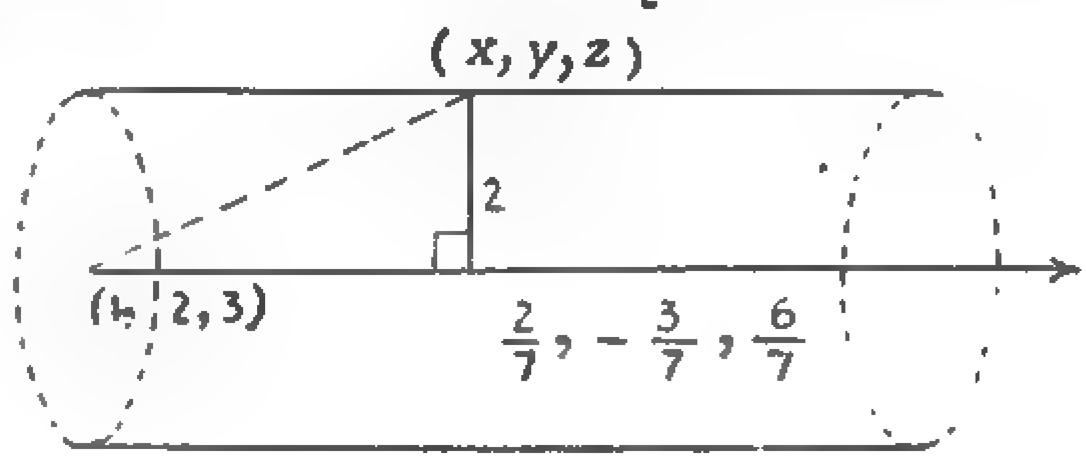
Right circular cylinder.

105. Right circular cylinder. Def. *A* **right circular cylinder** *is a surface generated by a straight line which is parallel to a fixed line, and is at a constant distance from it.*

The fixed line is called the **axis,** *and the constant distance the* **radius** *of the cylinder.*

EXAMPLES

1. Find the equation to the right circular cylinder of radius 2 whose axis passes through (1, 2, 3) and has direction-cosines proportional to (2, −3, 6). [*P. U. 1940*



The direction-cosines of the axis are proportional to 2, −3, 6.

Dividing by

$$\sqrt{(2)^2+(-3)^2+(6)^2}$$
$$=\sqrt{4+9+36}=\sqrt{49}=7,$$

the actual direction-cosines are

$$\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}.$$

Let (x, y, z) be *any* pt. on the cylinder.

Then the $\perp$ distance of (x, y, z) from the axis $= 2$ ... (1)

[Radius (Def. Art. 105)]

Now the distance between one pt. (1, 2, 3) on the axis and the given pt. $(x, y, z)=\sqrt{(x-1)^2+(y-2)^2+(z-3)^2}$.

The projection of this distance on the axis [direction-cosines $\frac{2}{7}, -\frac{3}{7}, \frac{6}{7}$]

$$=(x-1)\left(\frac{2}{7}\right)+(y-2)\left(-\frac{3}{7}\right)+(z-3)\frac{6}{7}$$

[$(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$ (Art. 14)]

$$=\frac{2x-3y+6z-14}{7}$$

$$\therefore \quad \perp = \sqrt{(\text{distance})^2 - (\text{projection})^2}$$

$$= \left\{(x-1)^2+(y-2)^2+(z-3)^2 - \left(\frac{2x-3y+6z-14}{7}\right)^2\right\}^{\frac{1}{2}}$$

$\therefore$ from (1),

$$\left\{(x-1)^2+(y-2)^2+(z-3)^2 - \left(\frac{2x-3y+6z-14}{7}\right)^2\right\}^{\frac{1}{2}} = 2$$

or, squaring,

$$(x-1)^2+(y-2)^2+(z-3)^2 - \left(\frac{2x-3y+6z-14}{7}\right)^2 = 4$$

or, multiplying thro' out by 49,

$$49\,[\,(x-1)^2+(y-2)^2+(z-3)^2\,] - (2x-3y+6z-14)^2 = 196$$

or $49\,(x^2-2x+1+y^2-4y+4+z^2-6z+9)$

$$-(4x^2+9y^2+36z^2+196-12xy+24zx-56x-36yz+84y-168z)^* = 196$$

or $45x^2+40y^2+13z^2+36yz-24zx+12xy$

$$-42x-280y-126z+294=0,$$

which is the required equation.

$$\begin{array}{r} 49 \\ 14 \\ \hline 686 \\ -392 \\ \hline 294 \end{array}$$

2. **Find the equation of the right circular cylinder of radius 3, whose axis passes through (1, −1, 2) and has direction-cosines proportional to (2, −1, 3).** [*P. U. 1937 S*]

3. **Standard form.** Find the Cartesian equation of the right circular cylinder whose axis is OZ and radius *a*. [*P(P). U. 1956S*]

[**N.B.** The equation of the right circular cylinder, whose axis is the z-axis and radius a, is $\mathbf{x^2+y^2=a^2}$.

****Note. Standard form.** The equation $x^2+y^2=a^2$ is the *simplest* form of the equation of a right circular cylinder, and may be called the **standard form.**]

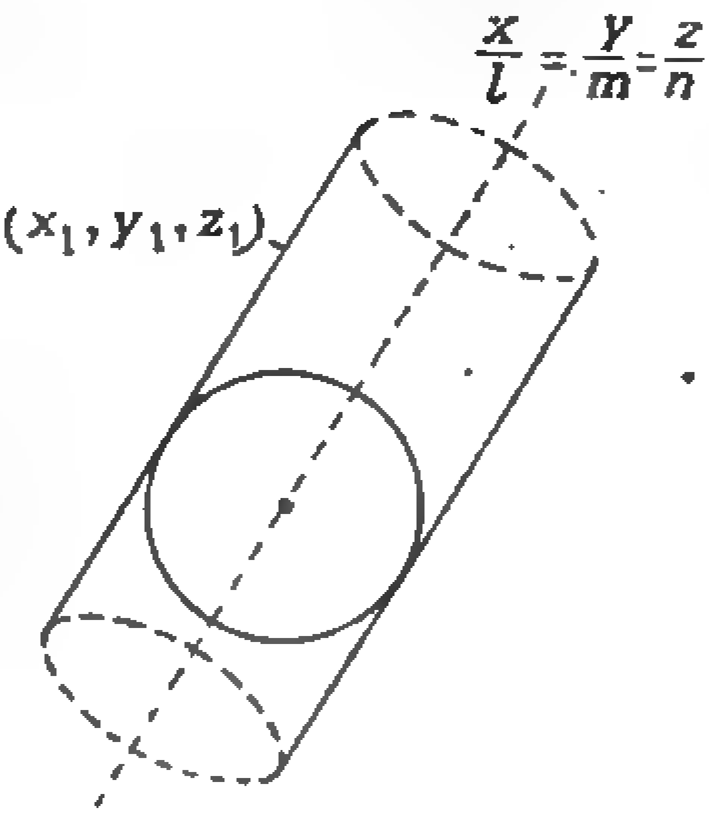
Enveloping cylinder.

106. Enveloping cylinder. Def. *The locus of the tangents to a sphere (or conicoid), which are parallel to a given line, is a cylinder called the* **enveloping cylinder.**

107. (*a*) *Equation of the enveloping cylinder of a sphere.* **To find the equation of the enveloping cylinder of the sphere $\mathbf{x^2+y^2+z^2=a^2}$, whose generators are parallel to the line**

$$\frac{\mathbf{x}}{\mathbf{l}} = \frac{\mathbf{y}}{\mathbf{m}} = \frac{\mathbf{z}}{\mathbf{n}}.$$

* How to write this step. See Note in Ex. 1, Art. 91.



The equation of the sphere is

$$x^2+y^2+z^2=a^2 \quad ...(1)$$

and the equations of the given line are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(2)$$

Let (x_1, y_1, z_1) *be any pt. on a tangent* $\parallel$ *to the line* (2). [*Note this step*]

Then the equations of the tangent are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Any pt. on this line is

$$(x_1+lr, y_1+mr, z_1+nr).$$

If it lies on the sphere (1), then

$$(x_1+lr)^2+(y_1+mr)^2+(z_1+nr)^2=a^2$$

or $r^2(l^2+m^2+n^2)+2r(lx_1+my_1+nz_1)+(x_1^2+y_1^2+z_1^2-a^2)=0 \quad ...(3)$

which is a quadratic in r.

$\because$ the line *touches* the sphere, the quadratic (3) has equal roots

$\therefore$ $4(lx_1+my_1+nz_1)^2=4(l^2+m^2+n^2)(x_1^2+y_1^2+z_1^2-a^2)$ [Cancel 4]

$\therefore$ the locus of (x_1, y_1, z_1) is [changing (x_1, y_1, z_1) to (x, y, z)],

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)$$

or $(x^2+y^2+z^2-a^2)(l^2+m^2+n^2)=(lx+my+nz)^2 \quad ...(4)$

which is the required equation of the enveloping cylinder.

****Abridged notation.** If $S=x^2+y^2+z^2-a^2$, so that $S=0$ is the equation of the sphere,

$s_1=l^2+m^2+n^2$, so that s_1 is the result of substituting the co-ordinates of the pt. (l,m,n) in S, *the constant term* $(-a^2)$ *being omitted,*

$t=xl+ym+zn$, so that $t=0$ is the equation of the tangent plane at (l, m, n), *the constant term* $(-a^2)$ *being omitted,*

then from (4), the equation of the enveloping cylinder is

$$\mathbf{Ss_1=t^2}.$$

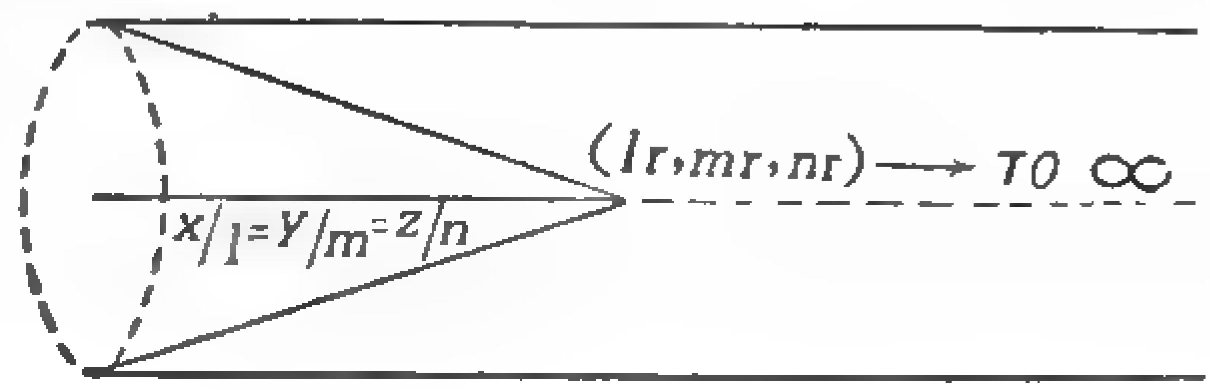
[Compare and contrast with the equation $(SS_1=T^2)$ of the enveloping cone (Art. 94).]

Caution. The above equation of the enveloping cylinder $(Ss_1=t^2)$ is applicable only if the equation of the sphere (or conicoid) is given *in the standard form, viz.,* $x^2+y^2+z^2=a^2$ (or $ax^2+by^2+cz^2=1$).

Thus it is *not* applicable if the equation of the sphere is $(x-a)^2+(y-b)^2+(z-c)^2=r^2$. (See Misc. Ex. 9, Chap. IX.)

Note. Important. Cylinder as the limiting form of cone.

A cylinder, whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, *is the limiting form of the cone, whose vertex is* (lr, mr, nr), *when* $r\to\infty$.



107. (*b*) *Equation of the enveloping cylinder of a sphere.* **To deduce from the equation of the enveloping cone of the sphere** $\mathbf{x^2+y^2+z^2=a^2}$, **the equation of the enveloping cylinder whose generators are parallel to the line** $\frac{\mathbf{x}}{\mathbf{l}}=\frac{\mathbf{y}}{\mathbf{m}}=\frac{\mathbf{z}}{\mathbf{n}}$.

The equation of the sphere is $x^2+y^2+z^2=a^2$ ...(1)

[The enveloping cylinder, whose generators are to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$, is the limiting form of the enveloping cone, whose vertex is (lr, mr, nr), when $r\to\infty$.]

The equation of the enveloping cone of the sphere (1), whose vertex is (lr, mr, nr), is

$$(x^2+y^2+z^2-a^2)(l^2r^2+m^2r^2+n^2r^2-a^2)=(xlr+ymr+znr-a^2)^2$$

[$SS_1=T^2$ (Art. 94)]

Dividing thro' out by r^2, [*Note this step*]

$$(x^2+y^2+z^2-a^2)\left(l^2+m^2+n^2-\frac{a^2}{r^2}\right)=\left(lx+my+nz-\frac{a^2}{r}\right)^2.$$

Taking the limits when $r\to\infty$, so that $\frac{1}{r}\to 0$, $\frac{1}{r^2}\to 0$,

$$(x^2+y^2+z^2-a^2)(l^2+m^2+n^2)=(lx+my+nz)^2,$$

which is the required equation of the enveloping cylinder.

[**Rule to deduce from the equation of the enveloping cone of a sphere (or conicoid) the equation of the enveloping cylinder whose generators are parallel to the line** $\frac{\mathbf{x}}{\mathbf{l}}=\frac{\mathbf{y}}{\mathbf{m}}=\frac{\mathbf{z}}{\mathbf{n}}$.

(*i*) *Write down the equation of the enveloping cone of the sphere (or conicoid), whose vertex is* (lr, mr, nr).

(*ii*) *Divide thro' out by* r^2, *and take the limits when* $r\to\infty$, *so that* $\frac{1}{r}\to 0$, $\frac{1}{r^2}\to 0$.

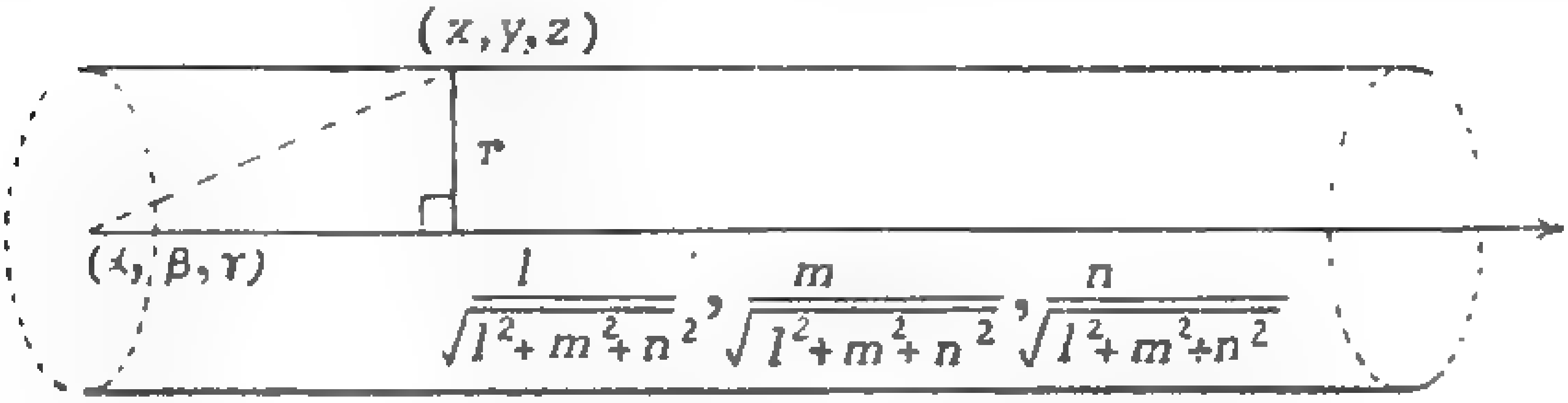
The resulting equation is the required equation of the enveloping cylinder.]

EXAMPLE

Find the equation of the enveloping cylinder of the sphere $x^2+y^2+z^2-2x+4y=1$, having its generators parallel to the line $x=y=z$. [*P. U. 1960 S*]

MISCELLANEOUS EXAMPLES ON CHAPTER IX

1. *Equation of a right circular cylinder. General form.* **Find the equation of the right circular cylinder whose radius is r and axis the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$.** [*P. U. 1960*]



The equations of the axis are $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ ...(1)

Let (x, y, z) be *any* pt. on the cylinder.

Then the $\perp$ distance of (x, y, z) from the axis (1)

$=r$ ...(2)

Now the distance between one pt. (α, β, γ) on the axis and the given pt. $(x, y, z)=\sqrt{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2}$.

The projection of this distance on the axis (1)

$$\left[\text{direction-cosines } \frac{l}{\sqrt{l^2+m^2+n^2}}, \frac{m}{\sqrt{l^2+m^2+n^2}}, \frac{n^*}{\sqrt{l^2+m^2+n^2}}\right]$$

$$=(x-\alpha)\frac{l}{\sqrt{l^2+m^2+n^2}}+(y-\beta)\frac{m}{\sqrt{l^2+m^2+n^2}}+(z-\gamma)\frac{n}{\sqrt{l^2+m^2+n^2}}$$

$[\,(x_2-x_1)l+(y_2-y_1)m+(z_2-z_1)n$ (Art. 14) $]$

$$=\frac{(x-\alpha)l+(y-\beta)m+(z-\gamma)n}{\sqrt{l^2+m^2+n^2}}.$$

$\therefore \quad \perp=\sqrt{(\text{distance})^2-(\text{projection})^2}$

$$=\left\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\left[\frac{(x-\alpha)l+(y-\beta)m+(z-\gamma)n}{\sqrt{l^2+m^2+n^2}}\right]^2\right\}^{\frac{1}{2}}.$$

$\therefore$ from (2),

$$\left\{(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\left[\frac{(x-\alpha)l+(y-\beta)m+(z-\gamma)n}{\sqrt{l^2+m^2+n^2}}\right]^2\right\}^{\frac{1}{2}}=r$$

or, squaring,

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\frac{[\,(x-\alpha)l+(y-\beta)m+(z-\gamma)n]^2}{l^2+m^2+n^2}=r^2$$

*See Note in Ex. 15, Art. 43, (c).

or, multiplying thro' out by $l^2+m^2+n^2$,

$$(l^2+m^2+n^2)\,[(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2]-[(x-\alpha)l+(y-\beta)m+(z-\gamma)n]^2 = (l^2+m^2+n^2)r^2,$$

which is the required equation.

2. Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2}=y-2=\frac{z-3}{2}$. [*J. & K.U. 1952*]

3. Find the equation of the right circular cylinder whose axis is $x=2y=-z$ and radius 4. [*P. U. 1957 S*]

4. Find the equation of the right circular cylinder whose guiding circle is $x^2+y^2+z^2=9$, $x-y+z=3$. [*P. U. 1959 S*]

[The axis of the cylinder is the line thro' the centre of the sphere $\perp$ to the plane, and
radius of the cylinder=radius of the circle.]

****5.** Obtain the equation of the right circular cylinder whose guiding curve is the circle through the points (1,0,0), (0,1,0), (0,0,1).
[*P. U. M. P. 1942*]

6. *Equation to cylinder with generators parallel to given line and given conic for base.* **Find the equation to the cylinder whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and base the conic $f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0$, $z=0$.**
[*P. U. 1952 S*]

The equations of the given line are $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ...(1)

and those of the conic are

$$ax^2+2hxy+by^2+2gx+2fy+c=0,\ z=0 \quad ...(2)$$

Let (x_1, y_1, z_1) be any pt. on a generator to the line (1).
[*Note this step*]

Then the equations of the generator are $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$.

It meets $z=0$, where $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{-z_1}{n}$

or $$x-x_1=-\frac{l}{n}z_1,\ y-y_1=-\frac{m}{n}z_1.$$

$$\therefore \quad x=x_1-\frac{l}{n}z_1,\quad y=y_1-\frac{m}{n}z_1.$$

Substituting these values of x, y in (2),

$$a\left(x_1-\frac{l}{n}z_1\right)^2+2h\left(x_1-\frac{l}{n}z_1\right)\left(y_1-\frac{m}{n}z_1\right)+b\left(y_1-\frac{m}{n}z_1\right)^2$$
$$+2g\left(x_1-\frac{l}{n}z_1\right)+2f\left(y_1-\frac{m}{n}z_1\right)+c=0.$$

$\therefore$ the locus of (x_1, y_1, z_1) is [changing (x_1, y_1, z_1) to (x, y, z)],

$$a\left(x-\frac{l}{n}z\right)^2+2h\left(x-\frac{l}{n}z\right)\left(y-\frac{m}{n}z\right)+b\left(y-\frac{m}{n}z\right)^2$$
$$+2g\left(x-\frac{l}{n}z\right)+2f\left(y-\frac{m}{n}z\right)+c=0$$

or, multiplying thro' out by n^2,

$$a(nx-lz)^2+2h(nx-lz)(ny-mz)+b(ny-mz)^2$$
$$+2gn(nx-lz)+2fn(ny-mz)+cn^2=0,$$

which is the required equation of the cylinder.

7. Find the equation of the surface generated by a straight line which is parallel to the line $y=mx$, $z=nx$, and intersects the ellipse

$$x^2/a^2+y^2/b^2=1, \quad z=0.$$

8. **Find the equation of the cylinder whose generators are parallel to the line $x=-\frac{y}{2}=\frac{z}{3}$ and whose guiding curve is the ellipse $x^2+2y^2=1$, $z=3$.** [*J. & K. U. 1955*]

9. Find the equation of the right circular cylinder which envelopes a sphere of centre (a, b, c) and radius r and has its generators parallel to the direction (l, m, n). [*P(P). U. 1947*]

10. **Locus of a line intersecting two given curves.** *A straight line is always parallel to the yz-plane and intersects the curves $x^2+y^2=a^2$, $z=0$, and $x^2=az$, $y=0$; prove that it generates the surface*

$$x^4y^2=(x^2-az)^2(a^2-x^2).$$ [*Ag. U. 1938*]

Let (x_1, y_1, z_1) be any pt. on the line, and let the equations of the line be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \dots (1)$$

$\because$ it is $\parallel$ to the yz-plane, *i.e.*, $x=0$

$\therefore$ it is $\perp$ to the normal to the plane, (direction-cosines 1, 0, 0),

$\therefore$ $l=0$.

Substituting this value of l in (1), the equations of the line become

$$\frac{x-x_1}{0}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \dots (2)$$

It intersects the curve $x^2+y^2=a^2$, $z=0$ ... (3)

The line (2) meets $z=0$, where $\frac{x-x_1}{0}=\frac{y-y_1}{m}=\frac{-z_1}{n}$

or $x-x_1=0$, $y-y_1=-\frac{m}{n}z_1$

or $x=x_1$, $y=y_1-\frac{m}{n}z_1$.

Substituting these values of x, y in (3),

$$x_1^2+\left(y_1-\frac{m}{n}z_1\right)^2=a^2 \quad ...(4)$$

Again the line (2) intersects the curve $x^2=az,\ y=0$ ... (5)

The line (2) meets $y=0$, where $\dfrac{x-x_1}{0}=\dfrac{-y_1}{m}=\dfrac{z-z_1}{n}$

or $x-x_1=0,\ z-z_1=-\dfrac{n}{m}y_1$,

or $x=x_1,\ z=z_1-\dfrac{n}{m}y_1$.

Substituting these values of x, z in (5),

$$x_1^2=a\left(z_1-\frac{n}{m}y_1\right) \quad ...(6)$$

Eliminating m, n from (4) and (6) [by substituting the value of $\dfrac{m}{n}$ from (6) in (4)],

$$x_1^2+\left(y_1-\frac{ay_1z_1}{az_1-x_1^2}\right)^2=a^2$$

or $$\frac{x_1^4y_1^2}{(az_1-x_1^2)^2}=a^2-x_1^2$$

or $$x_1^4y_1^2=(az_1-x_1^2)^2(a^2-x_1^2)$$

$$\frac{x_1^2}{a}=z_1-\frac{n}{m}y_1$$

or $$\frac{n}{m}y_1=z_1-\frac{x_1^2}{a}=\frac{az_1-x_1^2}{a}$$

or $$\frac{n}{m}=\frac{az_1-x_1^2}{ay_1}$$

or $$\frac{m}{n}=\frac{ay_1}{az_1-x_1^2}.$$

$\therefore$ the locus of (x_1, y_1, z_1) is [changing (x_1, y_1, z_1) to (x, y, z)]

$$x^4y^2=(az-x^2)^2(a^2-x^2)$$

or $$x^4y^2=(x^2-az)^2(a^2-x^2) \quad ...(7)$$

which is the required equation of the surface.

[**Check.** The equation of the surface (7) is satisfied by the equations of the given curves (3) and (5) thus :

Putting $z=0$ in (7), $x^4y^2=x^4(a^2-x^2)$ or, cancelling x^4, and transposing, $x^2+y^2=a^2$, which is true from (3).

Again, putting $y=0$ in (7), $0=(x^2-az)^2(a^2-x^2)$, or $x^2=az$, which is true from (5).]

CHAPTER X

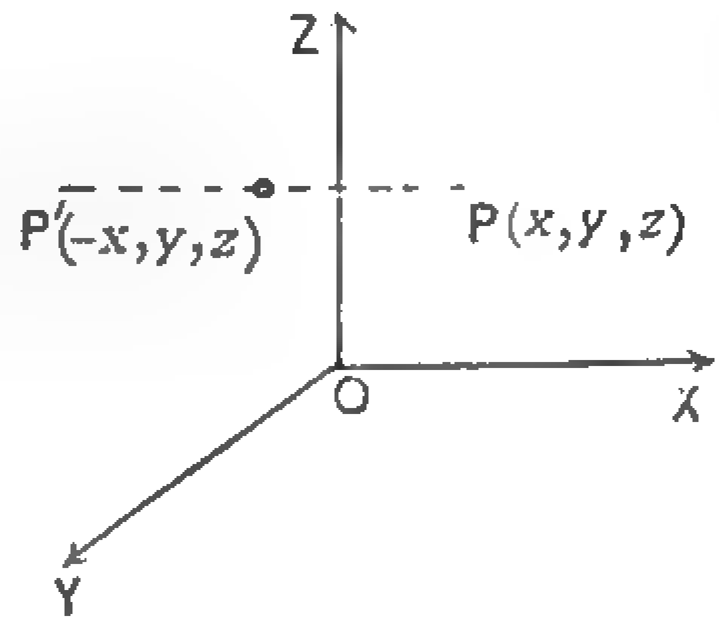
[☞ **This chapter should be omitted by the B. A. (Pass Course) students of the Punjab University.**]

TRACING THE LOCI OF NINE STANDARD EQUATIONS

Symmetry about a co-ordinate plane.

108. Rule to find whether a surface is symmetrical (*i*) about the yz-plane, (*ii*) about the zx-plane, (*iii*) about the xy-plane.

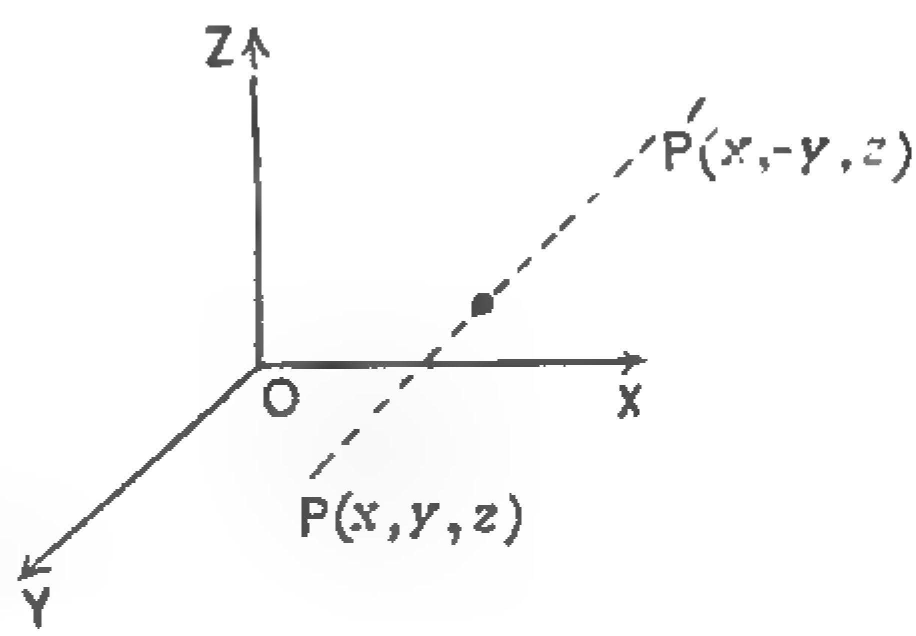
(*i*) **Symmetry about the yz-plane.**



If on changing x to −x, the equation remains unchanged, i.e., if only even powers of x occur in the equation, the surface is symmetrical about the yz-plane.*

Thus the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is symmetrical about the yz-plane.

(*ii*) **Symmetry about the zx-plane.**



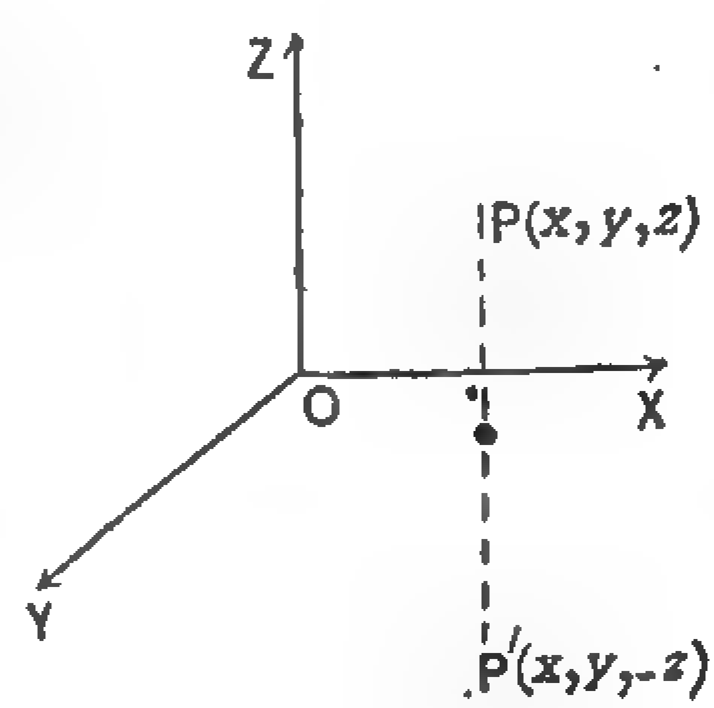
If on changing y to −y, the equation remains unchanged, i.e., if only even powers of y occur in the equation, the surface is symmetrical about the zx-plane.

Thus the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is symmetrical about the zx-plane.

(*iii*) **Symmetry about the xy-plane.**



If on changing z to −z, the equation remains unchanged, i.e., if only even powers of z occur in the equation, the surface is symmetrical about the xy-plane.

Thus the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is symmetrical about the xy-plane.

---

* *i.e., no odd powers of x occur.*

Central conicoids.

109. (a) **Ellipsoid.** *To trace the locus of the equation*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

The equation of the surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

(*i*) **Symmetry.** The surface is symmetrical about the yz-plane.

[(Art. 108, (*i*)), $\because$ only even powers of x occur in (1)]

It is symmetrical about the zx-plane.

[(Art. 108, (*ii*)), $\because$ only even powers of y occur in (1)]

It is symmetrical about the xy-plane.

[(Art. 108, (*iii*)), $\because$ only even powers of z occur in (1)]

(*ii*) **Sections by co-ordinate planes.** The surface meets the yz-plane where, putting $x=0$ in (1), $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which is an ellipse *in that plane.*

It meets the zx-plane where, putting $y=0$ in (1), $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$, which is an ellipse *in that plane.*

It meets the xy-plane where, putting $z=0$ in (1), $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is an ellipse *in that plane.*

(*iii*) **Axes-intersections.** The surface meets the x-axis where, putting $y=0$, $z=0$ in (1), $\frac{x^2}{a^2}=1$, or $x^2=a^2$, or $x=\pm a$, *i.e.*, in the pts. A$(a, 0, 0)$, A$'(-a, 0, 0)$.

It meets the y-axis where, putting $z=0$, $x=0$ in (1), $\frac{y^2}{b^2}=1$, or $y^2=b^2$, or $y=\pm b$, *i.e.*, in the pts. B$(0, b, 0)$, B$'(0, -b, 0)$.

It meets the z-axis where, putting $x=0$, $y=0$ in (1), $\frac{z^2}{c^2}=1$, or $z^2=c^2$, or $z=\pm c$, *i.e.*, in the pts. C$(0, 0, c)$, C$'(0, 0, -c)$.

(*iv*) **Generated by a variable curve.** The surface meets the plane $z=k$ where [putting $z=k$ in (1)],

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{k^2}{c^2}=1, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2}.$$

$\therefore$ the surface is generated by a variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2},\ z=k \quad ...(2) \quad [\ k \text{ varies }]$$

whose plane is $\parallel$ to the xy-plane, and centre $(0, 0, k)$ moves on the z-axis.

The ellipse (2) is real only if $1-\frac{k^2}{c^2}$ is +ve, or $\frac{k^2}{c^2}<1$, or $k^2<c^2$, or k is numerically $<c$, *i.e.*, k lies between c and $-c$.

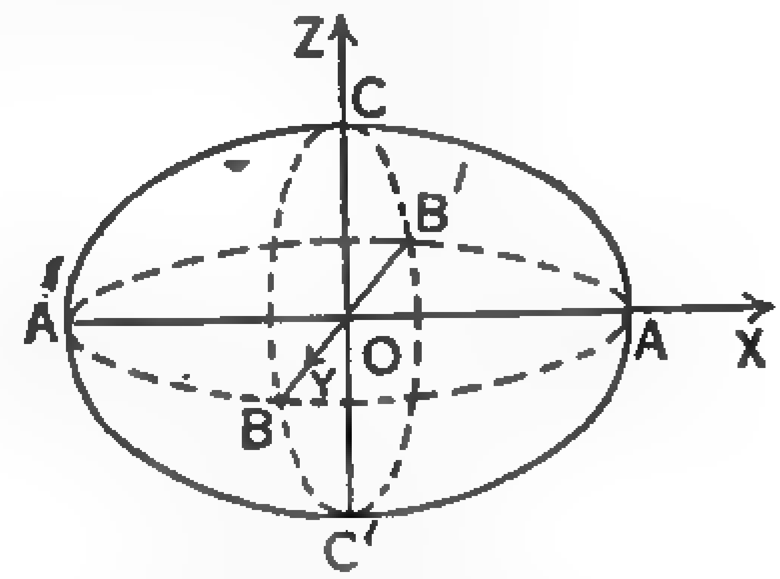
∴ the surface lies between the planes $z=c$ and $z=-c$.

Similarly it lies between the planes $x=a$ and $x=-a$, and between the planes $y=b$ and $y=-b$.

∴ the surface is limited in every direction.

Hence the shape of the surface is that shown in the Fig.*

It is called an **ellipsoid**.†
(Its shape is like that of an egg.)



[**Aid to memory.** Remember the word 'SAG', *S* is the first letter of the word *Symmetry*, and of *Sections by co-ordinate planes*, *A* of *Axes-intersections*, and *G* of *Generated by a variable curve*. The same Aid to memory will be useful for Arts. 109 (*b*), 109 (*c*), 109 (*d*), 111 (*a*), and 111 (*b*).

Note. Virtual ellipsoid. Def. The surface, whose equation is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$ ...(1), is called a **virtual ellipsoid.**

Every pt. on it is *imaginary*. For, from (1), x, y, z cannot all be real [∵ if x, y, z are all real, then L.H.S. of (1) is +ve, and R.H.S. is −ve, which is impossible].

Its equation resembles that of the *ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Hence it is also called an **imaginary ellipsoid.**

109. (*b*) **Hyperboloid of one sheet.** *To trace the locus of the equation* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

***How to draw the Fig.** Draw two ⊥ lines OX, OZ.

(*i*) On the x-axis take two pts. A, A′ on opposite sides of O, such that OA=OA′=a.

On the z-axis take two pts. C, C′ on opposite sides of O, such that OC=OC′=c.

In the zx-plane draw an ellipse ACA′C′.

(*ii*) Thro' O draw B′OB, the y-axis, and on it take two pts. B, B′ on opposite sides of O, such that OB=OB′=b.

In the xy-plane draw an ellipse AB′A′B.

(*iii*) In the yz-plane draw an ellipse B′CBC′.

†So called, ∵ all plane sections of the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ are *ellipses*.

The equation of the surface is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ ...(1)

(*i*) **Symmetry.** The surface is symmetrical about the yz-plane.

[(Art. 108, (*i*)) $\because$ only even powers of x occur in (1)]

It is symmetrical about the zx-plane.

[(Art. 108, (*ii*)) $\because$ only even powers of y occur in (1)]

It is symmetrical about the xy-plane.

[(Art. 108, (*iii*)) $\because$ only even powers of z occur in (1)]

(*ii*) **Sections by co-ordinate planes.** The surface meets the yz-plane where, putting $x=0$ in (1), $\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, which is a hyperbola *in that plane.*

It meets the zx-plane where, putting $y=0$ in (1), $\dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}=1$, which is a hyperbola *in that plane.*

It meets the xy-plane where, putting $z=0$ in (1), $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, which is an ellipse *in that plane.*

(*iii*) **Axes-intersections.** The surface meets the x-axis where, putting $y=0$, $z=0$ in (1), $\dfrac{x^2}{a^2}=1$, or $x^2=a^2$, or $x=\pm a$, *i.e.*, in the pts. A$(a, 0, 0)$, A$'(-a, 0, 0)$.

It meets the y-axis where, putting $z=0$, $x=0$ in (1), $\dfrac{y^2}{b^2}=1$, or $y^2=b^2$, or $y=\pm b$, *i.e.*, in the pts. B$(0, b, 0)$, B$'(0, -b, 0)$.

It meets the z-axis where, putting $x=0, y=0$ in (1), $-\dfrac{z^2}{c^2}=1$, or $z^2=-c^2$, or $z=\pm ic$, *i.e.*, in imaginary pts.

(*iv*) **Generated by a variable curve.** The surface meets the plane $z=k$ where [putting $z=k$ in (1)],

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{k^2}{c^2}=1, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}.$$

$\therefore$ the surface is generated by a variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2},\ z=k \ \dots (2) \quad [\,k \text{ varies}\,]$$

whose plane is $\parallel$ to the xy-plane, and centre $(0, 0, k)$ moves on the z-axis.

The ellipse (2) is real whether k is +ve or −ve, and its semi-axes

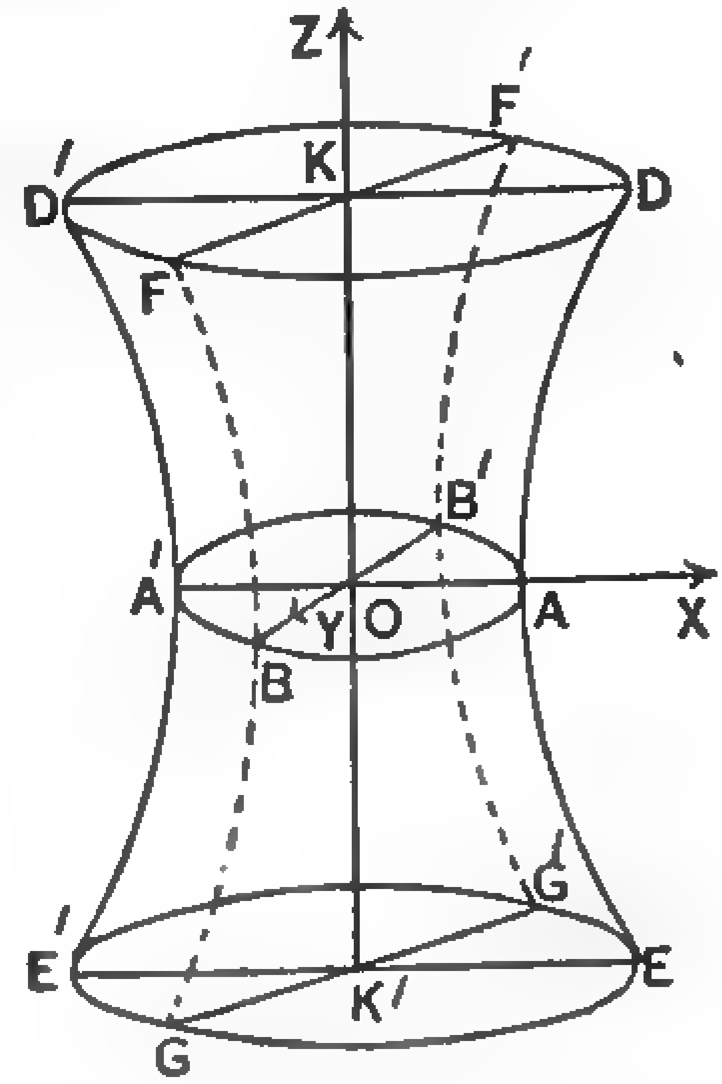
$a\sqrt{1+\dfrac{k^2}{c^2}}$, $b\sqrt{1+\dfrac{k^2}{c^2}}$

increase as k increases, and $\to\infty$ as $k\to\infty$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2}$$

$$\text{or } \frac{x^2}{a^2\left(1+\dfrac{k^2}{c^2}\right)}+\frac{y^2}{b^2\left(1+\dfrac{k^2}{c^2}\right)}=1$$

$\therefore$ the surface extends to infinity on both sides of the xy-plane.



Hence the shape of the surface is that shown in the Fig.*

It is called a **hyperboloid of one sheet.**

(Notice *one* $-ve$ *sign* in the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.)

(Its shape is like that of a 'murha' (मूढ़ा) with elliptic ends.)

109. (*c*) **Hyperboloid of two sheets.** *To trace the locus of the equation* $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

The equation of the surface is $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ ...(1)

(*i*) **Symmetry.** The surface is symmetrical about the yz-plane.
[(Art. 108, (*i*)) $\because$ only even powers of x occur in (1)]
It is symmetrical about the zx-plane.
[(Art. 108, (*ii*)) $\because$ only even powers of y occur in (1)]
It is symmetrical about the xy-plane.
[(Art. 108, (*iii*)) $\because$ only even powers of z occur in (1)]

(*ii*) **Sections by co-ordinate planes.** The surface meets the yz-plane where, putting $x=0$ in (1), $-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, or $\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$, which is an imaginary† ellipse *in that plane.*

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*How to draw the Fig. Draw two $\perp$ lines OX, OZ.

(*i*) On the x-axis take two pts. A, A′ on opposite sides of O, such that OA = OA′ = a.

In the zx-plane draw a hyperbola DAE, D′A′E′ (equal branches).

(*ii*) Thro' O draw B′OB, the y-axis, and on it take two pts. B, B′ on opposite sides of O, such that OB = OB′ = b.

In the xy-plane draw an ellipse AB′A′B.

(*iii*) Join DD′, EE′ to meet the z-axis in K, K′.

Thro' K, K′ draw F′KF, G′K′G $\parallel$ to B′OB, such that KF = KF′ = K′G = K′G′.

In a plane $\parallel$ to the xy-plane and above it draw an ellipse DF′D′F.

(*Continued on the next page*)

It meets the zx-plane where, putting $y=0$ in (1),
$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$, which is a hyperbola *in that plane.*

It meets the xy-plane where, putting $z=0$ in (1),
$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, which is a hyperbola *in that plane.*

(*iii*) **Axes-intersections.** The surface meets the x-axis where, putting $y=0, z=0$ in (1), $\frac{x^2}{a^2}=1$, or $x^2=a^2$, or $x=\pm a$, *i.e.*, in the pts. $A(a, 0, 0)$, $A'(-a, 0, 0)$.

It meets the y-axis where, putting $z=0$, $x=0$ in (1),
$-\frac{y^2}{b^2}=1$, or $y^2=-b^2$, or $y=\pm ib$, *i.e.*, in imaginary pts.

It meets the z-axis where, putting $x=0$, $y=0$ in (1),
$-\frac{z^2}{c^2}=1$, or $z^2=-c^2$, or $z=\pm ic$, *i.e.*, in imaginary pts.

(*iv*) **Generated by a variable curve.** The surface meets the plane $x=k$* where [putting $x=k$ in (1)],

$$\frac{k^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \text{ or } \frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1.$$

∴ the surface is generated by a variable ellipse

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1, \; x=k \; ...(2) \quad [k \text{ varies}]$$

whose plane is ∥ to the yz-plane, and centre $(k, 0, 0)$ moves on the x-axis.

The ellipse (2) is real only if $\frac{k^2}{a^2}-1$ is +ve, or $\frac{k^2}{a^2}>1$, or $k^2>a^2$,

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(*Continued from the last page*)

Again in a plane ∥ to the xy-plane and below it draw an ellipse EG'E'G.
(*iv*) In the yz-plane draw a hyperbola F'B'G', FBG.

† **Virtual ellipse.** Def. The curve, whose equation, in Analytical Plane Geometry, is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1$ ...(1), is called a **virtual ellipse.**

Every point on it is *imaginary.* For, from (1), x and y cannot both be real [∵ if x and y are both real, then L.H.S. of (1) is +ve, and R.H.S. is −ve, which is impossible].

Its equation resembles that of the *ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Hence it is also called an **imaginary ellipse.**

*__Why this step.__ In the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, the coefficients of y^2, z^2 are both −ve, and ∴ by transposition, both become +ve, so in order to get an *ellipse*, we consider the plane $x=k$ (and *not* $z=k$ as in Art. 109, (*b*)).

or k is numerically $>a$, *i.e.*, k does not lie between a and $-a$.

$\therefore$ no portion of the surface lies between the planes $x=a$ and $x=-a$.

The semi-axes of the ellipse (2), *viz.*,

$$b\sqrt{\frac{k^2}{a^2}-1},\ c\sqrt{\frac{k^2}{a^2}-1}$$

increase as k (numerically $>a$) increases, and $\to\infty$ as $k\to\infty$.

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1$$

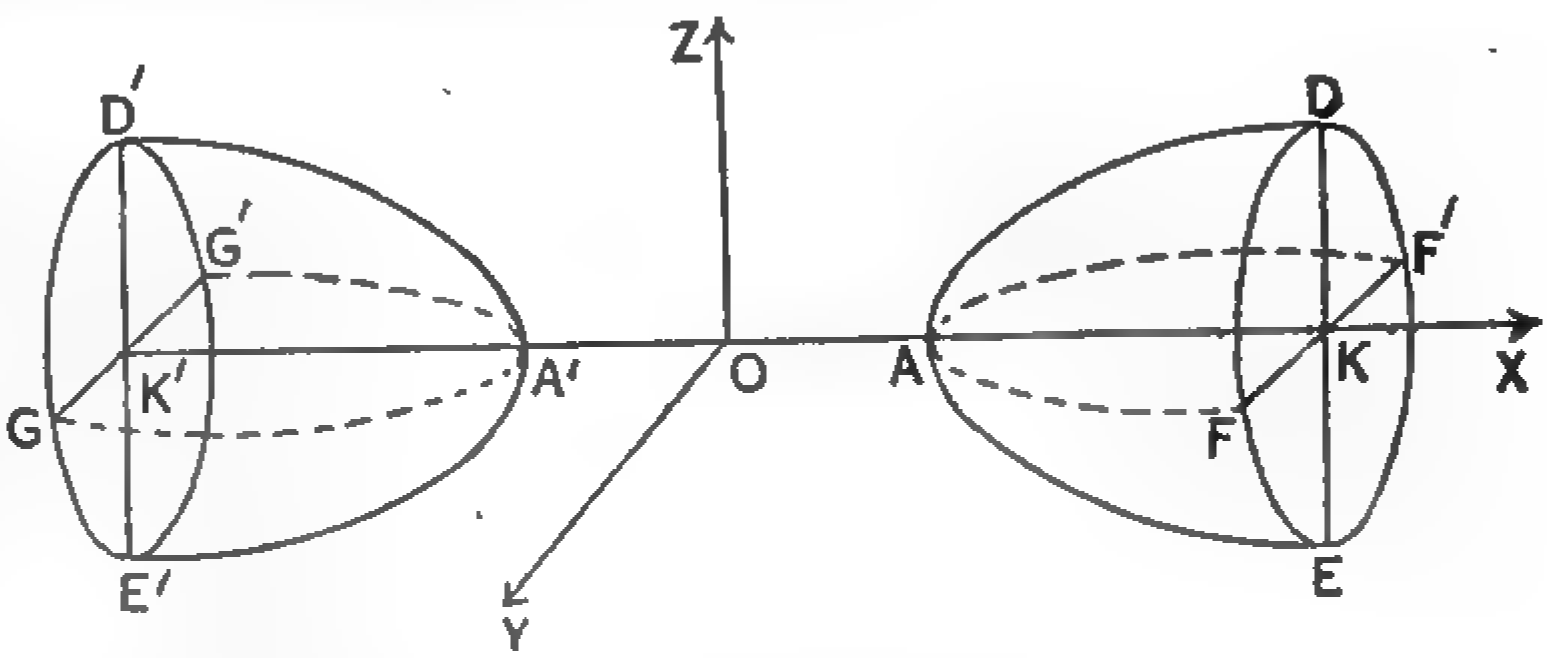
$$\text{or } \frac{y^2}{b^2\left(\frac{k^2}{a^2}-1\right)}+\frac{z^2}{c^2\left(\frac{k^2}{a^2}-1\right)}=1$$

$\therefore$ the surface extends to infinity on both sides of the yz-plane (to the right of A and to the left of A′).

Hence the shape of the surface is that shown in the Fig.*

It consists of two detached portions.

It is called a **hyperboloid of two sheets.**



(Notice *two* $-$*ve signs* in the equation $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.)

(Its shape is like that of two 'tabla's' (तबले) placed as shown in the Fig.)

109. (*d*) **Cone.** *To trace the locus of the equation*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0.$$

The equation of the surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ ...(1)

(*i*) **Symmetry.** The surface s symmetrical about the yz-plane. [(Art. 108, (*i*)) $\because$ only even powers of x occur in (1)]

***How to draw the Fig.** Draw two $\perp$ lines OX, OZ.

(*i*) On the x-axis take two pts. A, A′ on opposite sides of O, such that OA=OA′=a.

In the zx-plane draw a hyperbola DAE, D′A′E′ (equal branches).

(*ii*) Join DE, D′E′ to meet the x-axis in K, K′ Thro' O draw OY, the y-axis. Thro' K, K′ draw F′KF, G′K′G $\parallel$ to the y-axis, such that

KF=KF′=K′G=K′G′.

In a plane $\parallel$ to the yz-plane and to its right draw an ellipse F′DFE.

Again in a plane $\parallel$ to the yz-plane and to its left draw an ellipse G′D′GE′.

(*iii*) In the xy-plane draw a hyperbola F′AF, G′A′G.

It is symmetrical about the zx-plane.

[(Art. 108, (*ii*)) $\because$ only even powers of y occur in (1)]

It is symmetrical about the xy-plane.

[(Art. 108, (*iii*)) $\because$ only even powers of z occur in (1)]

(*ii*) **Sections by co-ordinate planes.** The surface meets the yz-plane where, putting $x=0$ in (1), $\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$, or $\frac{y^2}{b^2}=\frac{z^2}{c^2}$,

or $y^2=\frac{b^2}{c^2}z^2$, or $y=\pm\frac{b}{c}z$, which are two st. lines *in that plane* (on opposite sides of the z-axis and making equal angles with it).

It meets the zx-plane where, putting $y=0$ in (1), $\frac{x^2}{a^2}-\frac{z^2}{c^2}=0$, or $\frac{x^2}{a^2}=\frac{z^2}{c^2}$, or $x^2=\frac{a^2}{c^2}z^2$, or $x=\pm\frac{a}{c}z$, which are two st. lines *in that plane* (on opposite sides of the z-axis and making equal angles with it).

It meets the xy-plane where, putting $z=0$ in (1), $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$, which is a pt. ellipse *in that plane.*

(*iii*) **Axes-intersections.** The surface meets the x-axis where, putting $y=0, z=0$ in (1), $\frac{x^2}{a^2}=0$, or $x^2=0$, or $x=0, 0$, *i.e.*, in two coincident pts.

It meets the y-axis where, putting $z=0, x=0$ in (1), $\frac{y^2}{b^2}=0$, or $y^2=0$, or $y=0, 0$, *i.e.*, in two coincident pts.

It meets the z-axis where, putting $x=0, y=0$ in (1), $-\frac{z^2}{c^2}=0$, or $z^2=0$, or $z=0, 0$, *i.e.*, in two coincident pts.

(*iv*) **Generated by a variable curve.** The surface meets the plane $z=k$ where [putting $z=k$ in (1)],

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{k^2}{c^2}=0, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}.$$

$\therefore$ the surface is generated by a variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2},\ z=k \quad ...(2) \quad [k \text{ varies}]$$

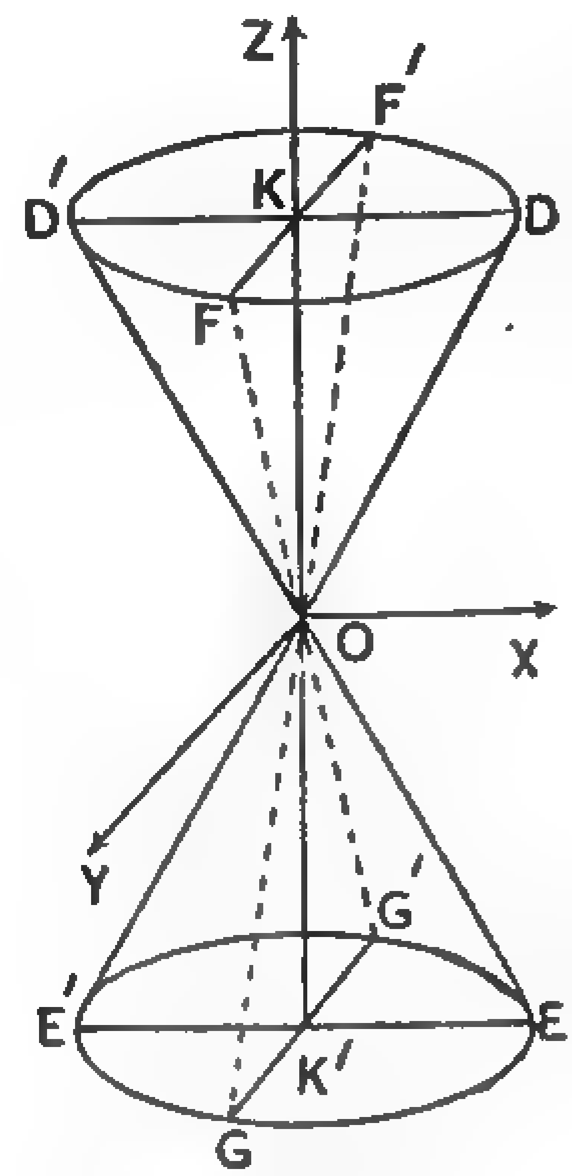
whose plane is to the xy-plane, and centre $(0, 0, k)$ moves on the z-axis.

The ellipse (2) is real whether k is +ve or −ve, and its semi-axes $\frac{ak}{c}, \frac{bk}{c}$ increase as k(+ve) increases, and $\to\infty$ as $k\to\infty$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$$

or $$\frac{x^2}{\frac{a^2k^2}{c^2}}+\frac{y^2}{\frac{b^2k^2}{c^2}}=1$$

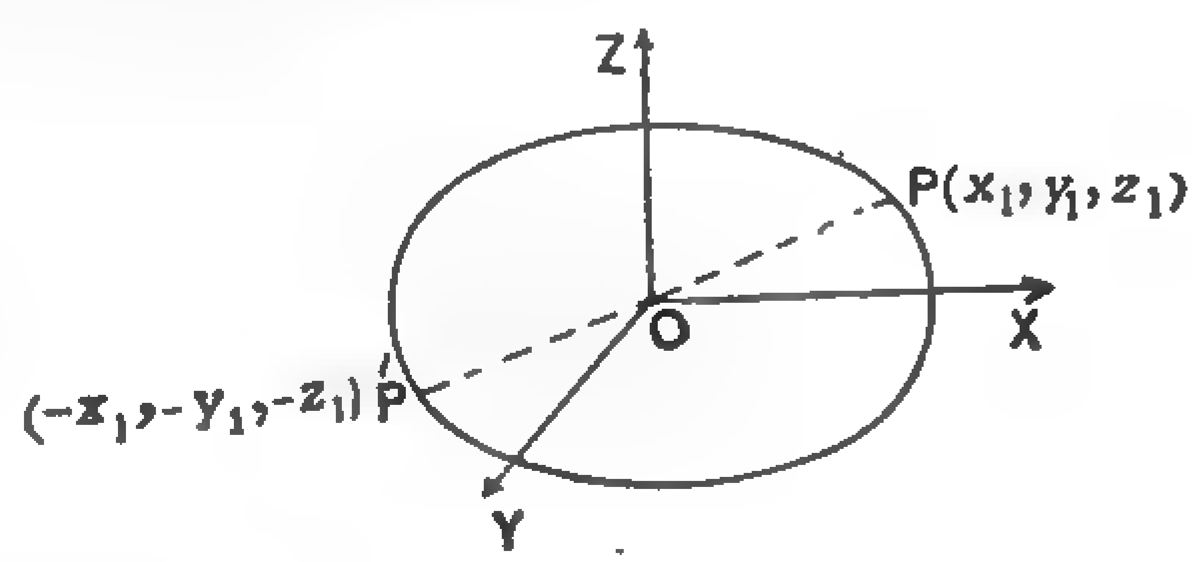
$\therefore$ the surface extends to infinity on both sides of the xy-plane.



Hence the shape of the surface is that shown in the Fig.*

It is called a **cone.**

110. Centre. *To prove that the origin bisects all chords of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, *which pass through it.*



The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots(1)$$

Let P (x_1, y_1, z_1) be *any* pt. on the ellipsoid (1). Then

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1 \quad \ldots(2)$$

Substituting the co-ordinates of the pt. P′$(-x_1, -y_1, -z_1)$ in

---

***How to draw the Fig.** Draw two $\perp$ lines OX, OZ.

(*i*) In the zx-plane draw two st. lines DOE′, D′OE on opposite sides of the z-axis and making equal angles with it, such that OD=OE′=OD′=OE.

(*ii*) Join DD′, EE′ to meet the z-axis in K, K′. Thro' O draw OY, the y-axis. Thro' K, K′ draw F′KF, G′K′G $\parallel$ to the y-axis, such that

KF=KF′=K′G=K′G′.

In a plane $\parallel$ to the xy-plane and above it draw an ellipse DF′D′F.

Again in a plane $\parallel$ to the xy-plane and below it draw an ellipse EG′E′G.

(*iii*) In the yz-plane draw two st. lines F′OG, FOG′.

(1), we get

$$\frac{(-x_1)^2}{a^2}+\frac{(-y_1)^2}{b^2}+\frac{(-z_1)^2}{c^2}=1$$

or $\quad \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}=1$, which is true from (2).

$\therefore$ P′ also lies on the ellipsoid.

Now the mid-pt. of PP′ is $\left[\dfrac{x_1+(-x_1)}{2}, \dfrac{y_1+(-y_1)}{2}, \dfrac{z_1+(-z_1)}{2}\right]$, or (0, 0, 0), *i.e.*, the origin.

$\therefore$ the origin bisects *all* chords of the ellipsoid, which pass thro' it.

Note 1. The origin is called the **centre** of the ellipsoid, and the ellipsoid is called a **central conicoid.**

Similarly the origin is the **centre** of the hyperboloid of one sheet $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, and of the hyperboloid of two sheets $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, which are also **central conicoids.**

Note 2. The equations of an ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, a hyperboloid of one sheet $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, and a hyperboloid of two sheets $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, are all included in the equation

$$ax^2+by^2+cz^2=1,$$

$\therefore$ the equation of a **central conicoid** is $\mathbf{ax^2+by^2+cz^2=1}$.

Note 3. Standard form. The equation $ax^2+by^2+cz^2=1$ is the *simplest* form of the equation of a central conicoid, and may be called the **standard form.**

EXAMPLES

1. Find the equation of the cone whose vertex is at the centre of an ellipsoid and which passes through all the points of intersection of (*i*) the ellipsoid and a given plane ;
(*ii*) the ellipsoid and a concentric sphere.

2. CP, CQ, CR are three central radii of an ellipsoid which are mutually at right angles to one another, show that the plane PQR touches a sphere. [*Ag. U. 1952*]

Paraboloids.

111. (*a*) **Elliptic paraboloid.** *To trace the locus of the equation*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}.$$

The equation of the surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$ ...(1)

(*i*) **Symmetry.** The surface is symmetrical about the yz-plane.

[(Art. 108, (*i*)) $\because$ only even powers of x occur in (1)]

It is symmetrical about the zx-plane.

[(Art. 108, (*ii*)) $\because$ only even powers of y occur in (1)]

(*ii*) **Sections by co-ordinate planes.** The surface meets the yz-plane where, putting $x=0$ in (1), $\frac{y^2}{b^2}=\frac{2z}{c}$, or $y^2=\frac{2b^2}{c}z$, which is an upward parabola *in that plane.* (*Assume that c is +ve.*)

It meets the zx-plane where, putting $y=0$ in (1), $\frac{x^2}{a^2}=\frac{2z}{c}$, or $x^2=\frac{2a^2}{c}z$, which is an upward parabola *in that plane.*

It meets the xy-plane where, putting $z=0$ in (1), $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$, which is a pt. ellipse *in that plane.*

(*iii*) **Axes-intersections.** The surface meets the x-axis where, putting $y=0$, $z=0$ in (1), $\frac{x^2}{a^2}=0$, or $x^2=0$, or $x=0, 0$, *i.e.*, in two coincident pts., $\therefore$ the surface *touches* the x-axis at $O(0, 0, 0)$.

It meets the y-axis where, putting $z=0$, $x=0$ in (1), $\frac{y^2}{b^2}=0$, or $y^2=0$, or $y=0, 0$, *i.e.*, in two coincident pts., $\therefore$ the surface *touches* the y-axis at $O(0, 0, 0)$.

$\therefore$ the surface *touches* the xy-plane at O.

It meets the z-axis where, putting $x=0, y=0$ in (1), $0=\frac{2z}{c}$, or $z=0$, *i.e.*, in the pt. $O(0, 0, 0)$.

(*iv*) **Generated by a variable curve.** The surface meets the plane $z=k$ where [putting $z=k$ in (1)], $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c}$.

$\therefore$ the surface is generated by a variable ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c},\ z=k \quad ...(2) \quad [k \text{ varies}]$$

whose plane is $\parallel$ to the xy-plane, and centre $(0, 0, k)$ moves on the z-axis.

The ellipse (2) is real only if k is +ve [$\because$ *c is +ve* (*See step* (*ii*))].

$\therefore$ the surface lies only above the xy-plane (*i.e.*, in the +ve direction of the z-axis).

The semi-axes of the ellipse (2), $a\sqrt{\frac{2k}{c}}$, $b\sqrt{\frac{2k}{c}}$ increase as k (+ve) increases, and $\to\infty$ as $k\to\infty$.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c}$$

or $$\frac{x^2}{\frac{2a^2k}{c}}+\frac{y^2}{\frac{2b^2k}{c}}=1$$

$\therefore$ the surface extends to infinity above the xy-plane (*i.e.*, in the +ve direction of the z-axis).

Hence the shape of the surface is that shown in the Fig.*

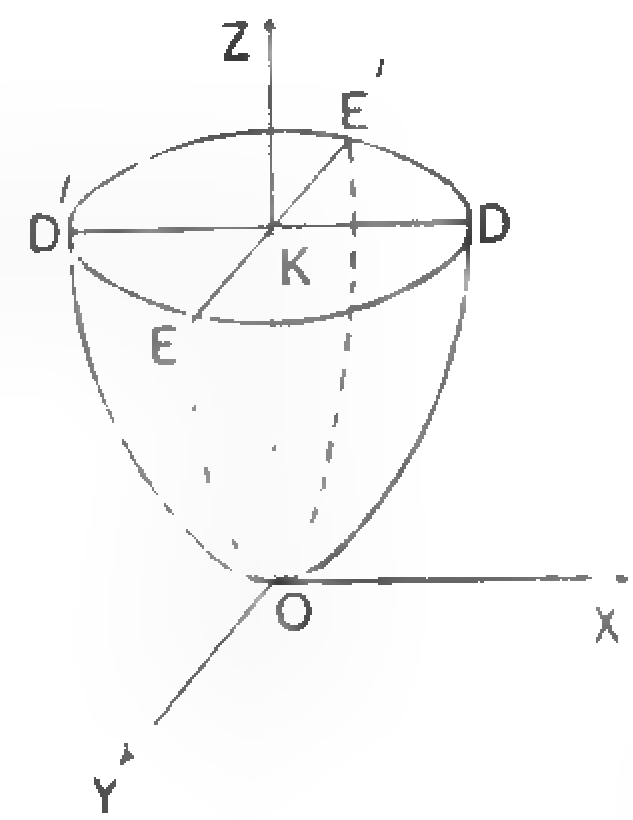
It is called an **elliptic paraboloid.**

(Its shape is like that of a 'tabla'' (तबला).)

Note. Why the name "elliptic" paraboloid. Since the paraboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=\dfrac{2z}{c}$ is generated by a variable *ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c},\ z=k \quad [\ k \text{ varies}\ ]$$

hence the name **"elliptic" paraboloid.**



111. (*b*) **Hyperbolic paraboloid.** *To trace the locus of the equation*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}.$$

The equation of the surface is $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=\dfrac{2z}{c}$ ...(1)

(*i*) **Symmetry.** The surface is symmetrical about the yz-plane.

[(Art. 108, (*i*)) $\because$ only even powers of x occur in (1)]

It is symmetrical about the zx-plane.

[(Art. 108, (*ii*)) $\because$ only even powers of y occur in (1)]

(*ii*) **Sections by co-ordinate planes.** The surface meets the yz-plane where, putting $x=0$ in (1), $-\dfrac{y^2}{b^2}=\dfrac{2z}{c}$, or $y^2=-\dfrac{2b^2}{c}z$, which is a downward parabola *in that plane.* (*Assume that* c *is* +*ve.*)

It meets the zx-plane where, putting $y=0$ in (1), $\dfrac{x^2}{a^2}=\dfrac{2z}{c}$, or $x^2=\dfrac{2a^2}{c}z$, which is an upward parabola *in that plane.*

It meets the xy-plane where, putting $z=0$ in (1),

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0, \text{ or } \frac{x^2}{a^2}=\frac{y^2}{b^2}, \text{ or } x^2=\frac{a^2}{b^2}y^2, \text{ or } x=\pm\frac{a}{b}y,$$

which are two st. lines (*not shown in the Fig.*) *in that plane.*

---

*How to draw the Fig. Draw two $\perp$ lines OX, OZ.

(*i*) In the zx-plane draw an upward parabola D'OD (arc OD = arc OD').

Thro' O draw OY, the y-axis.

(*ii*) Join DD' to meet the z-axis in K. Thro' K draw E'KE $\parallel$ to the y-axis, such that KE = KE'. In a plane $\parallel$ to the xy-plane and above it draw an ellipse DE'D'E.

(*iii*) In the yz-plane draw an upward parabola EOE'.

(*iii*) **Axes-intersections.** The surface meets the x-axis where, putting $y=0$, $z=0$ in (1), $\frac{x^2}{a^2}=0$, or $x^2=0$, or $x=0, 0$, *i.e.*, in two coincident pts.

$\therefore$ the surface *touches* the x-axis at the pt. O(0, 0, 0).

It meets the y-axis where, putting $z=0$, $x=0$ in (1),

$-\frac{y^2}{b^2}=0$, or $y^2=0$, or $y=0, 0$, *i.e.*, in two coincident pts.

$\therefore$ the surface *touches* the y-axis at the pt. O(0, 0, 0).

$\therefore$ the surface touches the xy-plane at O.

It meets the z-axis where, putting $x=0$, $y=0$ in (1),

$0=\frac{2z}{c}$, or $z=0$, *i.e.*, in the pt. O (0, 0, 0).

(*iv*) **Generated by a variable curve.** The surface meets the plane $z=k$ where [putting $z=k$ in (1)], $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c}$.

$\therefore$ the surface is generated by a variable hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c},\ z=k \quad ...(2) \quad [k \text{ varies}]$$

whose plane is $\parallel$ to the xy-plane, and centre $(0, 0, k)$ moves on the z-axis.

The hyperbola (2) has its transverse axis $\parallel$ to the x-axis if k is +ve, and $\parallel$ to the y-axis if k is $-$ve [$\because$ *c is +ve (See step (ii))*].

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c}$$

$$\text{or } \frac{x^2}{\frac{2a^2k}{c}}-\frac{y^2}{\frac{2b^2k}{c}}=1$$

The transverse semi-axis $a\sqrt{\frac{2k}{c}}$ increases as k (+ve) increases, and $\to\infty$ as $k\to\infty$.

$\therefore$ the surface extends to infinity above the xy-plane.

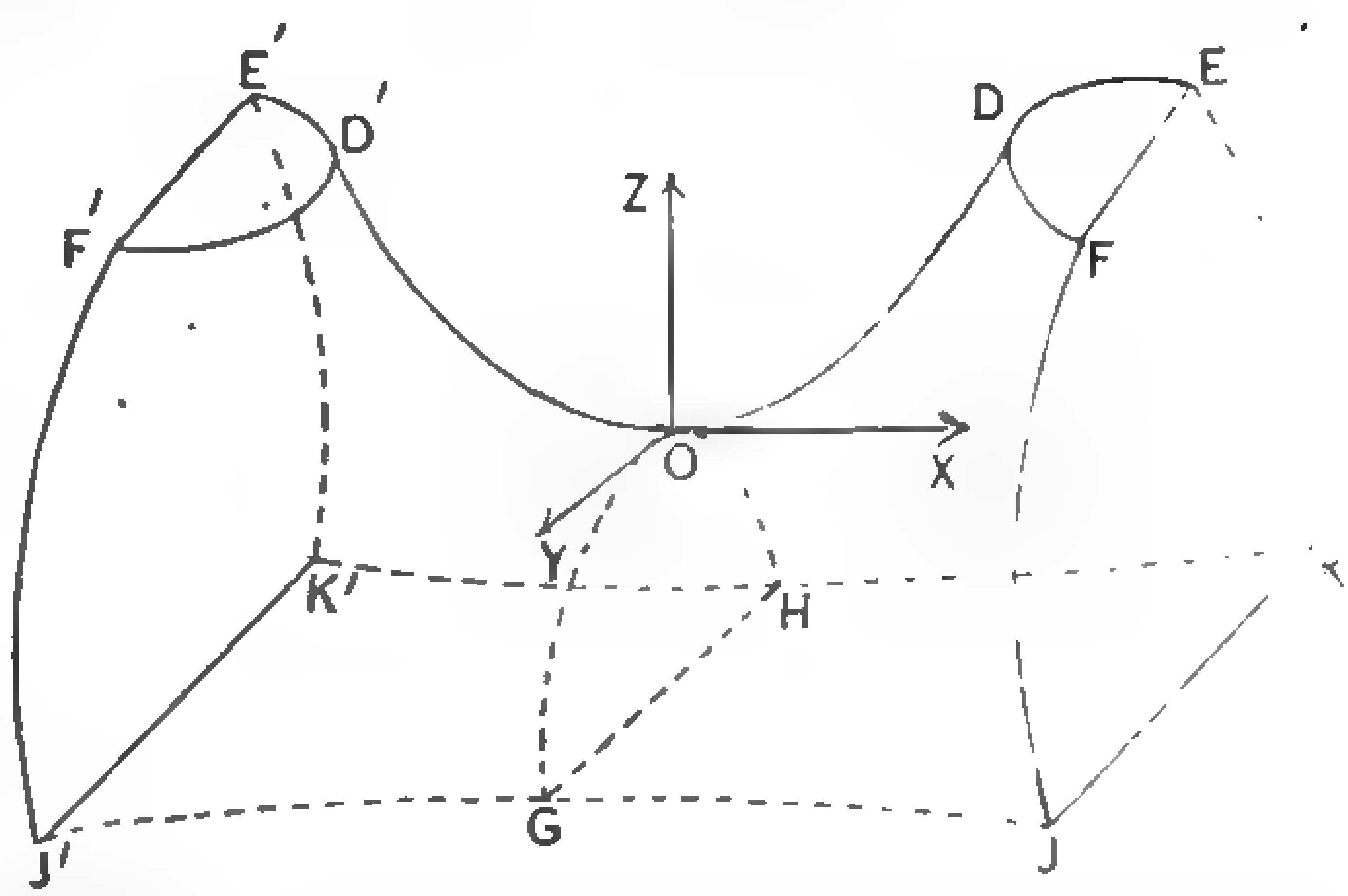
Similarly the surface extends to infinity below the xy-plane.

$\therefore$ the surface extends to infinity on both sides of the xy-plane.

Hence the shape of the surface is that shown in the Fig.*

It is called a **hyperbolic paraboloid.**

(Its shape is like that of a saddle.)



Note 1. Why the name "hyperbolic" paraboloid. Since the paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$ is generated by a variable *hyperbola*, $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c}$, $z=k$ [k varies], hence the name **"hyperbolic" paraboloid.**

Note 2. The equations of an elliptic paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$, and a hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$, are both included in the equation $ax^2+by^2=2z$,

$\therefore$ the equation of a paraboloid is $\mathbf{ax^2+by^2=2z}$.

Note 3. Standard form. The equation $ax^2+by^2=2z$ is the *simplest* form of the equation of a paraboloid, and may be called the **standard form.**

EXAMPLE

(*a*) What surface is represented by the equation $xy=cz$?

(*b*) A variable line intersects the x-axis, and the curve $x=y$, $y^2=cz$, and is parallel to the yz-plane; show that it generates the paraboloid

$$xy = cz.$$

***How to draw the Fig.** Draw two $\perp$ lines OX, OZ.

(*i*) In the zx-plane draw an upward parabola D'OD (arc OD = arc OD'). Thro' O draw OY, the y-axis. In a plane $\parallel$ to the xy-plane and above it thro' D, D' draw a hyperbola EDF, E'D'F' (equal branches), such that EF, E'F' are to the y-axis, and EF = E'F'.

(*ii*) In the yz-plane, thro' O draw a downward parabola GOH, such that HG is $\parallel$ to the y-axis.

(*iii*) In a plane $\parallel$ to the xy-plane and below it thro' G, H draw a hyperbola J'GJ, K'HK (transverse axis $\parallel$ to the y-axis), such that KJ, K'J' are $\parallel$ to HG, and KJ = K'J'.

(*iv*) Join FJ, F'J' by parallel and equal curves, also join EK, E'K' by parallel and equal curves.

Cylinders.

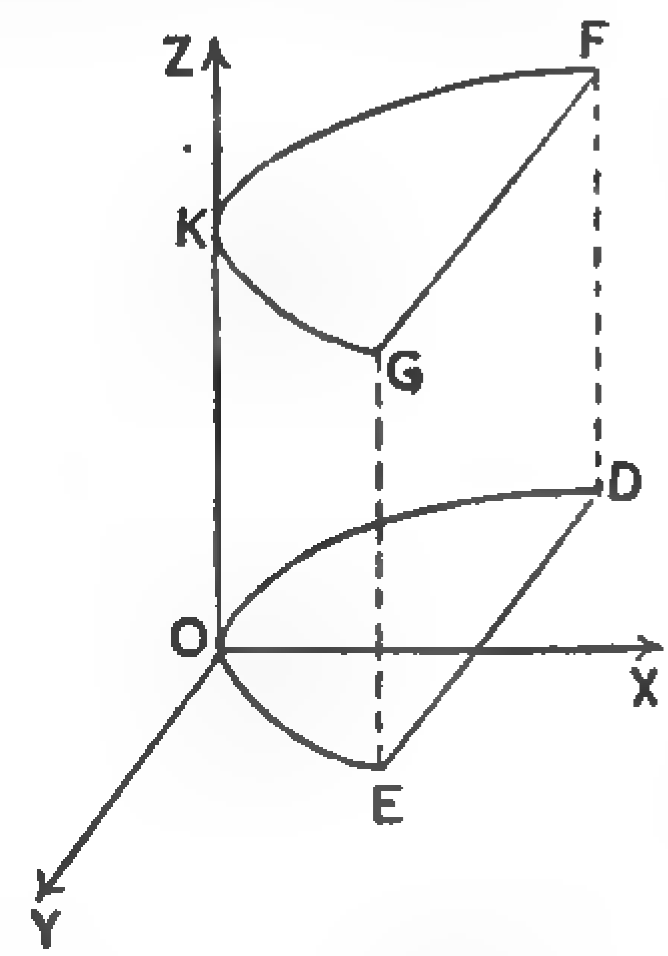
112. (*a*) **Parabolic cylinder.** *To trace the locus of the equation*

$$y^2 = 4ax.$$

The equation of the surface is $y^2=4ax$ ... (1)

Here z is absent.

$\therefore$ (1) represents a cylinder generated by a st. line which is $\parallel$ to the z-axis and intersects the curve whose equation *in the xy-plane* is $y^2=4ax$, *i.e.*, a parabola. [Art. 17]



Hence the shape of the surface is that shown in the Fig. *

It is called a **parabolic cylinder.**

112. (*b*) **Elliptic cylinder.** *To trace the locus of the equation*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The equation of the surface is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ... (1)

Here z is absent.

$\therefore$ (1) represents a cylinder generated by a st. line which is $\parallel$ to the z-axis and intersects the curve whose equation *in the xy-plane* is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, *i.e.*, an ellipse. [Art. 17]

---

*How to draw the Fig. Draw two $\perp$ lines OX, OZ, and OY the y-axis.

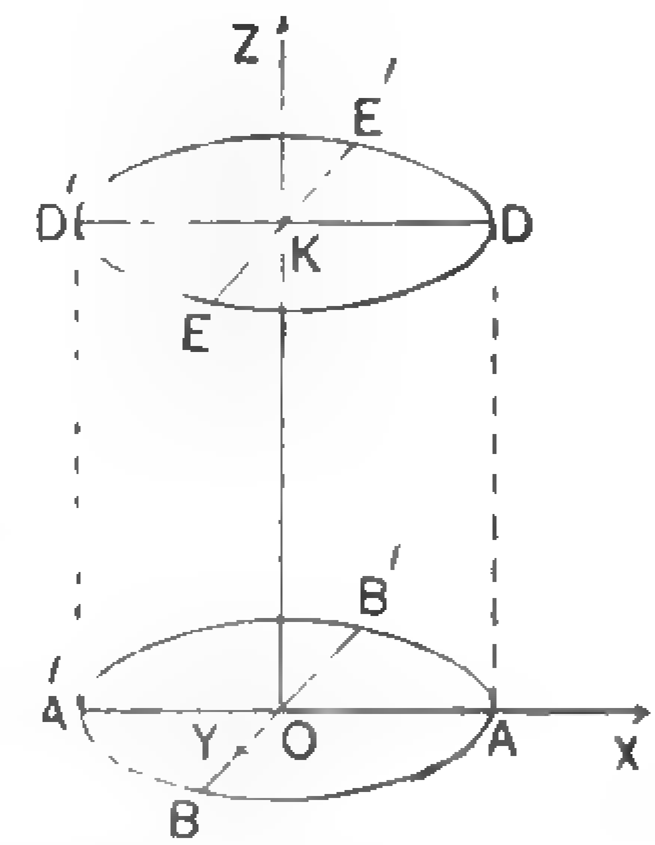
(*i*) In the xy-plane draw a right-handed parabola DOE, such that DE is $\parallel$ to the y-axis.

(*ii*) Thro' D, E draw DF, EG $\parallel$ to the z-axis, such that DF = EG, so that FG $\parallel$ and = DE. Take OK along the z-axis, such that OK=DF=EG.

(*iii*) In a plane $\parallel$ to the xy-plane and above it draw a parabola FKG.

Hence the shape of the surface is that shown in the Fig. *

It is called an **elliptic cylinder.**



112. (c) **Hyperbolic cylinder.** *To trace the locus of the equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

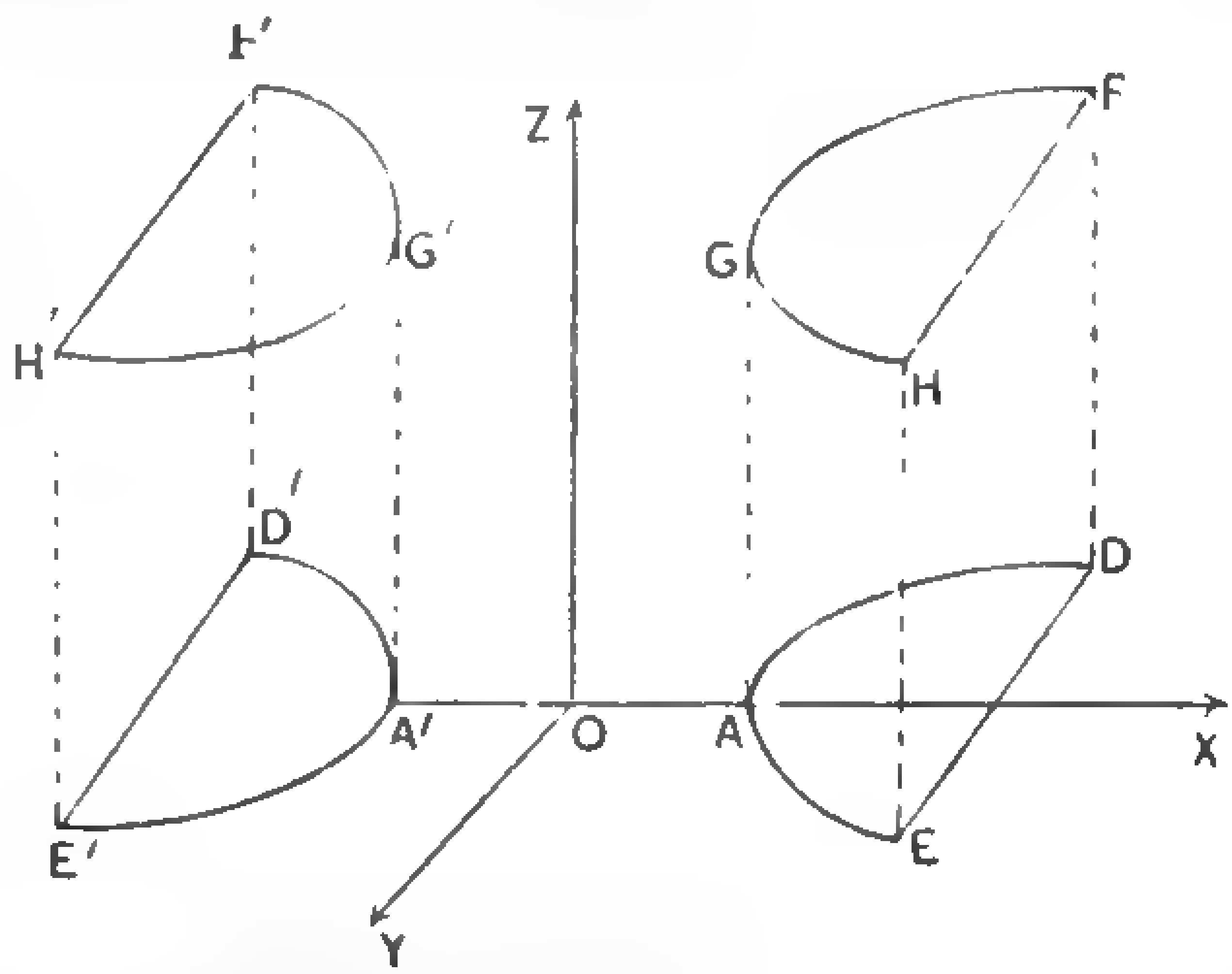
The equation of the surface is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ ... (1)

Here z is absent.

$\therefore$ (1) represents a cylinder generated by a st. line which is to the z-axis and intersects the curve whose equation *in the xy-plane* is $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$, *i.e.,* a hyperbola. [Art. 17]

Hence the shape of the surface is that shown in the Fig. †

It is called a **hyperbolic cylinder.**



***How to draw the Fig.** Draw two $\perp$ lines OX, OZ.

(*i*) On the x-axis take two pts. A, A' on opposite sides of O, such that OA=OA'=a.

Thro' O draw B'OB, the y-axis, and on it take two pts. B, B' on opposite sides of O, such that OB=OB'=b.

(*Continued on the next page*)

MISCELLANEOUS EXAMPLES ON CHAPTER X

1. If three fixed points of a straight line are on three given mutually perpendicular planes, show that any other point of the line describes an ellipsoid.

[Take the three given mutually $\perp$ planes as co-ordinate planes. Let A, B, C be the three fixed pts. of the line on the *yz-*, *zx-*, *xy*-planes, and P (x_1, y_1, z_1) any other (marked) pt. of the line. Let PA$=a$, PB$=b$, PC$=c$.

Let the equations of the line (thro' P) be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r, \text{ (l, m, n actual direction-cosines).}$$

It meets the *yz*-plane where, putting $x=0$, $r=a$,

$$-\frac{x_1}{l}=a,$$

$$\therefore\ l=-\frac{x_1}{a}.$$

Similarly $m=-\dfrac{y_1}{b}$, $n=-\dfrac{z_1}{c}$.

Square and add.]

2. Prove that the locus of a point, the sum of whose distances from the points $(a, 0, 0)$, $(-a, 0, 0)$ is constant $(=2k)$, is the ellipsoid of revolution $\dfrac{x^2}{k^2}+\dfrac{y^2+z^2}{k^2-a^2}=1$.

(*Continued from the last page*)

In the *xy*-plane draw an ellipse AB′A′B.

(*ii*) Thro' A, A′ draw AD, A′D′ $\parallel$ to the *z*-axis, such that AD$=$A′D′. Join DD′ to meet the *z*-axis in K. Thro' K draw E′KE $\parallel$ to B′OB, such that

KE$=$KE′$=$OB$=$OB′.

In a plane $\parallel$ to the *xy*-plane and above it draw an ellipse DE′D′E.

† **How to draw the Fig.** Draw two $\perp$ lines OX, OZ.

(*i*) On the *x*-axis take two pts. A, A′ on opposite sides of O, such that OA$=$OA′$=a$. Thro' O draw OY, the *y*-axis.

In the *xy*-plane draw a hyperbola DAE, D′A′E′ (equal branches), such that DE, D′E′ are $\parallel$ to the *y*-axis, and DE$=$D′E′.

(*ii*) Thro' D, A, E draw DF, AG, EH $\parallel$ to the *z*-axis, such that DF$=$AG$=$EH, so that FH is $\parallel$ and $=$DE.

Thro' D′, A′, E′ draw D′F′, A′G′, E′H′ $\parallel$ to the *z*-axis, such that D′F′$=$A′G′$=$E′H′$=$DF$=$AG$=$EH, so that F′H′ is $\parallel$ and $=$D′E′.

(*iii*) In a plane $\parallel$ to the *xy*-plane and above it draw a hyperbola FGH, F′G′H′.

[**Note. Central conicoid of revolution. Def.** If in the equation of a central conicoid $ax^2+by^2+cz^2=1$, two of the three coefficients are equal (*i.e.*, $a=b$, or $b=c$, or $c=a$), the **conicoid** is said to be **of revolution.**

Paraboloid of revolution. Def. If in the equation of a paraboloid $ax^2+by^2=2z$, $a=b$, the **paraboloid** is said to be **of revolution.**]

3. (*a*) What surface is represented by the equation x^2+y^2-2az ?

(*b*) The tangents from a point P to a sphere are all equal to the distance of P from a fixed tangent plane to the sphere. Show that the locus of P is a paraboloid of revolution.

[Choose the axes as in Ex., Art. 77. The equation of the sphere is

$$x^2+y^2+(z-c)^2=c^2, \text{ or } x^2+y^2+z^2-2cz=0,$$

and that of the tangent plane is $z=0$.]

4. A point moves so that the line joining the feet of the perpendiculars from it to two given lines subtends a right angle at the mid-point of their shortest distance. Prove that its locus is a hyperbolic cylinder.

☞ Next three chapters (XI to XIII) should be omitted by the B. A. & B. Sc. Pass Course (old type) students of the Punjab University. They are meant only for the B.A. Honours (old type) students of the Punjab University.

CHAPTER XI

THE CONICOID

SECTION I

THE CENTRAL CONICOID

113. Diameter. Def. A chord of a central conicoid which passes through the centre is called a **diameter.**

EXAMPLE

Prove that the sum of the squares of the reciprocals of any three mutually perpendicular diameters of an ellipsoid is constant.

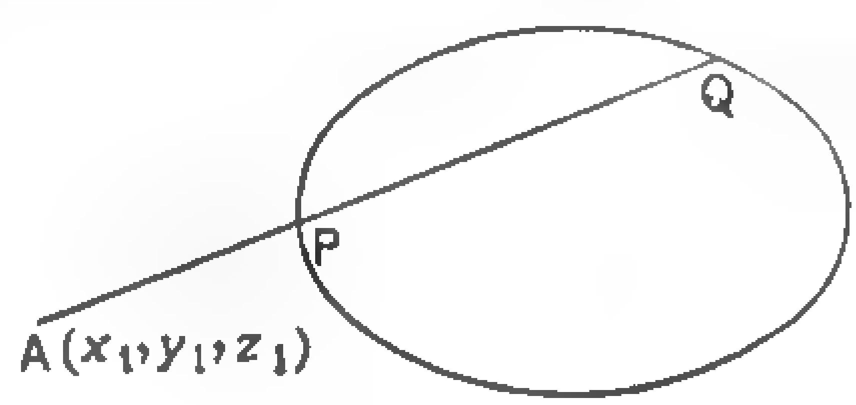
A central conicoid and a line.

114. *A line through a given point $A(x_1, y_1, z_1)$ meets a central conicoid $ax^2+by^2+cz^2=1$ in P and Q ; to find the lengths of AP and AQ.*

The equation of the central conicoid is $\quad ax^2+by^2+cz^2 = 1 \;\ldots\; (1)$

Let the equations of the line thro' $A(x_1, y_1, z_1)$ be

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n},$$



l, m, n being actual direction-cosines.

Any pt. on this line is $(x_1+lr, y_1+mr, z_1+nr) \;\ldots\; (2)$

If it lies on the conicoid (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2 = 1$$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)$
$= 0 \;\ldots\; (3)$

which is a quadratic in r, giving two values of r, the lengths of AP and AQ.

Cor. 1. Intersections of a line and a conicoid. *To find the points of intersection of the line and the conicoid.*

Substituting the two values of r found from (3), one by one, in (2), we get the two pts. of intersection.

Cor. 2. *All plane sections of a central conicoid are conics.*

Proof. $\because$ every st. line meets a central conicoid in two pts. [Cor. 1]

$\therefore$ every st. line lying in a particular plane meets the central conicoid and $\therefore$ the curve of intersection of the central conicoid and the plane in *two* pts.

$\therefore$ by Analytical Plane Geometry, the curve of intersection is a conic.

EXAMPLES

1. A line through a given point A meets a central conicoid in P, Q. If OR is the diameter parallel to APQ, prove that AP.AQ : OR^2 is constant.

****2.** A is a given point and POP′ any diameter of a central conicoid. If OQ and OQ′ are the diameters parallel to AP and AP′, prove that $\dfrac{AP^2}{OQ^2} + \dfrac{AP'^2}{OQ'^2}$ is constant.

[Let the equation of the central conicoid be

$$ax^2 + by^2 + cz^2 = 1 \quad \dots (1)$$

Let A be the pt. (α, β, γ), and $P(x_1, y_1, z_1)$, $P'(-x_1, -y_1, -z_1)$ the extremities of any diameter.

It will be found that the direction-cosines of AP are

$$l = \frac{x_1-\alpha}{AP}, \; m = \frac{y_1-\beta}{AP}, \; n = \frac{z_1-\gamma}{AP}.$$

$\therefore$ the direction-cosines of OQ ($\parallel$ to AP) are also l, m, n.

If $OQ = r$, the co-ordinates of Q are (lr, mr, nr).

$\because$ Q lies on the conicoid (1), $\therefore$?

It will be found that

$$\frac{AP^2}{OQ^2} = a(x_1-\alpha)^2 + b(y_1-\beta)^2 + c(z_1-\gamma)^2.$$

$\therefore$ changing (x_1, y_1, z_1) to $(-x_1, -y_1, -z_1)$,

$$\frac{AP'^2}{OQ'^2} = a(x_1+\alpha)^2 + b(y_1+\beta)^2 + c(z_1+\gamma)^2.$$

$$\therefore \quad \frac{AP^2}{OQ^2} + \frac{AP'^2}{OQ'^2} = ?$$

But $P(x_1, y_1, z_1)$ lies on the conicoid (1).]

3. Prove that the line joining the points $(1, -5, -6)$ and $(-4, 5, 4)$ meets the surface $2x^2+3y^2-z^2=1$ in coincident points.

115. *Equation of the tangent plane.* **To find the equation of the tangent plane at any point (x_1, y_1, z_1) of the central conicoid**

$$\mathbf{ax^2 + by^2 + cz^2 = 1.}$$

The equation of the central conicoid is

$$ax^2 + by^2 + cz^2 = 1 \quad \dots (1)$$

The equations of *any* line thro' (x_1, y_1, z_1) are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \dots (2)$$

Any pt. on this line is $(x_1 + lr,\ y_1 + mr,\ z_1 + nr)$.

If it lies on the conicoid (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1$$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)$

$$+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \dots (3)$$

which is a quadratic in r.

$\because$ (x_1, y_1, z_1) lies on the conicoid (1)

$\therefore$ $$ax_1^2+by_1^2+cz_1^2=1 \quad \dots (4)$$

$\therefore$ one root of the quadratic (3) is zero.

If the line *touches* the conicoid, the other root is also zero.

$\therefore$ coeff. of $r=0$,

i.e., $alx_1+bmy_1+cnz_1=0 \quad \dots (5)$

Eliminating l, m, n from (2) and (5) [by substituting their values from (2) in (5)], the locus of the tangent lines is

$$a(x-x_1)x_1+b(y-y_1)y_1+c(z-z_1)z_1=0$$

or $$axx_1+byy_1+czz_1-(ax_1^2+by_1^2+cz_1^2)=0 \quad \dots (6)$$

Adding (4) and (6),

$$\mathbf{axx_1+byy_1+czz_1=1,}$$

which is the required equation of the tangent plane.

[**Rule to write down the equation of the tangent plane at the point (x_1, y_1, z_1) of a central conicoid :** *See Rule of Art. 70, (b).*]

EXAMPLES

1. Find the equation of the tangent plane at the point (x', y', z') of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$. [*D.U.H. 1941*]

Show that the length of the perpendicular from the origin on the tangent plane at the point (x', y', z') is given by

$$\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}.$$

2. If P, Q are any two points on an ellipsoid, the plane through the centre and the line of intersection of the tangent planes at P, Q bisects PQ.

116. *Condition of tangency of a plane and a central conicoid.* **To find the condition that the plane $lx+my+nz=p$ should touch the central conicoid $ax^2+by^2+cz^2=1$.**

[**Method of point of contact.**]

The equation of the plane is $lx+my+nz=p \quad \dots (1)$

and that of the central conicoid is $ax^2+by^2+cz^2=1 \quad \dots (2)$

Let (x_1, y_1, z_1) be the pt. of contact.

The equation of the tangent plane at (x_1, y_1, z_1) to the conicoid (2) is $axx_1 + byy_1 + czz_1 = 1$ ... (3) [Art. 115]

$\because$ it is the same as the equation of the given plane (1)

$\therefore$ comparing coeffs. in (3) and (1),

$$\frac{ax_1}{l} = \frac{by_1}{m} = \frac{cz_1}{n} = \frac{1}{p}$$

$$\therefore\ x_1 = \frac{l}{ap},\ y_1 = \frac{m}{bp},\ z_1 = \frac{n}{cp}\ \ldots (4)$$

$\because$ (x_1, y_1, z_1) lies on the conicoid (2)

$\therefore$ $ax_1^2 + by_1^2 + cz_1^2 = 1$ ... (5)

Substituting the values of x_1, y_1, z_1 from (4) in (5),

$$a\frac{l^2}{a^2p^2} + b\frac{m^2}{b^2p^2} + c\frac{n^2}{c^2p^2} = 1$$

or
$$\mathbf{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2},$$

which is the required condition.

Cor. 1. *If the condition is satisfied, to find the point of contact.*

From (4), the pt. of contact is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$. [(x_1, y_1, z_1)]

Cor. 2. *To find the equations of the two tangent planes to the central conicoid $ax^2+by^2+cz^2=1$, parallel to the plane $lx+my+nz=0$.*

The equation of the central conicoid is $ax^2+by^2+cz^2=1$ ...(1)

The equation of *any* plane ∥ to the plane $lx+my+nz=0$, is

$$lx+my+nz=p \ \ldots (2)$$

If it touches the conicoid (1), then

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2, \text{ or } p = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}.$$

Substituting these values of p in (2),

$$lx+my+nz = \pm\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}},$$

which are the required equations of the two tangent planes.

EXAMPLES

1. Find the equations to the tangent planes to

$$\mathbf{2x^2 - 6y^2 + 3z^2 = 5}$$

which pass through the line $\mathbf{x+9y-3z=0=3x-3y+6z-5}$.

[*D. U. H. 1946*]

2. Prove that the equation to the two tangent planes to the conicoid $\mathbf{ax^2+by^2+cz^2=1}$ which pass through the line

$\mathbf{u \equiv lx+my+nz-p=0,\ u' \equiv l'x+m'y+n'z-p'=0}$, is

$$\mathbf{u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)}$$
$$\mathbf{+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0.}\quad [\textit{P. U. H. 1959}]$$

3. Tangent planes are drawn to the conicoid $ax^2+by^2+cz^2=1$ through the point (α, β, γ). Prove that the perpendiculars to them from the origin generate the cone $(\alpha x+\beta y+\gamma z)^2=\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}$.

[*P. U. H.*]

4. (*a*) **Show that the plane $\mathbf{lx+my+nz=p}$ will touch the ellipsoid $\mathbf{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1}$, if $\mathbf{a^2l^2+b^2m^2+c^2n^2=p^2}$.** [*B.H.U. 1932*]

(*b*) Obtain the tangent planes for the ellipsoid
$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$
which are parallel to $lx+my+nz=0$.

If $2r$ is the distance between the planes, show that a line through the origin and perpendicular to the planes lies on the cone
$$x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0.\quad [\textit{P. U. H. 1961}]$$

****5. If the line of intersection of two perpendicular tangent planes to the ellipsoid whose equation, referred to rectangular axes, is $\mathbf{x^2/a^2+y^2/b^2+z^2/c^2=1}$, passes through the fixed point (0, 0, k), show that it lies on the cone**
$$\mathbf{x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.}\quad [\textit{P.U.H. 1960}]$$

The equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

Let the equations of the line of intersection of two $\perp$ tangent planes, passing thro' $(0, 0, k)$, be
$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-k}{n}, \text{ or } \frac{x}{l}=\frac{y}{m}=\frac{z-k}{n} \quad ...(2)$$

Let the equation of a tangent plane thro' this line be
$$Ax+By+C(z-k)=0, \text{ or } Ax+By+Cz=Ck \quad ...(3)$$
where $Al+Bm+Cn=0$ ...(4) [Art. 44, (*b*)]

$\because$ the plane (3) touches the ellipsoid (1)

$\therefore$ $A^2a^2+B^2b^2+C^2c^2=C^2k^2$ $\left[\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2 \text{ (Art. 116)}\right]$

or $A^2a^2+B^2b^2+C^2(c^2-k^2)=0$ ...(5)

$\because$ the two tangent planes are $\perp$

$\therefore$ their normals are also $\perp$,

$\therefore$ the lines whose direction-cosines (A, B, C) are given by (4) and (5), are $\perp$

$\therefore$ $l^2(b^2+c^2-k^2)+m^2(c^2-k^2+a^2)+n^2(a^2+b^2)=0$ ...(6)

[$u^2(b+c)+v^2(c+a)+w^2(a+b)=0$ (Misc. Ex. 5, Chap. II)]

Eliminating l, m, n from (2) and (6) [by substituting their values from (2) in (6)], the line (2) lies on the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

Note. Important. *The condition, that the two lines whose direction-cosines* (l, m, n) *are given by*

$$ul+vm+wn=0,$$
$$al^2+bm^2+cn^2=0,$$

may be perpendicular, is

$$\mathbf{u^2(b+c)+v^2(c+a)+w^2(a+b)=0.}$$

[Misc. Ex. 5, Chap. II]

117. *Director sphere.* **To find the locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid.**

Let the equation of the central conicoid be $ax^2+by^2+cz^2=1$ ...(1)

Let the equation of one of the three mutually $\perp$ tangent planes to the conicoid (1) be

$$l_1x+m_1y+n_1z=p_1,$$

where $\dfrac{l_1^2}{a}+\dfrac{m_1^2}{b}+\dfrac{n_1^2}{c}=p_1^2$, or $p_1=\sqrt{\dfrac{l_1^2}{a}+\dfrac{m_1^2}{b}+\dfrac{n_1^2}{c}}$,

i.e.,
$$l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}} \quad ... (2)$$

Similarly let the equations of the other two tangent planes be

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad ... (3)$$

$$l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}} \quad ... (4)$$

[To find the locus of the pt. of intersection of the planes (2), (3), (4).]

Squaring (2), (3), (4), and adding vertically,

$$x^2(l_1^2+l_2^2+l_3^2)+y^2(m_1^2+m_2^2+m_3^2)+z^2(n_1^2+n_2^2+n_3^2)$$
$$+2yz(m_1n_1+m_2n_2+m_3n_3)+2zx(n_1l_1+n_2l_2+n_3l_3)+2xy(l_1m_1+l_2m_2+l_3m_3)$$
$$=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^2)+\frac{1}{c}(n_1^2+n_2^2+n_3^2)$$

[But l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 being the direction-cosines of three mutually $\perp$ lines, *viz.*, the normals to the three mutually $\perp$ tangent planes (2), (3), (4),

$l_1^2+l_2^2+l_3^2=1$, and so on ; $m_1n_1+m_2n_2+m_3n_3=0$, and so on

(Art. 58, (C) (D))]

or
$$\mathbf{x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}$$

which is the required locus.

It is a sphere, whose centre is the origin, *i.e.*, the centre of the conicoid. It is called the **director sphere** of the conicoid.

EXAMPLES

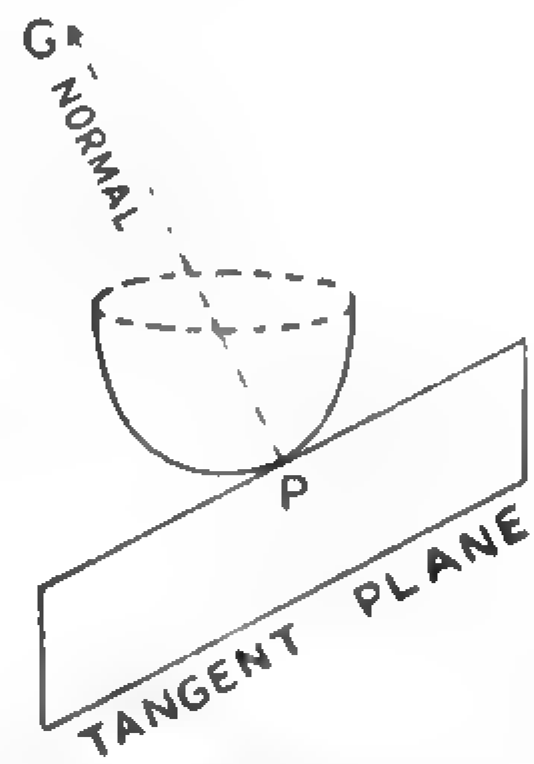
1. *Director sphere.* **Find the locus of the point of intersection of three tangent planes to an ellipsoid which are mutually at right angles.** [*P. U. H. 1938*]

2. Prove that the locus of points from which three mutually perpendicular planes can be drawn to touch the ellipse

$$x^2/a^2+y^2/b^2=1,\ z=0,$$

is the sphere $x^2+y^2+z^2=a^2+b^2$. [*P(P). U. 1956 S*]

Normals to an ellipsoid.



118. Normal. Def. The **normal** at any point P of a surface is the straight line through P perpendicular to the tangent plane at P.

119. *Equations of the normal.* **To find the equations of the normal at any point (x_1, y_1, z_1) of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.**

The equation of the ellipsoid is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1.$

The equation of the tangent plane at (x_1, y_1, z_1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad ...(1) \qquad \text{[Rule (Art. 70, (b))]}$$

$\therefore$ the direction-cosines of the normal are proportional to

$$\frac{x_1}{a^2},\ \frac{y_1}{b^2},\ \frac{z_1}{c^2}.$$

$\therefore$ the equations of the normal at (x_1, y_1, z_1) are

$$\frac{x-x_1}{\dfrac{x_1}{a^2}}=\frac{y-y_1}{\dfrac{y_1}{b^2}}=\frac{z-z_1}{\dfrac{z_1}{c^2}} \quad ...(2) \qquad \text{[Art. 37, Cor. 1]}$$

Abridged notation. If $F(x, y, z)=\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1$, so that $F(x, y, z)=0$ is the equation of the ellipsoid, then

$$\frac{\partial F}{\partial x}=\frac{1}{a^2}2x,\ \frac{\partial F}{\partial y}=\frac{1}{b^2}2y,\ \frac{\partial F}{\partial z}=\frac{1}{c^2}2z$$

$$\therefore \quad \frac{\partial F}{\partial x_1}=\frac{2x_1}{a^2},\ \frac{\partial F}{\partial y_1}=\frac{2y_1}{b^2},\ \frac{\partial F}{\partial z_1}=\frac{2z_1}{c^2}$$

$\therefore$ from (2), the equations of the normal are

$$\frac{x-x_1}{\frac{1}{2}\frac{\partial F}{\partial x_1}}=\frac{y-y_1}{\frac{1}{2}\frac{\partial F}{\partial y_1}}=\frac{z-z_1}{\frac{1}{2}\frac{\partial F}{\partial z_1}},\ \text{or}\ \frac{\mathbf{x-x_1}}{\frac{\partial \mathbf{F}}{\partial \mathbf{x_1}}}=\frac{\mathbf{y-y_1}}{\frac{\partial \mathbf{F}}{\partial \mathbf{y_1}}}=\frac{\mathbf{z-z_1}}{\frac{\partial \mathbf{F}}{\partial \mathbf{z_1}}}.$$

Note 1. The above equations of the normal hold even if $F(x, y, z)=0$ (instead of being the equation of an ellipsoid) is the equation of *any* conicoid.

Cor. To find the equations of the normal in the actual direction-cosines form.

If p is the $\perp$ from the centre (0, 0, 0) on the tangent plane (1),

then $$p=\frac{1}{\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}}. \qquad \Bigg|\ \frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1=0$$

Now the direction-cosines of the normal are proportional to

$$\frac{x_1}{a^2},\ \frac{y_1}{b^2},\ \frac{z_1}{c^2}. \qquad [\text{ Art. } 119\ ]$$

Dividing by $\sqrt{\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}}=\frac{1}{p}$, the actual direction-cosines are

$$\frac{\frac{x_1}{a^2}}{\frac{1}{p}},\ \frac{\frac{y_1}{b^2}}{\frac{1}{p}},\ \frac{\frac{z_1}{c^2}}{\frac{1}{p}},\ i.e.,\ \frac{px_1}{a^2},\ \frac{py_1}{b^2},\ \frac{pz_1}{c^2}.$$

$\therefore$ the equations of the normal at (x_1, y_1, z_1), in the actual direction-cosines form, are

$$\frac{\mathbf{x-x_1}}{\frac{\mathbf{px_1}}{\mathbf{a^2}}}=\frac{\mathbf{y-y_1}}{\frac{\mathbf{py_1}}{\mathbf{b^2}}}=\frac{\mathbf{z-z_1}}{\frac{\mathbf{pz_1}}{\mathbf{c^2}}}. \qquad [\text{ Art. } 37\ ]$$

Note 2. Which form of the equations of the normal to use and where ? *For problems relating to distances measured from P (x_1, y_1, z_1) along the normal at P, use the equations of the normal at P, in the actual direction-cosines form, viz.,*

$$\frac{\mathbf{x-x_1}}{\frac{\mathbf{px_1}}{\mathbf{a^2}}}=\frac{\mathbf{y-y_1}}{\frac{\mathbf{py_1}}{\mathbf{b^2}}}=\frac{\mathbf{z-z_1}}{\frac{\mathbf{pz_1}}{\mathbf{c^2}}},$$

and for other problems use the equations of the normal in the simpler form, *viz.,*

$$\frac{\mathbf{x}-\mathbf{x}_1}{\frac{\mathbf{x}_1}{\mathbf{a}^2}}=\frac{\mathbf{y}-\mathbf{y}_1}{\frac{\mathbf{y}_1}{\mathbf{b}^2}}=\frac{\mathbf{z}-\mathbf{z}_1}{\frac{\mathbf{z}_1}{\mathbf{c}^2}}.$$

EXAMPLES

1. The normal at any point P of the ellipsoid

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}+\frac{\mathbf{z}^2}{\mathbf{c}^2}=1$$

meets the principal planes* in $\mathbf{G_1, G_2, G_3}$. Show that

$$\mathbf{PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2}.$$

The equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Let P be the pt. (x_1, y_1, z_1). Then the equations of the normal at P, in the actual direction-cosines form, are

$$\frac{x-x_1}{\frac{px_1}{a^2}}=\frac{y-y_1}{\frac{py_1}{b^2}}=\frac{z-z_1}{\frac{pz_1}{c^2}}. \qquad \text{[Art. 119, Cor.]}$$

Any pt. on the normal is $\left(x_1+\frac{px_1}{a^2}r,\ y_1+\frac{py_1}{b^2}r,\ z_1+\frac{pz_1}{c^2}r\right)$.

If it lies on the yz-plane, *i.e.*, $x=0$, then

$$x_1+\frac{px_1}{a^2}r=0, \text{ or } 1+\frac{p}{a^2}r=0, \text{ or } r=-\frac{a^2}{p}.$$

$\therefore\ PG_1=-\frac{a^2}{p}$.

Similarly $PG_2=-\frac{b^2}{p}$, $PG_3=-\frac{c^2}{p}$.

$\therefore\ PG_1 : PG_2 : PG_3=-\frac{a^2}{p} : -\frac{b^2}{p} : -\frac{c^2}{p}=a^2 : b^2 : c^2$.

2. The normal at a point P on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the principal planes in G_1, G_2, G_3. If $PG_1^2+PG_2^2+PG_3^2=k^2$, find the locus of P. [*Ag. U. 1938*]

3. Find the length of the normal chord through $P(x_1, y_1, z_1)$ of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and prove that if it is equal to $4PG_3$, where G_3 is the point in which the normal chord meets the plane XOY, then P lies on the cone $\frac{x^2}{a^6}(2c^2-a^2)+\frac{y^2}{b^6}(2c^2-b^2)+\frac{z^2}{c^4}=0$.

4. The normals to an ellipsoid at the points P, P′ meet a principal plane in G, G′; show that the plane which bisects PP′ at right angles bisects GG′. [*D. U. H. 1933*]

* *i.e.*, the planes YOZ, ZOX, XOY.

120. *Number of normals from a given point to an ellipsoid.* **To prove that there are six points on an ellipsoid the normals at which pass through a given point (α, β, γ).**

Let the equation of the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

The equations of the normal at (x_1, y_1, z_1) are

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}. \qquad \text{[Art. 119]}$$

If it passes thro' (α, β, γ), then

$$\frac{\alpha-x_1}{\frac{x_1}{a^2}}=\frac{\beta-y_1}{\frac{y_1}{b^2}}=\frac{\gamma-z_1}{\frac{z_1}{c^2}}=\lambda \text{ (say)} \quad ...(2)$$

From the first and last members, $\alpha-x_1=\frac{x_1}{a^2}\lambda$

or $\alpha=x_1\left(1+\frac{\lambda}{a^2}\right)=\frac{x_1}{a^2}\left(a^2+\lambda\right)$

$$\left.\begin{aligned} &\text{or} \quad x_1=\frac{a^2\alpha}{a^2+\lambda}. \\ &\text{Similarly } y_1=\frac{b^2\beta}{b^2+\lambda},\ z_1=\frac{c^2\gamma}{c^2+\lambda}. \end{aligned}\right\} \quad ...(3)$$

$\because$ (x_1, y_1, z_1) lies on the ellipsoid (1),

$\therefore$ $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$ ... (4)

Substituting the values of x_1, y_1, z_1 from (3) in (4),

$$\frac{1}{a^2}\frac{a^4\alpha^2}{(a^2+\lambda)^2}+\frac{1}{b^2}\frac{b^4\beta^2}{(b^2+\lambda)^2}+\frac{1}{c^2}\frac{c^4\gamma^2}{(c^2+\lambda)^2}=1$$

or
$$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}+\frac{c^2\gamma^2}{(c^2+\lambda)^2}=1.$$

Clearing of fractions,

$$a^2\alpha^2(b^2+\lambda)^2(c^2+\lambda)^2+b^2\beta^2(c^2+\lambda)^2(a^2+\lambda)^2+c^2\gamma^2(a^2+\lambda)^2(b^2+\lambda)^2$$
$$=(a^2+\lambda)^2(b^2+\lambda)^2(c^2+\lambda)^2,$$

which is an equation of the *sixth* degree in λ, giving *six* values of λ.

Substituting these values of λ, one by one, in (3), we get six pts. (x_1, y_1, z_1) on the ellipsoid the normals at which pass thro' (α, β, γ).

Cor. 1. *The feet of the normals from (α, β, γ) to the ellipsoid lie on three cylinders which have a common curve of intersection.*

From (2) [changing (x_1, y_1, z_1) to (x, y, z)], the feet of the normals lie on

$$\frac{\alpha-x}{\frac{x}{a^2}}=\frac{\beta-y}{\frac{y}{b^2}}=\frac{\gamma-z}{\frac{z}{c^2}}$$

or
$$\frac{a^2(\alpha-x)}{x}=\frac{b^2(\beta-y)}{y}=\frac{c^2(\gamma-z)}{z},$$

i.e., on the three cylinders

$$\left.\begin{array}{l} b^2z(\beta-y)=c^2y(\gamma-z),\ c^2x(\gamma-z)=a^2z(\alpha-x) \\ a^2y(\alpha-x)=b^2x(\beta-y). \end{array}\right\} \ldots (5) \text{ [Art. 17]}$$

From (3) [changing (x_1, y_1, z_1) to (x, y, z)], the feet of the normals lie on the curve

$$x=\frac{a^2\alpha}{a^2+\lambda},\ y=\frac{b^2\beta}{b^2+\lambda},\ z=\frac{c^2\gamma}{c^2+\lambda} \quad \ldots(6)$$

where λ is a parameter.

$\therefore$ the three cylinders (5) have a common curve of intersection (6).

Cor. 2. *Cubic curve through the feet of the six normals from a point.* **The feet of the normals from (α, β, γ) to the ellipsoid are the six points of intersection of the ellipsoid and a certain cubic curve.**

The six feet of the normals from (α, β, γ) to the ellipsoid lie on the ellipsoid [Art. 120].

Also from (3) (Art. 120) [changing (x_1, y_1, z_1) to (x, y, z)], the feet of the normals lie on the curve

$$x=\frac{a^2\alpha}{a^2+\lambda},\ y=\frac{b^2\beta}{b^2+\lambda},\ z=\frac{c^2\gamma}{c^2+\lambda} \quad \ldots(6)$$

where λ is a parameter.

[To prove that the curve (6) is a cubic curve.]

The curve (6) meets an arbitrary plane

$$Ax+By+Cz+D=0 \ \ldots(7)$$

where, substituting the values of x, y, z from (6) in (7),

$$A\frac{a^2\alpha}{a^2+\lambda}+B\frac{b^2\beta}{b^2+\lambda}+C\frac{c^2\gamma}{c^2+\lambda}+D=0.$$

Clearing of fractions,

$$Aa^2\alpha(b^2+\lambda)(c^2+\lambda)+Bb^2\beta(c^2+\lambda)(a^2+\lambda)+Cc^2\gamma(a^2+\lambda)(b^2+\lambda)$$
$$+D(a^2+\lambda)(b^2+\lambda)(c^2+\lambda)=0,$$

which is a cubic in λ, giving three values of λ.

Substituting these values of λ, one by one, in (6), we get three pts. (x, y, z) in which the plane (7) meets the curve (6).

$\therefore$ the curve (6) is a cubic curve.

$\therefore$ the feet of the normals from (α, β, γ) to the ellipsoid are the six pts. of intersection of the ellipsoid and the cubic curve (6).

Cor. 3. *Quadric cone through the six normals from a point.* **The six normals from (α, β, γ) to the ellipsoid lie on a cone of the second degree.**

Let the equations of a normal from (α, β, γ) to the ellipsoid be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad ...(1)$$

Then $l=\frac{px_1}{a^2}$ (Art. 119, Cor.) $\quad$ [But $x_1=\frac{a^2\alpha}{a^2+\lambda}$ (Art. 120, (3))]

$$=\frac{p}{a^2}\frac{a^2\alpha}{a^2+\lambda}=\frac{p\alpha}{a^2+\lambda},$$

or $\quad a^2+\lambda=\frac{p\alpha}{l}. \quad \Big| \; b^2-c^2$

Similarly $\quad b^2+\lambda=\frac{p\beta}{m}, \quad \Big| \; c^2-a^2$

$\quad c^2+\lambda=\frac{p\gamma}{n}. \quad \Big| \; a^2-b^2$

Multiplying these equations respectively by b^2-c^2, c^2-a^2, a^2-b^2, and adding vertically,

$$0=\frac{p\alpha}{l}(b^2-c^2)+\frac{p\beta}{m}(c^2-a^2)+\frac{p\gamma}{n}(a^2-b^2)$$

or $\quad \frac{\alpha}{l}(b^2-c^2)+\frac{\beta}{m}(c^2-a^2)+\frac{\gamma}{n}(a^2-b^2)=0 \quad ...(2)$

Eliminating l, m, n from (1) and (2) [by substituting their values from (1) in (2)], the normals lie on the surface

$$\frac{\alpha}{x-\alpha}(b^2-c^2)+\frac{\beta}{y-\beta}(c^2-a^2)+\frac{\gamma}{z-\gamma}(a^2-b^2)=0$$

or, clearing of fractions,

$$\alpha(b^2-c^2)(y-\beta)(z-\gamma)+\beta(c^2-a^2)(z-\gamma)(x-\alpha)+\gamma(a^2-b^2)(x-\alpha)(y-\beta)=0,$$

which is a cone of the second degree.

EXAMPLES

1. (*i*) **Prove that the six normals from a point to an ellipsoid lie on a cone of the second degree.** [*Ag. U. 1957*]

(*ii*) **Prove also that the feet of the normals are the six points of intersection of the ellipsoid and a certain cubic curve which lies on the above cone.** [*Ag. U. 1944*]

[**Rule to prove that a given curve lies on a given surface :** Substitute in the equation of the surface the co-ordinates (in terms of a parameter) of any point on the curve, and show that the equation is satisfied for *all* values of the parameter.]

2. *Quadric cone through the feet of the six normals from a point.* **Prove that the feet of the six normals from (α, β, γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on the curve of intersection of the ellipsoid and the cone** $\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0.$

[*P. U. H. 1954*]

Cor. *The feet of the six normals from (α, β, γ) to an ellipsoid lie on a cone of the second degree in whose equation,*
coeff. of $x^2=0$, coeff. of $y^2=0$, coeff. of $z^2=0$, constant term $=0$.

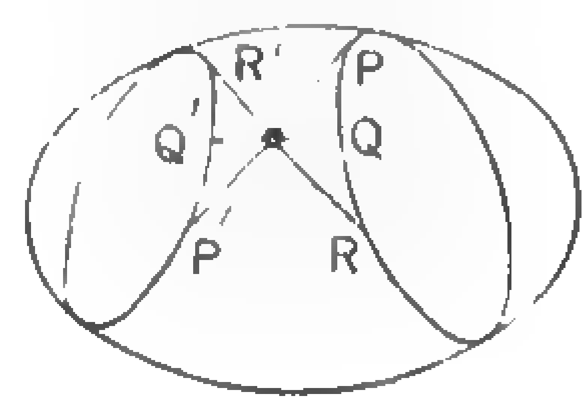
For, from Ex. 2, the equation of this cone is

$$a^2(b^2-c^2)\alpha yz+b^2(c^2-a^2)\beta zx+c^2(a^2-b^2)\gamma xy=0,$$

which is of the second degree, and in which

coeff. of $x^2=0$, coeff. of $y^2=0$, coeff. of $z^2=0$, constant term $=0$.

3. **If P, Q, R ; P′, Q′, R′ are the feet of the six normals from a point to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the plane PQR is given by $lx+my+nz=p$, then the plane P′Q′R′ is given by** $\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$ [*P. U. 1960*]



[The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad \ldots(1)$$

and that of the plane PQR is

$$lx+my+nz=p, \quad \text{or} \quad lx+my+nz-p=0 \quad \ldots(2)$$

Let the required equation of the plane P′Q′R′ be

$$l'x+m'y+n'z=p', \quad \text{or} \quad l'x+m'y+n'z-p'=0 \quad \ldots(3)$$

The combined equation of the planes (2) and (3) is

$$(lx+my+nz-p)(l'x+m'y+n'z-p')=0 \quad \ldots(4)$$

The equation of *any* conicoid thro' the pts. of intersection of the ellipsoid (1) and the pair of planes (4) is

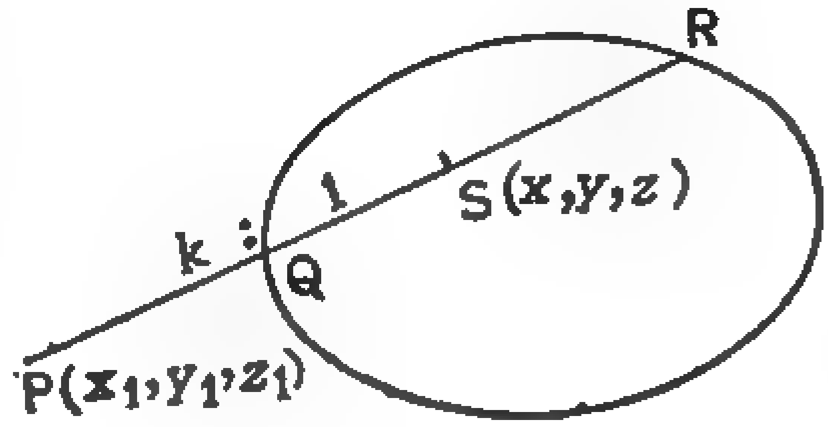
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+k(lx+my+nz-p)(l'x+m'y+n'z-p')=0.$$

If it is the same as the equation of the cone thro' the feet P,Q,R ; P′, Q′, R′ of the six normals from a pt. to the ellipsoid, then
coeff. of $x^2=0$, coeff. of $y^2=0$, coeff. of $z^2=0$, constant term $=0$.

(Ex. 2, Cor.)

Hence find the values of l', m', n', p' (in terms of k), and substitute these values in (3). Cancel k from the denominators.]

121. *Equation of the polar plane.* **To find the equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the central conicoid $ax^2+by^2+cz^2=1$.**



The equation of the central conicoid is $ax^2+by^2+cz^2 = 1$ ... (1)

Let P be the pt. (x_1, y_1, z_1), QR *any* chord of the conicoid thro' P, and $S(x, y, z)$ the harmonic conjugate of P w. r. t. Q and R.

[To find the locus of S.]

The pt. which divides PS in the ratio $k : 1$ is

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right).$$

If it lies on the conicoid (1), then

$$a\left(\frac{kx+x_1}{k+1}\right)^2+b\left(\frac{ky+y_1}{k+1}\right)^2+c\left(\frac{kz+z_1}{k+1}\right)^2=1$$

or, multiplying thro'out by $(k+1)^2$,

$$a(kx+x_1)^2+b(ky+y_1)^2+c(kz+z_1)^2 = (k+1)^2$$

or $$a(kx+x_1)^2+b(ky+y_1)^2+c(kz+z_1)^2 - (k+1)^2 = 0$$

or $$k^2(ax^2+by^2+cz^2-1)+2k(axx_1+byy_1+czz_1-1) + (ax_1^2+by_1^2+cz_1^2-1)= 0 \quad ... (2)$$

which is a quadratic in k.

$\because$ PS is divided harmonically, *i.e.*, internally and externally in the same ratio at Q and R [Def. (Art. 72)], the quadratic (2) has equal and opposite roots,

$\therefore$ sum of the roots $= 0$, $\therefore$ coeff. of $k = 0$

i.e., $axx_1+byy_1+czz_1-1 = 0$

or $\mathbf{axx_1+byy_1+czz_1 = 1}$,

which is the required equation of the polar plane. [Def. (Art. 73)]

[**Aid to memory. See Aid to memory in Art. 74.**]

Cor. 1. *If P is on the central conicoid, the polar plane of P is the tangent plane at P.*

For the equation of the polar plane of P (Art. 121) is the same as that of the tangent plane at P (Art. 115).

Cor. 2. Reciprocal property. *If the polar plane of a point P with respect to a central conicoid passes through a point Q, then the polar plane of Q passes through P.*

Let the equation of the central conicoid be $ax^2+by^2+cz^2=1$...(1)

Let P, Q be the pts. (x_1, y_1, z_1), (x_2, y_2, z_2).

The equation of the polar plane of P w. r. t. the conicoid (1) is

$$axx_1 + byy_1 + czz_1 = 1.$$

If it passes thro' Q, then

$$ax_2x_1+by_2y_1 + cz_2z_1= 1$$

or

$$ax_1x_2+by_1y_2 + cz_1z_2= 1.$$

The *symmetry* of this result shows that it is also the condition that the polar plane of Q should pass thro' P.

EXAMPLES

1. Find the equation of the polar plane of (x', y', z') with respect to the ellipsoid $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2=1.$ [*D. U. H. 1951*]

2. Prove that the six normals from a point P to an ellipsoid lie on a cone of the second degree.

Prove also that on this cone lie (*a*) the line joining P to the origin (*b*) the parallels through P to the axes, (*c*) the perpendicular from P to its polar plane. [*Ag. U. 1951*]

3. *Pole of a given plane.* **Find the pole of the plane**

$$\mathbf{lx+my+nz=p,}$$

with respect to the central conicoid $ax^2+by^2+cz^2=1$.

The equation of the plane is $lx+my+nz=p$ ...(1)

and that of the central conicoid is $ax^2+by^2+cz^2=1$ ...(2)

Let (x_1, y_1, z_1) be the required pole.

The equation of the polar plane of (x_1, y_1, z_1) w.r.t. the conicoid (2) is $axx_1+byy_1+czz_1=1$ ...(3) [Art. 121]

∵ it is the same as the equation of the given plane (1)

∴ comparing coeffs. in (3) and (1),

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{cz_1}{n}=\frac{1}{p}$$

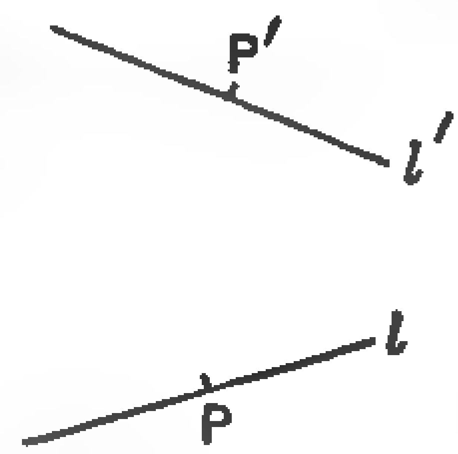
$$\therefore \quad x_1=\frac{l}{ap}, \; y_1=\frac{m}{bp}, \; z_1=\frac{n}{cp}.$$

∴ the required pole is $\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$.

4. If P, Q, R ; P′, Q′, R′ are the feet of the six normals from a point to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and S, S′ are the poles of the planes PQR, P′Q′R′, and O the centre, prove that

$$SS'^2-OS^2-OS'^2=2(a^2+b^2+c^2).$$

122. Polar lines. *If the polar plane of any point on a line l with respect to a central conicoid passes through a line l', then the polar plane of any point on l' passes through l.*



Let P be *any* pt. on l, and P′ *any* pt. on l'.

If the polar plane of P w.r.t. the central conicoid passes thro' l', then $\because$ P′ is a pt. on l'

$\therefore$ the polar plane of P passes thro' P′.

$\therefore$ by the reciprocal property (Art. 121, Cor. 2), the polar plane of P′ passes thro' P.

But P is *any* pt. on l.

$\therefore$ the polar plane of P′ passes thro' l.

Note. Polar lines. Def. Two lines, which are such that the polar plane of any point on each w.r.t. a conicoid passes through the other, are called **polar lines** w.r.t. the conicoid.

123. *Equations of the polar of a line.* **To find the equations of the polar of the line $\mathbf{\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}}$ with respect to the central conicoid $\mathbf{ax^2+by^2+cz^2=1}$.**

The equations of the line are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(1)$$

and the equation of the central conicoid is $ax^2+by^2+cz^2=1$ ...(2)

Any pt. on the line (1) is (x_1+lr, y_1+mr, z_1+nr).

The equation of the polar plane of this pt. w.r.t. the conicoid (2)

is $\quad ax(x_1+lr)+by(y_1+mr)+cz(z_1+nr) \quad =1$ [Art. 121]

or $\quad ax(x_1+lr)+by(y_1+mr)+cz(z_1+nr)-1=0$

or $axx_1+byy_1+czz_1-1+r(alx+bmy+cnz)=0$,

which, for *all* values of r, passes thro' the line

$axx_1+byy_1+czz_1-1=0,\ alx+bmy+cnz=0$ [Art. 44, (*a*)]

$\therefore$ the required equations of the polar are

$$\mathbf{axx_1+byy_1+czz_1=1,\ alx+bmy+cnz=0.}$$

[**Rule to write down the equations of the polar of the line $\mathbf{\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}}$ w.r.t. a central conicoid (equation in the standard form) :**

(*i*) *Write down the equation of the polar plane of (x_1, y_1, z_1) w.r.t. the central conicoid.*

(*ii*) *Write down the equation of the polar plane of (l, m, n), and omit the constant term.*

(*iii*) *The two equations are the required equations of the polar.*]

EXAMPLES

1. Define polar lines with respect to a conicoid.

[*P. U. H. 1959*]

Obtain the equations to the polar of the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

with regard to $ax^2+by^2+cz^2=1$. [*B. U. H. 1948*]

2. Find the locus of straight lines drawn through a fixed point (α, β, γ) at right angles to their polars with respect to $ax^2+by^2+cz^2=1$; rectangular axes. [*B. U. H. 1948*]

3. Find the conditions that the lines

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$$

should be polar with respect to the conicoid $ax^2+by^2+cz^2=1$.

[The equations of the polar of the line (1) w.r.t. the conicoid are

$$axx_1+byy_1+czz_1=1,\ axl_1+bym_1+czn_1=0 \ \ldots(3)$$

[Rule (Art. 123)]

If it is the same as the line (2), then *any* pt. on the line (2), *viz.*, $(x_2+l_2r,\ y_2+m_2r,\ z_2+n_2r)$ lies on the line (3) for *all* values of r.]

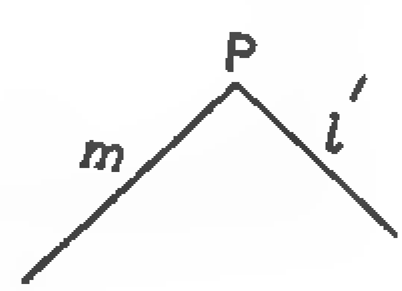
4. Find the condition that the line $\dfrac{x-x_1}{l_1}=\dfrac{y-y_1}{m_1}=\dfrac{z-z_1}{n_1}$

should intersect the polar of the line $\dfrac{x-x_2}{l_2}=\dfrac{y-y_2}{m_2}=\dfrac{z-z_2}{n_2}$

with respect to the conicoid $ax^2+by^2+cz^2=1$.

5. Prove that if the polar of a line l with respect to a conicoid intersects a line l', then the polar of l' intersects l. [*P.U.H. 1938*]

[**Conjugate lines. Def.** Two lines, which are such that the polar of each w. r. t. a conicoid intersects the other, are called **conjugate lines** w. r. t. the conicoid.]



Let m be the polar of l intersecting l' in P, and m' be the polar of l'

Consider the polar plane of P.

$\because$ m, l are polar lines

$\therefore$ the polar plane of P(a pt. on m) passes thro' l [Def. (Note, Art. 122)].

Again $\because$ l', m' are polar lines

$\therefore$ the polar plane of P (a pt. on l') passes thro' m'.

$\therefore$ the polar plane of P passes thro' l, m'

i.e., l, m' are coplanar.

$\therefore$ m', the polar of l', intersects l.

6. (*a*) **If P, Q are the points (x_1, y_1, z_1), (x_2, y_2, z_2), the polar of PQ with respect to $ax^2+by^2+cz^2 = 1$ is given by**

$$axx_1+byy_1+czz_1=1, \quad axx_2+byy_2+czz_2 = 1.$$

(*b*) If P and Q are two points on a conicoid, the polar of PQ with respect to the conicoid is the line of intersection of the tangent planes at P and Q.

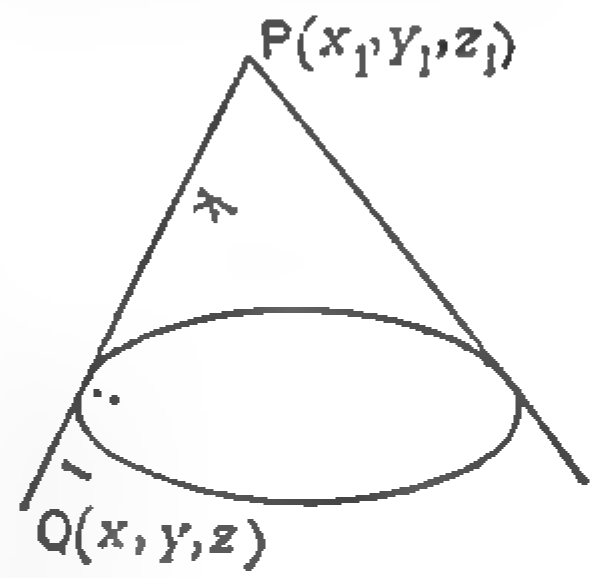
7. Find the equations to the polar of the line $-2x=5y-1=2z$ with respect to the conicoid $2x^2-y^2+2z^2=1$. Prove that it meets the conicoid in two points P and Q, and verify that the tangent planes at P and Q pass through the given line.

8. Find the equations of the polar line of

$$\frac{x+1}{2}=\frac{y-2}{3}=z+3$$

with respect to the sphere $x^2+y^2+z^2=1$. [*P.U. 1956*]

124. *Equation of the (tangent cone or) enveloping cone**. **To find the equation of the (tangent cone or) enveloping cone from the point (x_1, y_1, z_1) to the central conicoid $ax^2+by^2+cz^2 = 1$.**



The equation of the central conicoid is

$$ax^2+by^2+cz^2=1 \quad \dots (1)$$

Let P be the pt. (x_1, y_1, z_1).

Let $Q(x, y, z)$ be *any* pt. on a tangent from P to the conicoid.

The pt. which divides PQ in the ratio $k:1$ is

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right).$$

If it lies on the conicoid (1), then

$$a\left(\frac{kx+x_1}{k+1}\right)^2+b\left(\frac{ky+y_1}{k+1}\right)^2+c\left(\frac{kz+z_1}{k+1}\right)^2=1$$

or, multiplying thro' out by $(k+1)^2$,

$$a(kx+x_1)^2+b(ky+y_1)^2+c(kz+z_1)^2=(k+1)^2$$

or $$a(kx+x_1)^2+b(ky+y_1)^2+c(kz+z_1)^2-(k+1)^2=0$$

or $$k^2(ax^2+by^2+cz^2-1)+2k(axx_1+byy_1+czz_1-1)$$
$$+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \dots (2)$$

which is a quadratic in k.

$\because$ PQ *touches* the conicoid, the quadratic (2) has equal roots

$\therefore$ $4(axx_1+byy_1+czz_1-1)^2=4(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)$

or $(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)=(axx_1+byy_1+czz_1-1)^2 \dots (3)$

which is the required equation of the enveloping cone.

* See Def. Art. 93.

Abridged notation. If $S = ax^2+by^2+cz^2-1$, so that $S=0$ is the equation of the central conicoid,

$S_1=ax_1^2+by_1^2+cz_1^2-1$, so that S_1 is the result of substituting the co-ordinates of the pt. (x_1, y_1, z_1) in S,

$T=axx_1+byy_1+czz_1-1$, so that $T=0$ is the equation of the tangent plane at (x_1, y_1, z_1), then, from (3), the equation of the enveloping cone is $$SS_1 = T^2.$$

[**Check.** See **Check** in Art. 94.]

Note 1. Important. *The above equation of the enveloping cone* ($SS_1=T^2$) *holds even if* $S=0$ *(instead of being the equation of a central conicoid) is the equation of any conicoid.*

Note 2. Compare, in Analytical Plane Geometry, the equation of the pair of tangents ($SS_1=T^2$) from the point (x_1, y_1) to the conic $S = 0$.

EXAMPLES

1. Find the locus of the tangents drawn from a given point (α, β, γ) to the conicoid $ax^2+by^2+cz^2 = 1$. [*P. U. H. 1938*]

Find the locus of points from which three mutually perpendicular tangent lines can be drawn to the surface
$$ax^2 + by^2 + cz^2 = 1.$$ [*P. U. H.*]

2. The section of the enveloping cone of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ whose vertex is P by the plane $z=0$ is (i) a parabola, (ii) a rectangular hyperbola. Find the locus of P.

[*D. U. H. 1958*]

3. Find the locus of a luminous point if the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ casts a circular shadow on the plane $z=0$.

[*Ag. U. 1943*]

4. The plane $z=a$ meets any enveloping cone of the sphere $x^2+y^2+z^2=a^2$ in a conic which has a focus at the point $(0, 0, a)$.

[*P. U. H. 1938*]

5. Show that the lines drawn through the point (α, β, γ) whose direction-cosines satisfy $al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0$, generate the cone

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2 +2f(y-\beta)(z-\gamma)+2g(z-\gamma)(x-\alpha)+2h(x-\alpha)(y-\beta)=0.$$

Show that the equation of the cone whose vertex is the origin and generators parallel to the generators of the above cone is
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

Cor. Important. *If the equation*
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$
represents a cone, the equation of the cone whose vertex is the origin and

generators parallel to those of the above cone is

$$\mathbf{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0},$$

i.e., **second degree terms$=0$.**

Note. Compare, in Analytical Plane Geometry, the following: If the equation $ax^2+2hxy+by^2+2gx+2fy+c=0$ represents two straight lines, the equation of the lines through the origin parallel to them is $\mathbf{ax^2+2hxy+by^2=0}$, *i.e.,* **second degree terms$=0$.**

6. Lines drawn from the centre of a central conicoid parallel to the generators of the enveloping cone whose vertex is P generate a cone which intersects the conicoid in two conics whose planes are parallel to the polar plane of P.

Let the equation of the central conicoid be $ax^2+by^2+cz^2=1$ ...(1)

Let P be the pt. (x_1, y_1, z_1).

The equation of the enveloping cone of the conicoid (1), whose vertex is P, is

$$(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)=(axx_1+byy_1+czz_1-1)^2$$

[$SS_1=T^2$ (Art. 124)]

or $(ax^2+by^2+cz^2)(ax_1^2+by_1^2+cz_1^2-1)-(ax_1^2+by_1^2+cz_1^2-1)$

$$-[(axx_1+byy_1+czz_1)^2-2(axx_1+byy_1+czz_1)+1]=0 \quad ...(2)$$

The equation of the cone generated by lines drawn from the centre of the conicoid (1), *i.e.,* from the origin and $\parallel$ to the generators of the enveloping cone (2), is

$$(ax^2+by^2+cz^2)(ax_1^2+by_1^2+cz_1^2-1)-(axx_1+byy_1+czz_1)^2=0.$$

[Second degree terms$=0$ (Ex. 5, Cor.)]

It meets the conicoid (1) where, putting $ax^2+by^2+cz^2=1$,

$$(ax_1^2+by_1^2+cz_1^2-1)-(axx_1+byy_1+czz_1)^2=0,$$

or $(axx_1+byy_1+czz_1)^2=(ax_1^2+by_1^2+cz_1^2-1)$

or $axx_1+byy_1+czz_1=\pm\sqrt{ax_1^2+by_1^2+cz_1^2-1}$, *i.e.,* in two conics [Art. 114, Cor. 2]

whose planes are $\parallel$ to the plane $axx_1+byy_1+czz_1=1$, the polar plane of P w.r.t. the conicoid (1).

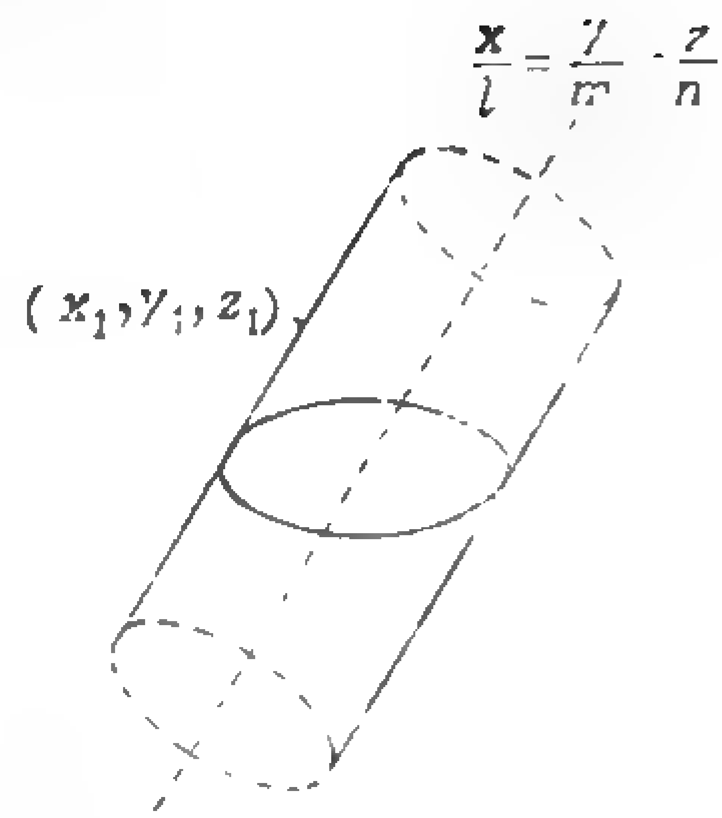
125. *Equation of the enveloping cylinder*.* **To find the equation of the enveloping cylinder of the central conicoid $\mathbf{ax^2+by^2+cz^2=1}$, whose generators are parallel to the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$.**

---

*See Def. Art. 106.

The equation of the central conicoid is $ax^2+by^2+cz^2=1$ ...(1)
and the equations of the given line are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(2)$$



Let (x_1, y_1, z_1) *be any pt. on a tangent* || *to the line* (2). [*Note this step*]

Then the equations of the tangent are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Any pt. on this line is

(x_1+lr, y_1+mr, z_1+nr).

If it lies on the conicoid (1), then

$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)$
$+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(3)

which is a quadratic in r.

$\because$ the line *touches* the conicoid, the quadratic (3) has equal roots

$\therefore$ $4(alx_1+bmy_1+cnz_1)^2=4(al^2+bm^2+cn^2)(ax_1^2+by_1^2+cz_1^2-1)$ ...(4)

[Cancel 4]

$\therefore$ the locus of (x_1, y_1, z_1) is [changing (x_1, y_1, z_1) to (x, y, z)],

$(alx+bmy+cnz)^2=(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1)$

or $(ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)=(alx+bmy+cnz)^2$ ...(5)

which is the required equation of the enveloping cylinder.

****Abridged notation.*** If $S=ax^2+by^2+cz^2-1$, so that $S=0$ is the equation of the central conicoid,

$s_1=al^2+bm^2+cn^2$, so that s_1 is the result of substituting the co-ordinates of the pt. (l, m, n) in S, *the constant term* (-1) *being omitted,*

$t=axl+bym+czn$, so that $t=0$ is the equation of the tangent plane at (l, m, n), *the constant term* (-1) *being omitted,*

then, from (5), the equation of the enveloping cylinder is

$$\mathbf{Ss_1=t^2}.$$

[Compare and contrast with the equation $(SS_1=T^2)$ of the enveloping cone (Art. 124).]

Caution. See **Caution** in Art. 107, (*a*).

****Cor. Condition of tangency of a line and a central conicoid.***

The condition, that the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ *may touch the central conicoid* $ax^2+by^2+cz^2=1$, *is*

$$(ax_1^2+by_1^2+cz_1^2-1)(al^2+bm^2+cn^2)=(alx_1+bmy_1+cnz_1)^2.$$

[From (4)]

EXAMPLES

1. What is meant by an enveloping cylinder of an ellipsoid ?
[*Ag. U. 1948*]

Find the locus of the tangents to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

which are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. [*P.U.H. 1951*]

2. **Deduce from the equation of the enveloping cone of the central conicoid** $ax^2+by^2+cz^2=1$, **the equation of the enveloping cylinder whose generators are parallel to the line** $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

[Proceed as in Art. 107, (*b*).]

3. **Find the equation of the cone whose vertex is** (x_1, y_1, z_1) **and which envelops the ellipsoid** $x^2/a^2+y^2/b^2+z^2/c^2=1$. **Deduce the equation of the enveloping cylinder whose generators are parallel to** $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. [*P. U. H. 1950*]

4. **Prove that the enveloping cylinders of the ellipsoid**

$$x^2/a^2+y^2/b^2+z^2/c^2=1,$$

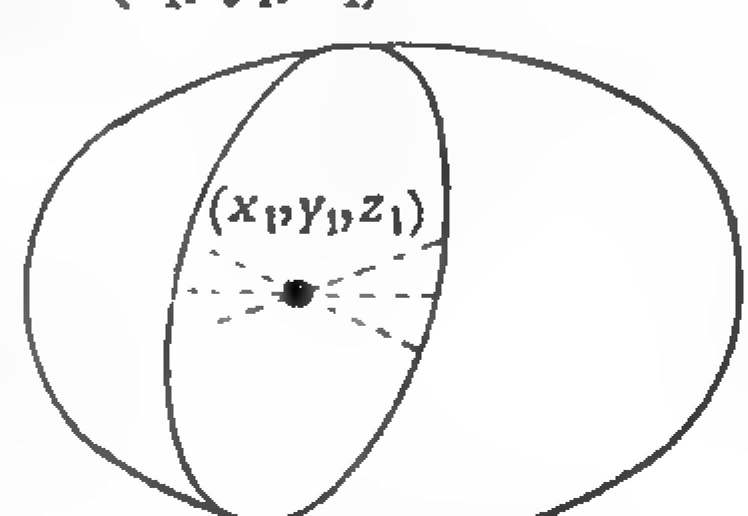
whose generators are parallel to the lines

$$\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c},$$

meet the plane $z=0$ **in circles.** [*D. U. H. 1954*]

5. Prove that the polar of a line AB with respect to a central conicoid is the line of intersection of the planes of contact of the enveloping cone whose vertex is A and the enveloping cylinder whose generators are parallel to AB. [*P. U. H. 1958*]

126. *Equation of the plane of the section with a given centre.* **To find the equation of the plane of the section of the central conicoid** $ax^2+by^2+cz^2=1$, **whose centre is** (x_1, y_1, z_1).



The equation of the central conicoid is

$$ax^2+by^2+cz^2=1 \quad ...(1)$$

[To find the locus of the chords of the conicoid (1) whose mid-pt. is (x_1, y_1, z_1).]

Let (x_1, y_1, z_1) be the mid-pt. of *any* chord.

Let the equations of the chord [thro' (x_1, y_1, z_1)] be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(2)$$

Any pt. on this line is $(x_1+lr,\ y_1+mr,\ z_1+nr)$.

If it lies on the conicoid (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2 = 1$$

or $$r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)$$
$$+(ax_1^2+by_1^2+cz_1^2-1) = 0 \quad \ldots (3)$$

which is a quadratic in r.

$\because$ (x_1, y_1, z_1) is the mid-pt. of the chord, the quadratic (3) has equal and opposite roots,

$\therefore$ sum of the roots = 0, $\therefore$ coeff. of $r = 0$,

i.e., $$alx_1+bmy_1+cnz_1 = 0 \quad \ldots (4)$$

Eliminating l, m, n from (2) and (4) [by substituting their values from (2) in (4)], the locus of the chords is

$$a(x-x_1)x_1+b(y-y_1)y_1+c(z-z_1)z_1 = 0$$

or $$axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2 \quad \ldots (5)$$

which is the required equation of the plane of the section.

Abridged notation. If $S=ax^2+by^2+cz^2-1$, so that $S=0$ is the equation of the central conicoid,

$S_1=ax_1^2+by_1^2+cz_1^2-1$, so that S_1 is the result of substituting the co-ordinates of the pt. (x_1, y_1, z_1) in S,

$T=axx_1+byy_1+czz_1-1$, so that $T=0$ is the equation of the tangent plane at (x_1, y_1, z_1), then, from (5), adding (-1) to both sides, the equation of the plane of the section is

$$axx_1 + byy_1 + czz_1-1 = ax_1^2 + by_1^2 + cz_1^2-1,$$

or $$T = S_1.$$

[**Check.** The equation of the plane of the section (5) is satisfied by the co-ordinates of the given pt. (x_1, y_1, z_1) thus,

$$ax_1^2 + by_1^2 + cz_1^2 = ax_1^2 + by_1^2 + cz_1^2.\ ]$$

Note 1. Compare, in Analytical Plane Geometry, the equation of the chord $(T=S_1)$ of the conic $S=0$, whose mid-point is (x_1, y_1).

Note 2. Important. The above equation of the plane of the section $(T=S_1)$ holds even if $S=0$ (instead of being the equation of a central conicoid) is the equation of *any* conicoid.

EXAMPLES

1. Find the equation of the plane which cuts the conicoid $ax^2+by^2+cz^2=1$ in a conic of which the point (α, β, γ) is the centre. [*Bar. U. 1954*]

Find the equation to the plane which cuts $x^2+4y^2-5z^2 = 1$ in a conic whose centre is at the point (2,3,4). [*P.U.B. Sc.H. 1951*]

2. (*a*) **Find the locus of chords of the ellipsoid**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

which are bisected at (f, g, h). [*P. U. H. 1951*]

(*b*) **The locus of the centres of all plane sections of a conicoid which pass through a fixed point is a conicoid.**

[(*b*) **Rule to find the locus of the centre of a section of a conicoid, which (section) satisfies a given condition :**

(*i*) *Let* (x_1, y_1, z_1) *be the centre of a section. Write down the equation of the plane of the section* $(T=S_1)$.

(*ii*) *Make the section satisfy the given condition.*

(*iii*) *Find the locus of* (x_1, y_1, z_1) [*by changing* (x_1, y_1, z_1) *to* (x, y, z)]. *This is the required locus.*

Note. *For problems relating to a conicoid (whose equation is not given), let the equation of the (central) conicoid be* $\mathbf{ax^2+by^2+cz^2=1}$.]

3. **Prove that the locus of the centres of parallel plane sections of a conicoid is a diameter.** [*P. U. H. 1950*]

Prove also that the tangent plane at an extremity of the diameter is parallel to the plane sections.

4. The locus of the centres of sections of a central conicoid which are parallel to a given line is a plane.

5. **Find the locus of centres of sections of** $\mathbf{ax^2+by^2+cz^2=1}$ **which touch** $\alpha\mathbf{x^2}+\beta\mathbf{y^2}+\gamma\mathbf{z^2=1}$. [*P. U. H. 1958*]

6. **Find the locus of the centres of sections of a conicoid that are at a constant distance from the centre.** [*P. U. H.*]

7. **Show that the centres of sections of a conicoid that pass through a given line lie on a conic.** [*Bar. U. 1953*]

[**Note. Equations of a conic.** Two equations, one of a conicoid and the other of a plane, together represent a conic (Art. 19). (See Art. 114, Cor. 2.)]

8. *Equation of the plane of the section of a sphere with a given centre.* **Find the equation of that section of a sphere**

$$\mathbf{x^2+y^2+z^2=a^2}$$

of which a given internal point $(\mathbf{x_1, y_1, z_1})$ **is the centre.**

[*P. U. 1939 S*]

9. **Converse problem.** Find the centre of the conic given by the equations $2x-2y-5z+5=0,\ 3x^2+2y^2-15z^2=4$.

[*P. U. B.Sc. H. 1955*]

127. *To prove that the plane YOZ bisects all chords of the central conicoid* $ax^2+by^2+cz^2=1$, *parallel to OX.*

The equation of the central conicoid is $ax^2+by^2+cz^2=1$ ...(1)

Let the equations of *any* chord ∥ to OX $(y=0, z=0)$ be $y=\mu,\ z=\nu$.

It meets the conicoid (1) where [putting $y=\mu$, $z=\nu$ in (1)],

$$ax^2+b\mu^2+c\nu^2=1, \text{ or } x^2=\frac{1-b\mu^2-c\nu^2}{a}, \text{or } x=\pm\sqrt{\frac{1-b\mu^2-c\nu^2}{a}},$$

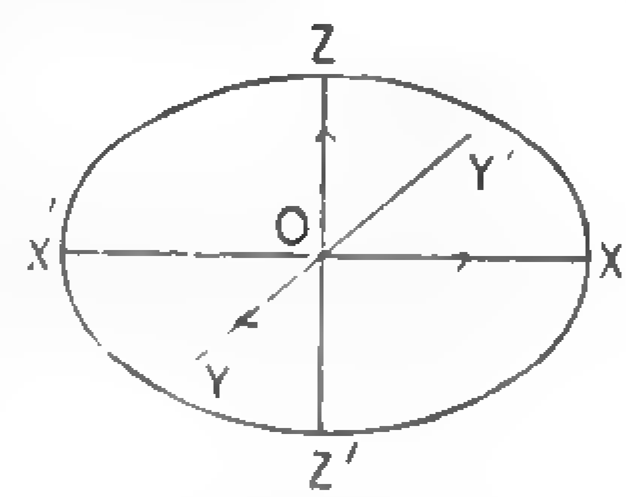
i.e., in the pts. $P\left(\sqrt{\frac{1-b\mu^2-c\nu^2}{a}}, \mu, \nu\right)$, $P'\left(-\sqrt{\frac{1-b\mu^2-c\nu^2}{a}}, \mu, \nu\right)$.

The mid-pt. of PP′ is $(0, \mu, \nu)$. [Art. 5, Cor.]
which lies on the plane YOZ $(x=0)$.

$\therefore$ the plane YOZ bisects all chords of the conicoid, to OX.

Similarly the plane ZOX bisects all chords of the conicoid, to OY, and the plane XOY bisects all chords of the conicoid, to OZ.

128. (*a*) **Diametral plane. Def.** *The plane which bisects a system of parallel chords of a conicoid is called a* **diametral plane.**



Thus the plane YOZ is the diametral plane of the conicoid $ax^2+by^2+cz^2=1$, which bisects chords parallel to OX (Art. 127) or, more shortly, the plane YOZ is the **diametral plane of OX.**

Similarly the plane ZOX is the diametral plane of OY, and the plane XOY is the diametral plane of OZ.

(*b*) **Conjugate diametral planes and conjugate diameters.**

Def. 1. Conjugate diametral planes. Three planes, which are such that each bisects chords parallel to the line of intersection of the other two, are called **conjugate diametral planes** of the central conicoid.

Thus the planes YOZ, ZOX, XOY are **conjugate diametral planes** of the central conicoid $ax^2+by^2+cz^2=1$ (Art. 127).

Def. 2. Conjugate diameters. *Three diameters, which are such that the plane through any two bisects chords parallel to the third, are called* **conjugate diameters** *of the central conicoid.*

Thus X′OX, Y′OY, Z′OZ are **conjugate diameters** of the central conicoid $ax^2+by^2+cz^2=1$ (Art. 127).

(*c*) **Principal planes and principal axes.**

Def. 1. Principal planes. Diametral planes, which are perpendicular to the chords which they bisect, are called **principal planes.**

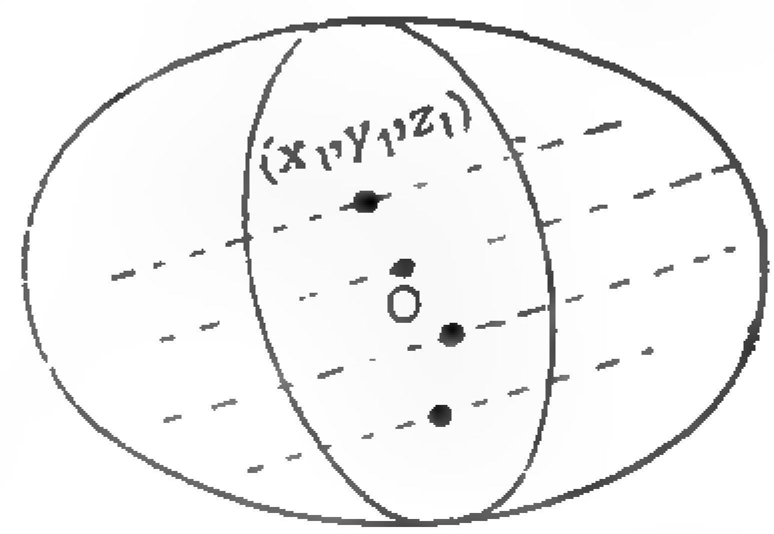
Thus the planes YOZ, ZOX, XOY are **principal planes** of the central conicoid $ax^2+by^2+cz^2=1$.

Def. 2. Principal axes. The lines of intersection of principal planes taken two by two, are called **principal axes.**

Thus X′OX, Y′OY, Z′OZ are the **principal axes** of the central conicoid $ax^2+by^2+cz^2=1$.

Cor. (The axes being rectangular) *The equation* $ax^2+by^2+cz^2=1$ *represents a central conicoid referred to its principal axes as the co-ordinate axes.*

129. Equation of the diametral plane. *To find the equation of the diametral plane of the central conicoid* $ax^2+by^2+cz^2=1$, *which bisects chords parallel to the line* $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$.



The equation of the central conicoid is

$$ax^2+by^2+cz^2=1 \quad ...(1)$$

and the equations of the line are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(2)$$

[To find the locus of the mid-pts. of chords of the conicoid (1), ∥ to the line (2). (Def. Art. 128, (*a*))]

Let (x_1, y_1, z_1) be the mid-pt. of *any* chord ∥ to the line (2).

Then the equations of the chord are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the conicoid (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1$$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(3)

which is a quadratic in r.

∵ (x_1, y_1, z_1) is the mid-pt. of the chord, the quadratic (3) has equal and opposite roots, ∴ sum of the roots $=0$, ∴ coeff. of $r=0$, *i.e.*, $alx_1+bmy_1+cnz_1=0$

∴ the locus of (x_1, y_1, z_1) is [changing (x_1, y_1, z_1) to (x, y, z)],

$$\mathbf{alx+bmy+cnz=0} \quad ...(4)$$

which is the required equation of the diametral plane.

Cor. *A diametral plane of a central conicoid passes through the centre of the conicoid.*

For, from (4), the diametral plane passes thro' (0, 0, 0), the centre of the conicoid.

Abridged notation. If $F(x, y, z)=ax^2+by^2+cz^2-1$, so that $F(x, y, z)=0$ is the equation of the central conicoid, then

$$\frac{\partial F}{\partial x}=a.2x,\ \frac{\partial F}{\partial y}=b.2y,\ \frac{\partial F}{\partial z}=c.2z$$

$\therefore$ from (4), the equation of the diametral plane is

$$l.\tfrac{1}{2}\frac{\partial F}{\partial x}+m.\tfrac{1}{2}\frac{\partial F}{\partial y}+n.\tfrac{1}{2}\frac{\partial F}{\partial z}=0$$

or
$$\mathbf{l}\frac{\partial \mathbf{F}}{\partial \mathbf{x}}+\mathbf{m}\frac{\partial \mathbf{F}}{\partial \mathbf{y}}+\mathbf{n}\frac{\partial \mathbf{F}}{\partial \mathbf{z}}=\mathbf{0}.$$

Note. The above equation of the diametral plane

$$\left(l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0\right)$$

holds even if $F(x, y, z)=0$ (instead of being the equation of a central conicoid) is the equation of *any* conicoid.

EXAMPLES

1. Find the equation of the diametral plane of the line $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$ with regard to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

Prove that the diametral plane of any diameter of an ellipsoid is parallel to the tangent plane at an extremity of that diameter.

[(*b*) Let the equation of the ellipsoid be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ ...(1)

Let $P(x_1, y_1, z_1)$ be one extremity of the diameter. Then the equations of the diameter OP are $\dfrac{x}{x_1}=\dfrac{y}{y_1}=\dfrac{z}{z_1}$. Find the equation of its diametral plane and that of the tangent plane at $P(x_1, y_1, z_1)$ to the ellipsoid (1), and show that they are ||. Find the equation of the tangent plane at the other extremity $P'(-x_1, -y_1, -z_1)$ of the diameter, and show that the diametral plane of OP is also || to this tangent plane.]

2. For the conicoid $x^2-2y^2+3z^2+4=0$, find (*i*) the equations of the tangent plane and the normal at $(-1, 2, 1)$; (*ii*) equation of the diametral plane bisecting chords whose direction-cosines are proportional to $(1, 2, 2)$, and (*iii*) pole of the plane $x+y-2z=1$.

[*P. U. B.Sc. H. 1950*]

3. Find the locus of the mid-points of chords of the conicoid $ax^2+by^2+cz^2=1$ which pass through the point (α, β, γ). [*P. U. H.*]

[The equation of the conicoid is $ax^2+by^2+cz^2=1$ ... (1)

Let (x_1, y_1, z_1) be the mid-pt. of *any* chord. Let the equations of the chord [thro' (x_1, y_1, z_1)] be $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ ...(2)

Proceed as in Art. 129. It will be found that

$$alx_1+bmy_1+cnz_1=0 \quad ...(3)$$

If the chord (2) passes thro' (α, β, γ), then

$$\frac{\alpha-x_1}{l}=\frac{\beta-y_1}{m}=\frac{\gamma-z_1}{n} \quad ...(4)$$

Eliminate l, m, n from (3) and (4) [by substituting their values from (4) in (3).

Note. Important. *For problems relating to the locus of the mid-points of chords of a central conicoid $ax^2+by^2+cz^2=1$, let (x_1, y_1, z_1) be the mid-point of* **any** *chord.*

Let the equations of the chord [thro' (x_1, y_1, z_1)] be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Then $alx_1+bmy_1+cnz_1=0$. (As in Art. 129)]

Conjugate diametral planes and conjugate diameters of an ellipsoid.

130. *If P, Q are points on an ellipsoid (centre O), and the diametral plane of OP passes through Q, then the diametral plane of OQ passes through P.*

Let the equation of the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ... (1)

Let P, Q be the pts. (x_1, y_1, z_1), (x_2, y_2, z_2).

The equations of OP are O(0, 0, 0) P(x_1, y_1, z_1)

$$\frac{x-0}{x_1-0}=\frac{y-0}{y_1-0}=\frac{z-0}{z_1-0} \quad \left[\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1} \text{ (Art. 40)}\right]$$

or $$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1} \quad ...\ (2)$$

$\therefore$ the equation of the diametral plane of OP w.r.t. the ellipsoid (1) is $x_1\frac{2x}{a^2}+y_1\frac{2y}{b^2}+z_1\frac{2z}{c^2}=0$ $\left[l\frac{\partial F}{\partial x}+m\frac{\partial F}{\partial y}+n\frac{\partial F}{\partial z}=0\right.$ (Art. 129), here $l=x_1$, $m=y_1$, $n=z_1$, $\left.F(x, y, z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right]$

or $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0.$$

If it passes thro' Q, then

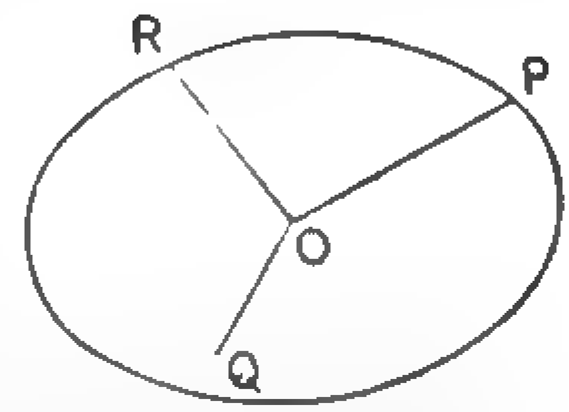
$$\frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}=0, \text{ or } \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0.$$

The *symmetry* of this result shows that it is also the condition that the diametral plane of OQ should pass thro' P.

Cor. Conjugate diametral planes and conjugate semi-diameters of an ellipsoid. (*See Fig. on the next page.*) If P is a point on an ellipsoid, and Q a point on the diametral plane of OP and on the ellipsoid, and OR the line of intersection of the diametral

planes of OP and OQ (R on the ellipsoid), then the planes QOR, ROP, POQ are **conjugate diametral planes,** and OP, OQ, OR are **conjugate semi-diameters.**

[**Proof.** The plane QOR is the diametral plane of OP.



[$\because$ the diametral plane of OP passes thro' Q, and also thro' R (Given)]

Again the plane ROP is the diametral plane of OQ.

[$\because$ the diametral plane of OQ passes thro' R (Given). Also the diametral plane of OQ passes thro' P since the diametral plane of OP passes thro' Q (Given)]

Also the plane POQ is the diametral plane of OR.

[$\because$ the diametral plane of OR passes thro' P since the diametral plane of OP passes thro' R (Given). Similarly the diametral plane of OR passes thro' Q]]

$\therefore$ the planes QOR, ROP, POQ are conjugate diametral planes. [Def. (Art. 128, (*b*))]

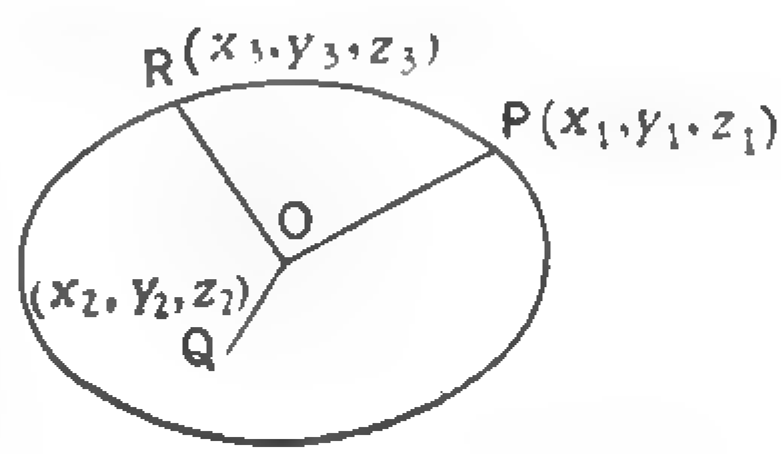
$\therefore$ OP, OQ, OR are conjugate semi-diameters. [Def. (Art. 128, (*b*))]

131. To find the relations between the co-ordinates of the extremities of three conjugate semi-diameters of the ellipsoid

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}+\frac{\mathbf{z}^2}{\mathbf{c}^2}=\mathbf{1}.$$

The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \dots (1)$$



Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the co-ordinates of the extremities P, Q, R of three conjugate semi-diameters OP, OQ, OR.

$\because$ P, Q, R lie on the ellipsoid (1)

$$\therefore \quad \left.\begin{aligned} \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}&=1,\\ \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}&=1,\\ \frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}&=1. \end{aligned}\right\} \dots (A)$$

The equation of the diametral plane of OP is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0. \qquad [\text{ Art. } 130]$$

$\because$ it passes thro' Q, R

$$\left.\begin{aligned} \therefore \qquad & \frac{x_2x_1}{a^2}+\frac{y_2y_1}{b^2}+\frac{z_2z_1}{c^2}=0, \\ & \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0, \\ \text{Similarly } & \frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0. \end{aligned}\right\} \quad \ldots \text{(B)}$$

From (A) and (B),

$$\left.\begin{aligned} & \frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c}; \\ & \frac{x_2}{a},\ \frac{y_2}{b},\ \frac{z_2}{c}; \\ & \frac{x_3}{a},\ \frac{y_3}{b},\ \frac{z_3}{c} \end{aligned}\right\}$$

are the direction-cosines of three mutually $\perp$ lines
[Art. 9, and Art. 13, (*a*), Cor. 3]

$\therefore$ (changing columns into rows),

$$\left.\begin{aligned} & \frac{x_1}{a},\ \frac{x_2}{a},\ \frac{x_3}{a}; \\ & \frac{y_1}{b},\ \frac{y_2}{b},\ \frac{y_3}{b}; \\ & \frac{z_1}{c},\ \frac{z_2}{c},\ \frac{z_3}{c} \end{aligned}\right\}$$

are also the direction-cosines of three mutually $\perp$ lines [Art. 58, II (*i*)]

$$\left.\begin{aligned} \therefore\ & \frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2}=1, \text{ or } \mathbf{x_1^2+x_2^2+x_3^2=a^2}; \\ & \frac{y_1^2}{b^2}+\frac{y_2^2}{b^2}+\frac{y_3^2}{b^2}=1, \text{ or } y_1^2+y_2^2+y_3^2=b^2; \\ & \frac{z_1^2}{c^2}+\frac{z_2^2}{c^2}+\frac{z_3^2}{c^2}=1, \text{ or } z_1^2+z_2^2+z_3^2=c^2. \end{aligned}\right\} \ldots\text{(C)}$$

$$\left.\begin{aligned} & \frac{y_1}{b}\cdot\frac{z_1}{c}+\frac{y_2}{b}\cdot\frac{z_2}{c}+\frac{y_3}{b}\cdot\frac{z_3}{c}=0, \text{ or } \mathbf{y_1z_1+y_2z_2+y_3z_3=0}; \\ & \frac{z_1}{c}\cdot\frac{x_1}{a}+\frac{z_2}{c}\cdot\frac{x_2}{a}+\frac{z_3}{c}\cdot\frac{x_3}{a}=0, \text{ or } z_1x_1+z_2x_2+z_3x_3=0; \\ & \frac{x_1}{a}\cdot\frac{y_1}{b}+\frac{x_2}{a}\cdot\frac{y_2}{b}+\frac{x_3}{a}\cdot\frac{y_3}{b}=0, \text{ or } x_1y_1+x_2y_2+x_3y_3=0. \end{aligned}\right\} \ldots\text{(D)}$$

(A), (B), (C), (D) are the required relations.

[Aid to memory for (A), (B), (C), (D).

Remember :
$$\begin{aligned} & \frac{x_1}{a},\ \frac{y_1}{b},\ \frac{z_1}{c}; \\ & \frac{x_2}{a},\ \frac{y_2}{b},\ \frac{z_2}{c}; \\ & \frac{x_3}{a},\ \frac{y_3}{b},\ \frac{z_3}{c} \end{aligned}$$

are the direction-cosines of three mutually $\perp$ lines.

$\therefore$ (changing columns into rows),

$$\frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a};$$
$$\frac{y_1}{b}, \frac{y_2}{b}, \frac{y_3}{b}$$
$$\frac{z_1}{c}, \frac{z_2}{c}, \frac{z_3}{c}$$

are also the direction-cosines of three mutually $\perp$ lines.

Now for each set, use $l^2+m^2+n^2=1$ (Art. 9), and $ll'+mm'+nn'=0$ (Art. 13, (*a*), Cor. 3).]

EXAMPLES

1. (*a*) Define conjugate diameters and conjugate diametral planes of an ellipsoid.

(*b*) If $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ are the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, find the equation to the plane PQR. [*Ag. U.*]

(*b*) The equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

Let the required equation of the plane PQR be

$$lx+my+nz=p \quad ...(2)$$

$\because$ it passes thro' $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$,

$$\therefore \quad lx_1+my_1+nz_1=p \quad ...(3) \qquad x_1$$
$$lx_2+my_2+nz_2=p \quad ...(4) \qquad x_2$$
$$lx_3+my_3+nz_3=p \quad ...(5) \qquad x_3$$

Multiplying (3), (4), (5) by x_1, x_2, x_3, and adding vertically,

$$l(x_1^2+x_2^2+x_3^2)+m(x_1y_1+x_2y_2+x_3y_3)+n(z_1x_1+z_2x_2+z_3x_3)=p(x_1+x_2+x_3)$$

[But (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) being the extremities of three conjugate semi-diameters of the ellipsoid (1),

$$\left.\begin{array}{l} x_1^2+x_2^2+x_3^2=a^2, \text{ and so on ;} \\ y_1z_1+y_2z_2+y_3z_3=0, \text{ and so on ;} \end{array}\right\} \text{(Art. 131, (C), (D))]}$$

$$\therefore \quad l(a^2)+m(0)+n(0)=p(x_1+x_2+x_3)$$

$$\therefore \quad l=p\left(\frac{x_1+x_2+x_3}{a^2}\right).$$

Similarly $m=p\left(\frac{y_1+y_2+y_3}{b^2}\right)$, $n=p\left(\frac{z_1+z_2+z_3}{c^2}\right)$.

Substituting these values of l, m, n (in terms of p) in (2),

$$p\left(\frac{x_1+x_2+x_3}{a^2}\right)x+p\left(\frac{y_1+y_2+y_3}{b^2}\right)y+p\left(\frac{z_1+z_2+z_3}{c^2}\right)z=p$$

or, dividing thro' out by p,

$$\mathbf{x\left(\frac{x_1+x_2+x_3}{a^2}\right)+y\left(\frac{y_1+y_2+y_3}{b^2}\right)+z\left(\frac{z_1+z_2+z_3}{c^2}\right)=1,}$$

which is the required equation.

2. If P, Q, R are the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, show that the pole of the plane PQR lies on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3$. [*Ag. U. 1954*]

3. If P, Q, R are the extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, prove that the plane PQR touches the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}$ at the centroid of the triangle PQR. [*P. U. H. 1959*]

4. If the cone $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$ passes through a set of conjugate diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

then $Aa^2+Bb^2+Cc^2=0$. [*D. U. H. 1935*]

5. If three conjugate diameters of an ellipsoid meet the director sphere in P, Q, R, prove that the plane PQR touches the ellipsoid. [*P. U.*]

[Let the equation of the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

The equation of the director sphere is $x^2+y^2+z^2=a^2+b^2+c^2$...(2)

Let the equation of the plane PQR be $lx+my+nz=p$ ...(3)

Then the equation of the cone, whose vertex is the origin and which passes thro' the pts. of intersection of the director sphere (2) and the plane (3), is

$$x^2+y^2+z^2=(a^2+b^2+c^2)\left(\frac{lx+my+nz}{p}\right)^2 \quad ...(4) \text{ (As in Ex. 1, Art. 87)}$$

$\because$ it passes thro' three conjugate semi-diameters (direction-cosines proportional to x_1, y_1, z_1 ; x_2, y_2, z_2 ; x_3, y_3, z_3),

$$\therefore \quad x_1^2+y_1^2+z_1^2 = (a^2+b^2+c^2)\left(\frac{lx_1+my_1+nz_1}{p}\right)^2. \qquad \text{[Art. 88]}$$

Write down two similar results for (x_2, y_2, z_2), (x_3, y_3, z_3), and add vertically.]

6. Prove that the locus of the point of intersection of three tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to conjugate diametral planes of $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$, is $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}$.

[*P. U. H. 1956*]

What does the theorem become when $\alpha=\beta=\gamma$?

****7. The locus of the foot of the perpendicular from the centre of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ to the plane through the extremities of three conjugate semi-diameters is**

$$a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2. \qquad [P.\ U.\ H.\ 1951]$$

[Let $N(x_0, y_0, z_0)$ be the foot of the $\perp$ from O on the plane PQR.

Then the equation of the plane PQR [thro' $N(x_0, y_0, z_0)$ and $\perp$ to ON (direction-cosines proportional to x_0, y_0, z_0)] is

$$x_0(x-x_0)+y_0(y-y_0)+z_0(z-z_0)=0$$

or $$xx_0+yy_0+zz_0=x_0^2+y_0^2+z_0^2 \quad \dots (1)$$

$\because$ it is the same as the equation of the plane PQR

$$x\left(\frac{x_1+x_2+x_3}{a^2}\right)+y\left(\frac{y_1+y_2+y_3}{b^2}\right)+z\left(\frac{z_1+z_2+z_3}{c^2}\right)=1 \quad \dots (2)$$

[Ex. 1, (b), Art. 131]

$\therefore$ comparing coeffs. in (1) and (2),

$$\frac{a^2x_0}{x_1+x_2+x_3}=\frac{b^2y_0}{y_1+y_2+y_3}=\frac{c^2z_0}{z_1+z_2+z_3}=\frac{x_0^2+y_0^2+z_0^2}{1}$$

$$\therefore \quad \frac{x_1+x_2+x_3}{a}=\frac{ax_0}{x_0^2+y_0^2+z_0^2} \quad \dots (3)$$

Similarly $$\frac{y_1+y_2+y_3}{b}=\frac{by_0}{x_0^2+y_0^2+z_0^2} \quad \dots (4),$$

$$\frac{z_1+z_2+z_3}{c}=\frac{cz_0}{x_0^2+y_0^2+z_0^2} \quad \dots (5)$$

Square (3), (4), (5), and add vertically.

Note. Important. For problems relating to the locus of the foot of the perpendicular from the origin on a plane, let $P(x_1, y_1, z_1)$ be the foot of the perpendicular. Then the equation of the plane [thro' $P(x_1, y_1, z_1)$ and $\perp$ to OP (direction-cosines proportional to x_1, y_1, z_1)] is $xx_1+yy_1+zz_1=x_1^2+y_1^2+z_1^2$.

In Ex. 7, in order to avoid confusion we have taken $N(x_0, y_0, z_0)$ to be the foot of the perpendicular.]

Three important properties of three conjugate semi-diameters of an ellipsoid.

132. (*a*) **To prove that the sum of the squares of three conjugate semi-diameters of an ellipsoid is constant.**

Let the equation of the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ be the extremities of three conjugate semi-diameters. Then O(0, 0, 0) $P(x_1, y_1, z_1)$

$$OP^2 = (x_1-0)^2 + (y_1-0)^2 + (z_1-0)^2$$

or $$OP^2 = x_1^2+y_1^2 + z_1^2.$$

Similarly $OQ^2 = x_2^2+y_2^2 + z_2^2,$

$OR^2 = x_3^2+y_3^2 + z_3^2.$

Adding vertically,

$OP^2+OQ^2+OR^2=(x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)+(z_1^2+z_2^2+z_3^2)$

[But (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) being the extremities of three conjugate semi-diameters of the ellipsoid, $x_1^2+x_2^2+x_3^2=a^2$, and so on (Art. 131, (C))]

$=a^2 + b^2 + c^2$, which is constant.

132. (*b*) To prove that the volume of the parallelepiped which has three conjugate semi-diameters of an ellipsoid for coterminous edges is constant.

Let the equation of the ellipsoid be $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ ... (1)

Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$, $R(x_3, y_3, z_3)$ be the extremities of three conjugate semi-diameters.

Then the volume of the parallelepiped which has OP, OQ, OR for coterminous edges is

V= 6 (volume of the tetrahedron OPQR) [From Mensuration]

$$= 6.\tfrac{1}{6}\begin{vmatrix} 0, & 0, & 0, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix} = -\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} \quad ...(2)$$

But $\begin{vmatrix} \frac{x_1}{a}, & \frac{y_1}{b}, & \frac{z_1}{c} \\ \frac{x_2}{a}, & \frac{y_2}{b}, & \frac{z_2}{c} \\ \frac{x_3}{a}, & \frac{y_3}{b}, & \frac{z_3}{c} \end{vmatrix} = \pm 1$ $\left(\begin{vmatrix} l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \\ l_3, & m_3, & n_3 \end{vmatrix} = \pm 1 \right)$

[$\because \dfrac{x_1}{a}, \dfrac{y_1}{b}, \dfrac{z_1}{c}$; etc., are the direction-cosines of three mutually $\perp$ lines (Art. 131 and Art. 58, Cor.)]

[Take $\dfrac{1}{a}$ common from the first column, $\dfrac{1}{b}$ from the second column, and $\dfrac{1}{c}$ from the third]

or $\dfrac{1}{a} \cdot \dfrac{1}{b} \cdot \dfrac{1}{c} \begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = \pm 1$, or $\begin{vmatrix} x_1, & y_1, & z_1 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \end{vmatrix} = \pm abc$

$\therefore$ from (2), $V = -(\pm abc) = \pm abc$.

$\therefore$ V (*in magnitude*) $= abc$, which is constant.

132. (*c*) **To prove that if OP, OQ, OR are three conjugate semi-diameters of an ellipsoid, and A_1, A_2, A_3 are the areas (of the triangles) QOR, ROP, POQ, then $A_1^2+A_2^2+A_3^2$ is constant.

Let the equation of the ellipsoid be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ ...(1)

Let P, Q, R, the extremities of three conjugate semi-diameters, be the pts. (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3).

$$\frac{1}{2}\left|\begin{array}{ccc} 0, & 0, & 0 \\ x_2, & y_2, & z_2 \\ x_3, & y_3, & z_3 \\ 0, & 0, & 0 \end{array}\right|$$

Then $A_1=\triangle\, QOR=\triangle\, OQR$ (*in magnitude*)

$$=\sqrt{[\tfrac{1}{2}(y_2z_3-y_3z_2)]^2+[\tfrac{1}{2}(z_2x_3-z_3x_2)]^2+[\tfrac{1}{2}(x_2y_3-x_3y_2)]^2}$$

$$[\sqrt{A_x^2+A_y^2+A_z^2} \quad \text{(Art. 33)}]$$

$\therefore\quad A_1^2=\frac{1}{4}\,[(y_2z_3-y_3z_2)^2+(z_2x_3-z_3x_2)^2+(x_2y_3-x_3y_2)^2]$ ...(2)

But in the determinant $\left|\begin{array}{ccc} \dfrac{x_1}{a}, & \dfrac{y_1}{b}, & \dfrac{z_1}{c} \\ \dfrac{x_2}{a}, & \dfrac{y_2}{b}, & \dfrac{z_2}{c} \\ \dfrac{x_3}{a}, & \dfrac{y_3}{b}, & \dfrac{z_3}{c} \end{array}\right|$

each constituent $=\pm$ (its co-factor) (Arts. 131 and 59)

$$\therefore\ \frac{x_1}{a}=\pm\left(\frac{y_2}{b}\cdot\frac{z_3}{c}-\frac{y_3}{b}\cdot\frac{z_2}{c}\right),\ \frac{y_1}{b}=\pm\left(\frac{z_2}{c}\cdot\frac{x_3}{a}-\frac{z_3}{c}\cdot\frac{x_2}{a}\right),$$

$$\frac{z_1}{c}=\pm\left(\frac{x_2}{a}\cdot\frac{y_3}{b}-\frac{x_3}{a}\cdot\frac{y_2}{b}\right)$$

or $y_2z_3-y_3z_2=\pm\dfrac{bc}{a}\,x_1,\ z_2x_3-z_3x_2=\pm\dfrac{ca}{b}\,y_1,$

$x_2y_3-x_3y_2=\pm\dfrac{ab}{c}\,z_1.$

$\therefore$ from (2),

$$A_1^2=\frac{1}{4}\left[\frac{b^2c^2}{a^2}\,x_1^2+\frac{c^2a^2}{b^2}\,y_1^2+\frac{a^2b^2}{c^2}\,z_1^2\right].$$

Similarly

$$A_2^2=\frac{1}{4}\left[\frac{b^2c^2}{a^2}\,x_2^2+\frac{c^2a^2}{b^2}\,y_2^2+\frac{a^2b^2}{c^2}\,z_2^2\right],$$

$$A_3^2=\frac{1}{4}\left[\frac{b^2c^2}{a^2}\,x_3^2+\frac{c^2a^2}{b^2}\,y_3^2+\frac{a^2b^2}{c^2}\,z_3^2\right].$$

Adding vertically,

$$A_1^2+A_2^2+A_3^2=\frac{1}{4}\left[\frac{b^2c^2}{a^2}\,(x_1^2+x_2^2+x_3^2)+\frac{c^2a^2}{b^2}\,(y_1^2+y_2^2+y_3^2)\right.$$

$$\left.+\frac{a^2b^2}{c^2}\,(z_1^2+z_2^2+z_3^2)\right]$$

[But (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) being the extremities of three conjugate semi-diameters of the ellipsoid (1), $x_1^2+x_2^2+x_3^2=a^2$, and so on (Art. 131, (C))]

$$= \tfrac{1}{4}\left[\frac{b^2c^2}{a^2}(a^2)+\frac{c^2a^2}{b^2}(b^2)+\frac{a^2b^2}{c^2}(c^2)\right]$$
$$= \tfrac{1}{4}[b^2c^2+c^2a^2+a^2b^2],$$

which is constant.

[Aid to memory for the results of Arts. 132 (a), (b), (c).

Remember OA$(=a)$, OB$(=b)$, OC$(=c)$ are three conjugate semi-diameters of the ellipsoid, *which are mutually* $\perp$.

Hence we have the results :

(a) $OP^2+OQ^2+OR^2=\text{constant}=OA^2+OB^2+OC^2$
$=a^2+b^2+c^2$.

(b) $\left.\begin{array}{l}\text{Volume of the parallelepiped whose}\\ \text{coterminous edges are OP, OQ, OR}\end{array}\right]$ is
V = constant
$=\left[\begin{array}{l}\text{Volume of the parallelepiped whose coterminous}\\ \text{edges are OA, OB, OC}\end{array}\right.$
$=abc$.

(c) If A_1, A_2, A_3 are the areas of the $\triangle$s QOR, ROP, POQ, then
$A_1^2+A_2^2+A_3^2=\text{constant}$
$=(\triangle BOC)^2+(\triangle COA)^2+(\triangle AOB)^2$
$=(\tfrac{1}{2}bc)^2+(\tfrac{1}{2}ca)^2+(\tfrac{1}{2}ab)^2$
$=\tfrac{1}{4}(b^2c^2+c^2a^2+a^2b^2)$.]

EXAMPLES

1. Show that the sum of the squares of the projections of three conjugate semi-diameters of an ellipsoid (i) on any line, (ii) on any plane is constant. [*P. U. H. 1961*]

2. If OP, OQ, OR are conjugate diameters of an ellipsoid, and p_1, p_2, p_3 ; π_1, π_2, π_3 are their projections on any two given lines, then show that $p_1\pi_1+p_2\pi_2+p_3\pi_3$ is constant. [*Ag. U. 1956*]

3. If P, Q, R are the extremities of three equal conjugate semi-diameters of an ellipsoid, and S is the pole of the plane PQR, then the tetrahedron PQRS has any pair of opposite edges at right angles.

4. If the axes are rectangular, find the locus of the equal conjugate diameters of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$. [*Ag. U.*]

The equation of the ellipsoid is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ ...(1)

Let the equations of OP, one of three equal conjugate semi-diameters (thro' the centre (0,0,0)) be $\dfrac{x-0}{l}=\dfrac{y-0}{m}=\dfrac{z-0}{n}$,

or
$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(2)$$
(l, m, n being actual direction-cosines)

Let OP$=r$, then the co-ordinates of P are (lr, mr, nr).　[Art. 8]

$\because$ P lies on the ellipsoid (1)

$$\therefore \quad \frac{l^2r^2}{a^2}+\frac{m^2r^2}{b^2}+\frac{n^2r^2}{c^2}=1, \text{ or } r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=1 \quad ...(3)$$

$\because$ OP $(=r)$ is one of three equal conjugate semi-diameters

$\therefore$ $r^2+r^2+r^2 = a^2+b^2+c^2$

[$OP^2+OQ^2+OR^2 = a^2+b^2+c^2$ (Art. 132, (*a*))]

or
$$3\,r^2 = a^2+b^2+c^2, \text{ or } r^2=\frac{a^2+b^2+c^2}{3}.$$

Substituting this value of r^2 in (3),
$$\left(\frac{a^2+b^2+c^2}{3}\right)\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)=1$$
$$= l^2+m^2+n^2 \quad ... (4)$$

Eliminating l, m, n from (2) and (4) [by substituting their values from (2) in (4)], the locus of the equal conjugate diameters is
$$\left(\frac{a^2+b^2+c^2}{3}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=x^2+y^2+z^2$$

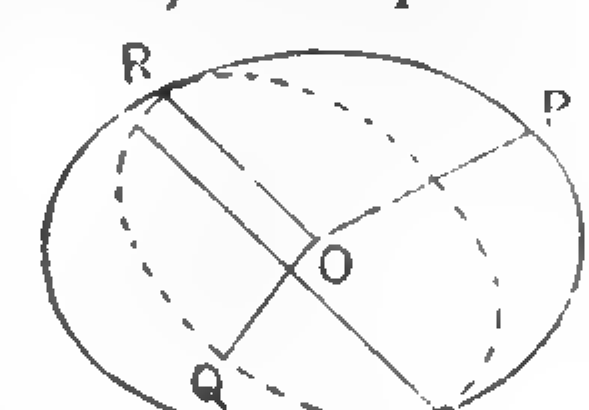
or
$$(a^2+b^2+c^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=3(x^2+y^2+z^2)$$

or
$$x^2\left[3-\frac{a^2+b^2+c^2}{a^2}\right]+y^2\left[3-\frac{a^2+b^2+c^2}{b^2}\right]+z^2\left[3-\frac{a^2+b^2+c^2}{c^2}\right]=0$$

or
$$\frac{x^2}{a^2}(2a^2-b^2-c^2)+\frac{y^2}{b^2}(2b^2-c^2-a^2)+\frac{z^2}{c^2}\left[\,2c^2-a^2-b^2\,\right]=0.$$

5. If λ, μ, ν are the angles between a set of equal conjugate diameters of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, then
$$\cos^2\lambda+\cos^2\mu+\cos^2\nu=\frac{3\,\Sigma\,(b^2-c^2)^2}{2(a^2+b^2+c^2)^2}. \quad [\textit{Ag. U. 1950}]$$

****133.** *If P is any point on an ellipsoid, and OQ, OR any two conjugate diameters of the ellipse in which the diametral plane of OP meets the ellipsoid, then OP, OQ, OR are conjugate diameters of the ellipsoid.*



$\because$ OQ, OR are conjugate diameters of the ellipse

$\therefore$ OQ bisects chords of the ellipse, $\parallel$ to OR ... (1)

But these chords are also chords of the ellipsoid

$\therefore$ the diametral plane of OR bisects these

chords ... (2) [Def. (Art. 128, (*a*))]

From (1) and (2), the diametral plane of OR passes thro' OQ, *i.e.*, thro' Q.

Also the diametral plane of OR passes thro' P

[(Art. 130) ∵ the diametral plane of OP, *viz.*, the plane QOR passes thro' R]

∴ the diametral plane of OR is the plane POQ.

Similarly the diametral plane of OQ is the plane ROP.

Also the diametral plane of OP is the plane QOR (Given)

∴ OP, OQ, OR are conjugate diameters of the ellipsoid.

[Def. (Art. 128, (*b*))]

EXAMPLES

**1. P is any point on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and 2α and 2β are the principal axes of the section of the ellipsoid by the diametral plane of OP. Prove that $OP^2=a^2+b^2+c^2-\alpha^2-\beta^2$, and that $\alpha\beta p=abc$, where p is the perpendicular from O to the tangent plane at P. [*Ag. U. 1956*]

**2. If P, (x_1, y_1, z_1) is a point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and (ξ_1, η_1, ζ_1), (ξ_2, η_2, ζ_2) are extremities of the principal axes of the section of the ellipsoid by the diametral plane of OP, prove that

$$\frac{\xi_1\xi_2}{a^2(b^2-c^2)}=\frac{\eta_1\eta_2}{b^2(c^2-a^2)}=\frac{\zeta_1\zeta_2}{c^2(a^2-b^2)},$$

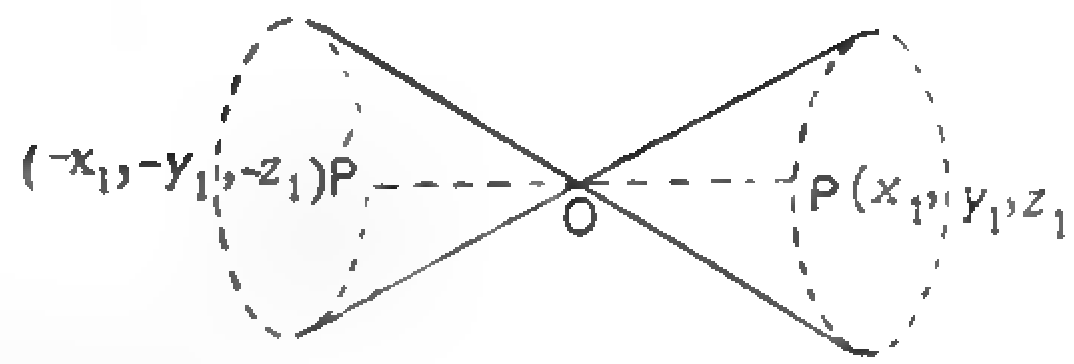
$$(b^2-c^2)\frac{x_1}{\xi_1}+(c^2-a^2)\frac{y_1}{\eta_1}+(a^2-b^2)\frac{z_1}{\zeta_1}=0. \qquad [\textit{B.U. 1942}]$$

SECTION II
THE CONE

134. Standard form. The equation $\mathbf{ax^2+by^2+cz^2=0}$, being a homogeneous equation of the second degree in x, y, z, represents a cone whose vertex is the origin (Art. 87, Cor.).

Note. Standard form. The equation $ax^2+by^2+cz^2=0$ is the *simplest* form of the equation of a cone, and may be called the **standard form.**

Cor. *A cone may be regarded as a central conicoid whose centre is the vertex.*



Let the equation of the cone be

$ax^2+by^2+cz^2=0$ ...(1)

Let $P(x_1, y_1, z_1)$ be *any* pt. on the cone (1).

Then $ax_1^2+by_1^2+cz_1^2 = 0$ ... (2)

Substituting the co-ordinates of the pt. $P'(-x_1, -y_1, -z_1)$ in (1), we get $a(-x_1)^2+b(-y_1)^2+c(-z_1)^2 = 0$

or $ax_1^2+by_1^2+cz_1^2 = 0$, which is true from (2).

∴ P' also lies on the cone.

Now the mid-pt. of PP' is $\left[\frac{x_1+(-x_1)}{2}, \frac{y_1+(-y_1)}{2}, \frac{z_1+(-z_1)}{2}\right]$

or (0, 0, 0), *i.e.*, the origin.

∴ the origin bisects *all* chords of the cone, which pass thro' it.

∴ it is the **centre** of the cone, and the cone is a **central conicoid.** [Defs. (Note 1, Art. 110)]

The co-ordinate planes are **conjugate diametral planes,** and the co-ordinate axes are **conjugate diameters.** [As in Arts. 127, 128]

135. As in the case of a central conicoid, the student can, and should prove the following results :

1. **Equation of the tangent plane.** *The equation of the tangent plane at any point* (x_1, y_1, z_1) *of the cone* $ax^2+by^2+cz^2=0$, *is*

$$\mathbf{axx_1+byy_1+czz_1=0.}$$

[**Rule to write down the equation of the tangent plane at the point** $(\mathbf{x_1, y_1, z_1})$ **of a cone (equation in the standard form)** : See Rule of Art. 70, (*b*).]

2. **Condition of tangency of a plane and a cone.** *The condition, that the plane* $lx+my+nz=0$ *should touch the cone* $ax^2+by^2+cz^2=0$, *is* $\dfrac{\mathbf{l^2}}{\mathbf{a}}+\dfrac{\mathbf{m^2}}{\mathbf{b}}+\dfrac{\mathbf{n^2}}{\mathbf{c}}=\mathbf{0}.$

Cor. *The condition, that the plane* $l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$ *should touch the cone* $a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$, *is*

$$\frac{\mathbf{l^2}}{\mathbf{a}}+\frac{\mathbf{m^2}}{\mathbf{b}}+\frac{\mathbf{n^2}}{\mathbf{c}}=\mathbf{0}.$$

[Change the origin to (α, β, γ).]

3. **Equation of the polar plane.** *The equation of the polar plane of the point* (x_1, y_1, z_1) *with respect to the cone* $ax^2+by^2+cz^2=0$, *is*

$$\mathbf{axx_1+byy_1+czz_1=0.}$$

[**Aid to memory.** See Aid to memory in Art. 74.]

4. **Equation of the pair of tangent planes from a point.** *The locus of the tangents drawn from* $P(x_1, y_1, z_1)$ *to the cone* $ax^2+by^2+cz^2=0$ *(vertex O) is the pair of tangent planes*

$$(\mathbf{ax^2+by^2+cz^2})(\mathbf{ax_1^2+by_1^2+cz_1^2}) = (\mathbf{axx_1+byy_1+czz_1})^2,$$

whose line of intersection is OP.

[Proceed as in Art. 94. Here the enveloping cone *degenerates* into a pair of planes.]

5. **Equation of the plane of the section with a given centre.** *The equation of the plane of the section of the cone* $ax^2+by^2+cz^2=0$, *whose centre is* (x_1, y_1, z_1), *is* $\mathbf{axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2}$, *i.e.,* $\mathbf{T=S_1}$.

6. **Equation of the diametral plane.** *The equation of the diametral plane of the cone* $ax^2+by^2+cz^2=0$, *which bisects chords parallel to the line* $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$, *is* $\mathbf{alx+bmy+cnz=0}$.

Note. See **Note** in Art. 129.

Cor. *The diametral plane of OP with respect to the cone* $ax^2+by^2+cz^2=0$ *is also the polar plane of P with respect to the cone.*

EXAMPLES

1. **Equation of the normal plane through a generator.** Find the equation of the normal plane of the cone $ax^2+by^2+cz^2=0$ through the generator $x/l=y/m=z/n$.

[**Normal plane. Def.** The **normal plane** through any generator OP of a cone (vertex O) is the plane through OP perpendicular to the tangent plane at any point of OP.]

2. Lines are drawn through the origin perpendicular to normal planes of the cone $ax^2+by^2+cz^2=0$. Show that they generate the cone

$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0. \qquad \text{[\textit{Bar. U. 1953}]}$$

3. Prove that if a plane cuts the cone $ax^2+by^2+cz^2=0$ in perpendicular generators, it touches the cone $\dfrac{x^2}{b+c}+\dfrac{y^2}{c+a}+\dfrac{z^2}{a+b}=0$.

4. Show that if a plane through the origin cuts the cones $ax^2+by^2+cz^2=0$, $\alpha x^2+\beta y^2+\gamma z^2=0$ in lines which form a harmonic pencil, it touches the cone $\dfrac{x^2}{b\gamma+c\beta}+\dfrac{y^2}{c\alpha+a\gamma}+\dfrac{z^2}{a\beta+b\alpha}=0$.

[*P. U. H. 1957*]

5. **Show that the perpendicular tangent planes to** $\mathbf{ax^2+by^2+cz^2=0}$ **intersect in generators of the cone**

$$\mathbf{a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0}. \qquad \text{[\textit{P. U. H. 1957}]}$$

[The equation of the cone is $ax^2+by^2+cz^2=0$ ...(1)

The equation of *any* tangent plane to the cone (1) is

$$ux+vy+wz=0 \quad ...(2)$$

where
$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=0 \quad ...(3)$$

Let the equations of the line of intersection of the two tangent planes, (thro' the origin) be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(4)$$

$\because$ it lies in the plane (2),

$\therefore$
$$ul+vm+wn=0 \quad ...(5)$$

The lines whose direction-cosines (u, v, w) are given by (3) and (5) are $\perp$.]

****6. Prove that the lines of intersection of pairs of tangent planes to the cone $ax^2+by^2+cz^2=0$ which touch along perpendicular generators lie on the cone**

$$\mathbf{a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0.} \qquad [\ \textit{Ag. U. 1955}\ ]$$

7. Through a fixed point $(d, 0, 0)$ pairs of perpendicular tangent lines are drawn to the conicoid $ax^2+by^2+cz^2=1$. Show that the plane through any pair touches the cone

$$\frac{(x-d)^2}{(ad^2-1)(b+c)}+\frac{y^2}{c(ad^2-1)-a}+\frac{z^2}{b(ad^2-1)-a}=0.$$

8. Show that the plane through a pair of equal conjugate diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ touches the cone

$$\Sigma\frac{x^2}{a^2(2a^2-b^2-c^2)}=0. \qquad [\ \textit{B. U. 1948}\ ]$$

[Proceed as in Ex. 4, Art. 132, (c), and prove that

$$\frac{l^2}{a^2}(2a^2-b^2-c^2)+\frac{m^2}{b^2}(2b^2-c^2-a^2)+\frac{n^2}{c^2}(2c^2-a^2-b^2)=0.$$

It will be found that the equation of the plane thro' two equal conjugate semi-diameters OQ, OR, *i.e.*, the plane QOR, the diametral plane of OP w.r.t. the ellipsoid, is $\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=0$.

It touches the cone $\Sigma\frac{x^2}{a^2(2a^2-b^2-c^2)}=0$, if

$$\Sigma\frac{l^2}{a^4}\cdot a^2(2a^2-b^2-c^2)=0,\ \textit{i.e.}, \text{ if } \Sigma\frac{l^2}{a^2}(2a^2-b^2-c^2)=0,$$

which is true (Proved above).]

9. If the two cones $ax^2+by^2+cz^2=0$, $\alpha x^2+\beta y^2+\gamma z^2=0$ have each sets of three mutually perpendicular generators, any two planes which pass through their four common generators are perpendicular.

10. Asymptotic cone. The locus of the asymptotes drawn from the origin to the conicoid $ax^2+by^2+cz^2=1$ is the **asymptotic cone $\mathbf{ax^2+by^2+cz^2=0}$.**

[**Asymptote. Def.** An **asymptote** (or, more fully, an **asymptotic line**) to a conicoid is a straight line which meets the conicoid in two points at infinity (*i.e.*, in two points which are at an infinite distance from the origin).]

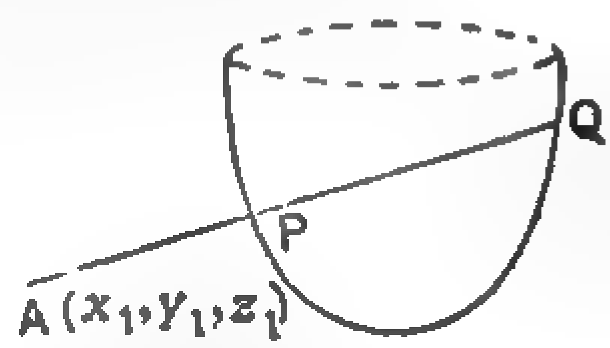
11. Prove that the hyperboloids

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \text{ and } -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

have the same asymptotic cone.

SECTION III
THE PARABOLOID
A paraboloid and a line.

136. *A line through a given point $A(x_1, y_1, z_1)$ meets a paraboloid $ax^2+by^2=2z$ in P and Q; to find the lengths of AP and AQ.*



The equation of the paraboloid is

$$ax^2+by^2=2z \quad \ldots(1)$$

Let the equations of the line thro' $A(x_1, y_1, z_1)$ be $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$,

l, m, n being actual direction-cosines.

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr) ...(2)

If it lies on the paraboloid (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2=2(z_1+nr)$$

or $a(x_1+lr)^2+b(y_1+mr)^2-2(z_1+nr)=0$

or $r^2(al^2+bm^2)+2r(alx_1+bmy_1-n)+(ax_1^2+by_1^2-2z_1)=0$ ...(3)

which is a quadratic in r, giving two values of r, the lengths of AP and AQ.

Cor. 1. Intersections of a line and a paraboloid. *To find the points of intersection of the line and the paraboloid.*

Substituting the two values of r found from (3), one by one, in (2), we get the two pts. of intersection.

Cor. 2. *All plane sections of a paraboloid are conics.*

Proof. $\because$ every st. line meets a paraboloid in two pts. [Cor. 1]

$\therefore$ every st. line lying in a particular plane meets the paraboloid and $\therefore$ the curve of intersection of the paraboloid and the plane in

two pts.

$\therefore$ by Analytical Plane Geometry, the curve of intersection is a conic.

Cor. 3. A line drawn through a point A parallel to OZ meets the paraboloid

$$\mathbf{ax^2 + by^2 = 2z}$$

in one point at an infinite distance from A, and in a point P.

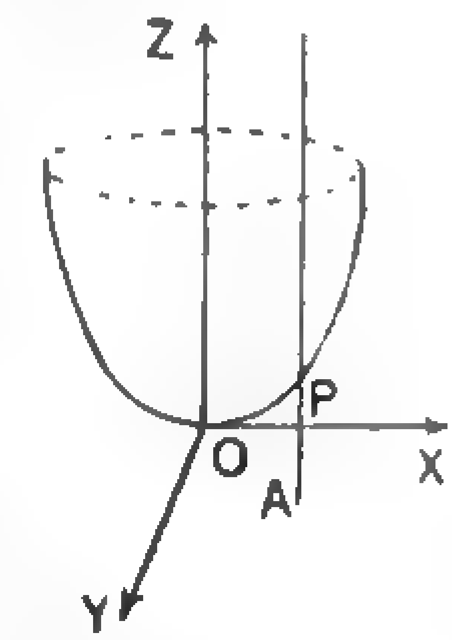
$\because$ the line is $\parallel$ to OZ (direction-cosines 0, 0, 1)

$\therefore$ $l=0$, $m=0$

$\therefore$ from (3) (Art. 136), $0.r^2+2r(-n)+(ax_1^2+by_1^2-2z_1)=0$

$\therefore$ one value of r is infinite, [$\because$ coeff. of $r^2=0$]

and the other value is $r=\dfrac{ax_1^2+by_1^2-2z_1}{2n}$, giving the length of AP.



Diameter. Def. A line drawn through a point A, which meets a paraboloid in one point at an infinite distance from A and in a point P is called a **diameter** of the paraboloid, and P is called the **extremity** of the diameter.

Thus a line $\parallel$ to OZ is a **diameter** of the paraboloid $ax^2+by^2=2z$. [Art. 136, Cor. 3]

Axis. The diameter, which is perpendicular to the tangent plane at its extremity, is called the **axis** of the paraboloid, and its extremity is called the **vertex** of the paraboloid.

Thus OZ is the **axis** of the paraboloid $ax^2+by^2=2z$ (Art. 111,(*a*) (*iii*)), and O is the **vertex**.

Cor. *A line parallel to the axis of a paraboloid is a* **diameter.**

137. *An elliptic paraboloid is a limiting form of an ellipsoid and a hyperbolic paraboloid is a limiting form of a hyperboloid of one sheet.*

[** **Proof.** Let the equation of the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \ldots (1)$$

Changing the origin to $(0, 0, -c)$, the equation (1) becomes

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{(z-c)^2}{c^2}=1, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-2\frac{z}{c}+1=1,$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{2z}{c},$$

or, multiplying thro' out by c, [To get $2z$ on the R. H. S.]

$$\frac{c}{a^2}x^2+\frac{c}{b^2}y^2+\frac{z^2}{c}=2z, \text{ or } \frac{x^2}{\frac{a^2}{c}}+\frac{y^2}{\frac{b^2}{c}}+\frac{z^2}{c}=2z \ldots (2)$$

Let a, b, c all $\to \infty$,

so that $\dfrac{a^2}{c}, \dfrac{b^2}{c}$ remain finite $= a'^2, b'^2$ (say).

Then (2) becomes

$$\frac{x^2}{a'^2} + \frac{y^2}{b'^2} = 2z \left[\because \frac{z^2}{c} \to 0 \text{ when } c \to \infty \right]$$

which is the equation of an elliptic paraboloid.

$\therefore$ an elliptic paraboloid is a limiting form of an ellipsoid.

Similarly a hyperbolic paraboloid $\left(\dfrac{x^2}{a'^2} - \dfrac{y^2}{b'^2} = 2z\right)$ *is a limiting form of a hyperboloid of one sheet* $\left(\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1\right)$.]

As in the case of a central conicoid the student can, and should prove the following results :

1. **Equation of the tangent plane.** *The equation of the tangent plane at any point* (x_1, y_1, z_1) *of the paraboloid* $ax^2+by^2=2z$, *is*

$$\mathbf{axx_1 + byy_1 = z + z_1.}$$

[**Rule to write down the equation of the tangent plane at the point (x_1, y_1, z_1) of a paraboloid :** See Rule of Art. 70, (*b*).]

2. **Condition of tangency of a plane and a paraboloid.** *The condition, that the plane* $lx+my+nz = p$ *should touch the paraboloid* $ax^2+by^2=2z$, *is* $\dfrac{\mathbf{l^2}}{\mathbf{a}} + \dfrac{\mathbf{m^2}}{\mathbf{b}} = \mathbf{-2np}.$

Cor. 1. *If the condition is satisfied, the point of contact is*

$$\left(-\frac{l}{an}, -\frac{m}{bn}, -\frac{p}{n}\right).$$

Cor. 2. *The equation of the tangent plane to the paraboloid* $ax^2+by^2 = 2z$, *parallel to the plane* $lx+my+nz=0$, *is*

$$2n(lx+my+nz) + \frac{l^2}{a} + \frac{m^2}{b} = 0.$$

3. **Equation of the polar plane.** *The equation of the polar plane of the point* (x_1, y_1, z_1) *with respect to the paraboloid* $ax^2+by^2=2z$, *is*

$$\mathbf{axx_1 + byy_1 = z + z_1.}$$

[**Aid to memory.** See Aid to memory in Art. 74.]

4. *Equation of the (tangent cone or) enveloping cone.*

The locus of the tangents from a point (α, β, γ) to the paraboloid $ax^2+by^2=2z$ is given by

$$\mathbf{(ax^2+by^2-2z)(a\alpha^2+b\beta^2-2\gamma) = (a\alpha x+b\beta y-z-\gamma)^2.}$$

Abridged notation. If $S=ax^2+by^2-2z$, so that $S=0$ is the equation of the paraboloid,

$S_1 = ax_1^2 + by_1^2 - 2z_1$, so that S_1 is the result of substituting the co-ordinates of the pt. (x_1, y_1, z_1) in S.

$T = axx_1 + byy_1 - (z + z_1)$, so that $T = 0$ is the equation of the tangent plane at (x_1, y_1, z_1), then the equation of the enveloping cone is

$$SS_1 = T^2.$$

5. **Equation of the plane of the section with a given centre.** *The equation of the plane of the section of the paraboloid* $ax^2 + by^2 = 2z$, *whose centre is* (x_1, y_1, z_1), *is* $\mathbf{axx_1 + byy_1 - z = ax_1^2 + by_1^2 - z_1}$.

Abridged notation. If $S = ax^2 + by^2 - 2z$, so that $S = 0$ is the equation of the paraboloid,

$S_1 = ax_1^2 + by_1^2 - 2z_1$, so that S_1 is the result of substituting the co-ordinates of the pt. (x_1, y_1, z_1) in S,

$T = axx_1 + byy_1 - (z + z_1)$, so that $T = 0$ is the equation of the tangent plane at (x_1, y_1, z_1), then, from the result of (5), adding $(-z_1)$ to both sides, the equation of the plane of the section is

$$axx_1 + byy_1 - (z + z_1) = ax_1^2 + by_1^2 - 2z_1$$

or

$$T = S_1. \qquad \text{[See Note 2, Art. 126.]}$$

EXAMPLES

1. **Find the equation of the tangent plane at the point** (α, β, γ) **on the paraboloid** $\mathbf{ax^2 + by^2 = 2z}$. [*P. U. H.*]

If P, Q are any two points on a paraboloid, and the tangent planes at P, Q meet in the line RS; then prove that the plane through RS and the mid-point of PQ is parallel to the axis of the paraboloid.

2. Show that the plane $8x - 6y - z = 5$ touches the paraboloid

$$\frac{x^2}{2} - \frac{y^2}{3} = z,$$

and find the co-ordinates of the point of contact. [*P(P). U. 1957S*]

3. Prove that the paraboloids

$$\frac{x^2}{a_1^2} + \frac{y^2}{b_1^2} = \frac{2z}{c_1}, \frac{x^2}{a_2^2} + \frac{y^2}{b_2^2} = \frac{2z}{c_2}, \frac{x^2}{a_3^2} + \frac{y^2}{b_3^2} = \frac{2z}{c_3}$$

have a common tangent plane if

$$\begin{vmatrix} a_1^2, & a_2^2, & a_3^2 \\ b_1^2, & b_2^2, & b_3^2 \\ c_1, & c_2, & c_3 \end{vmatrix} = 0.$$

4. Prove that two conjugate points on a diameter of a paraboloid are equidistant from the extremity of the diameter.

[**Conjugate points. Def.** *Two points, which are such that the polar plane of each with respect to a conicoid passes through the other, are called* **conjugate points** *with respect to the conicoid.*]

5. *Condition of tangency of a line and a paraboloid.* **Find the

condition that the line $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ may touch the paraboloid $ax^2+by^2=2z$.

Hence or otherwise find the equation of the enveloping cone having its vertex at (α, β, γ). [*D. U. H. 1936*]

6. Find the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2z$. [*P. U. H. 1956*]

7. Prove that the locus of the centres of a system of parallel plane sections of a paraboloid is a diameter.

Prove also that the tangent plane at the extremity of the diameter is parallel to the plane sections.

138. To find the locus of the point of intersection of three mutually perpendicular tangent planes to a paraboloid.

Let the equation of the paraboloid be $ax^2+by^2=2z$ ...(1)

Let the equation of one of the three mutually $\perp$ tangent planes to the paraboloid (1) be

$l_1x+m_1y+n_1z=p_1$, [l_1, m_1, n_1 being actual direction-cosines]

where $\dfrac{l_1^2}{a}+\dfrac{m_1^2}{b}=-2n_1p_1$ (Art. 137, (2)), or $p_1=-\dfrac{1}{2n_1}\left[\dfrac{l_1^2}{a}+\dfrac{m_1^2}{b}\right]$,

i.e., $$l_1x+m_1y+n_1z=-\frac{1}{2n_1}\left[\frac{l_1^2}{a}+\frac{m_1^2}{b}\right]$$

or, multiplying thro' out by n_1,

$$n_1l_1x+m_1n_1y+n_1^2z=-\frac{1}{2}\left[\frac{l_1^2}{a}+\frac{m_1^2}{b}\right] \quad ...(2)$$

Similarly let the equations of the other two tangent planes be

$$n_2l_2x+m_2n_2y+n_2^2z=-\frac{1}{2}\left[\frac{l_2^2}{a}+\frac{m_2^2}{b}\right] \quad ..(3)$$

$$n_3l_3x+m_3n_3y+n_3^2z=-\frac{1}{2}\left[\frac{l_3^2}{a}+\frac{m_3^2}{b}\right] \quad ...(4)$$

[To find the locus of the pt. of intersection of the planes (2), (3), (4).]

Adding (2), (3), (4) vertically,

$$x(n_1l_1+n_2l_2+n_3l_3)+y(m_1n_1+m_2n_2+m_3n_3)+z(n_1^2+n_2^2+n_3^2)$$
$$=-\frac{1}{2}\left[\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^2)\right]$$

[But l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 being the direction-cosines of three mutually $\perp$ lines, *viz.*, the normals to the three mutually $\perp$ tangent planes (2), (3), (4),

$l_1^2+l_2^2+l_3^2=1$, and so on ; $m_1n_1+m_2n_2+m_3n_3=0$, and so on (Art. 58, (C), (D))]

or $$z=-\frac{1}{2}\left[\frac{1}{a}+\frac{1}{b}\right],$$
which is the required locus.

It is a plane ∥ to the xy-plane, *i.e.*, ⊥ to the z-axis, *i.e.*, ⊥ to the axis of the paraboloid.

Normals to an elliptic paraboloid.

139. Equations of the normal. *To find the equations of the normal at any point* (x_1, y_1, z_1) *of the elliptic paraboloid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$.

The equation of the elliptic paraboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$.

The equation of the tangent plane at (x_1, y_1, z_1) is
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=z+z_1 \qquad \text{[Rule (Art. 70, (b))]}$$
or $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-z-z_1=0 \quad ...(1)$$

∴ the direction-cosines of the normal are proportional to
$$\frac{x_1}{a^2},\ \frac{y_1}{b^2},\ -1.$$

∴ the equations of the normal at (x_1, y_1, z_1) are
$$\frac{\mathbf{x}-\mathbf{x}_1}{\frac{\mathbf{x}_1}{\mathbf{a}^2}}=\frac{\mathbf{y}-\mathbf{y}_1}{\frac{\mathbf{y}_1}{\mathbf{b}^2}}=\frac{\mathbf{z}-\mathbf{z}_1}{-1} \quad ...(2) \qquad \text{[Art. 37, Cor. 1]}$$

Abridged notation. If $F(x, y, z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-2z$, so that $F(x, y, z)=0$ is the equation of the elliptic paraboloid, then
$$\frac{\partial F}{\partial x}=\frac{1}{a^2}2x,\ \frac{\partial F}{\partial y}=\frac{1}{b^2}2y,\ \frac{\partial F}{\partial z}=-2$$
$$\therefore \quad \frac{\partial F}{\partial x_1}=\frac{2x_1}{a^2},\ \frac{\partial F}{\partial y_1}=\frac{2y_1}{b^2},\ \frac{\partial F}{\partial z_1}=-2$$

∴ from (2), the equations of the normal are
$$\frac{x-x_1}{\frac{1}{2}\frac{\partial F}{\partial x_1}}=\frac{y-y_1}{\frac{1}{2}\frac{\partial F}{\partial y_1}}=\frac{z-z_1}{\frac{1}{2}\frac{\partial F}{\partial z_1}}, \text{ or } \frac{\mathbf{x}-\mathbf{x}_1}{\frac{\partial \mathbf{F}}{\partial \mathbf{x}_1}}=\frac{\mathbf{y}-\mathbf{y}_1}{\frac{\partial \mathbf{F}}{\partial \mathbf{y}_1}}=\frac{\mathbf{z}-\mathbf{z}_1}{\frac{\partial \mathbf{F}}{\partial \mathbf{z}_1}}.$$
[See Note 1, Art. 119.]

140. *Number of normals from a given point to an elliptic paraboloid.* **To prove that there are five points on an elliptic paraboloid the normals at which pass through a given point** (α, β, γ).

Let the equation of the elliptic paraboloid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z \quad ...(1)$

The equations of the normal at (x_1, y_1, z_1) are

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{-1}. \qquad \text{[Art. 139]}$$

$$\left[\frac{x-x_1}{\frac{\partial F}{\partial x_1}}=\frac{y-y_1}{\frac{\partial F}{\partial y_1}}=\frac{z-z_1}{\frac{\partial F}{\partial z_1}}, \text{ here } F(x, y, z)=\frac{x^2}{a^2}+\frac{y^2}{b^2}-2z\right]$$

If it passes thro' (α, β, γ), then

$$\frac{\alpha-x_1}{\frac{x_1}{a^2}}=\frac{\beta-y_1}{\frac{y_1}{b^2}}=\frac{\gamma-z_1}{-1}=\lambda \text{ (say).}$$

From the first and last members, $\alpha-x_1=\frac{x_1}{a^2}\lambda$

or $\alpha=x_1\left(1+\frac{\lambda}{a^2}\right)=\frac{x_1}{a^2}(a^2+\lambda)$

$$\left.\begin{aligned} &\text{or} \quad x_1=\frac{a^2\alpha}{a^2+\lambda}. \\ &\text{Similarly} \quad y_1=\frac{b^2\beta}{b^2+\lambda}, \; z_1=\gamma+\lambda. \end{aligned}\right\} \quad \ldots (2) \qquad \Big| \; \gamma-z_1=-\lambda$$

$\because$ (x_1, y_1, z_1) lies on the paraboloid (1),

$\therefore$ $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=2z_1$ ... (3)

Substituting the values of x_1, y_1, z_1 from (2) in (3),

$$\frac{1}{a^2}\frac{a^4\alpha^2}{(a^2+\lambda)^2}+\frac{1}{b^2}\frac{b^4\beta^2}{(b^2+\lambda)^2}=2(\gamma+\lambda)$$

or
$$\frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}=2(\gamma+\lambda).$$

Clearing of fractions

$$a^2\alpha^2(b^2+\lambda)^2+b^2\beta^2(a^2+\lambda)^2=2(\gamma+\lambda)(a^2+\lambda)^2(b^2+\lambda)^2,$$

which is an equation of the *fifth* degree in λ, giving *five* values of λ.

Substituting these values of λ, one by one, in (2), we get five pts. (x_1, y_1, z_1) on the paraboloid, the normals at which pass thro' (α, β, γ).

EXAMPLES

1. (*i*) *Quadric cone through the five normals from a point.*

Prove that the normals from (α, β, γ) to the paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$$

lie on the cone $\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$ [*P. U. H. 1961*]

(*ii*) *Cubic curve through the feet of the five normals from a point.*

Prove that the feet of the normals from any point to the paraboloid lie on a cubic curve which lies on the above cone.

[*P. U. H. 1953*]

(*iii*) Prove that the perpendicular from (α, β, γ) to its polar plane lies on the above cone.

2. **Prove that in general three normals can be drawn from a given point to the paraboloid of revolution $x^2+y^2=2az$.**

Prove also that if the point lies on the surface

$$27a(x^2+y^2)+8(a-z)^3 = 0,$$

two of the three normals coincide. [*Ag. U. 1948*]

141. *To prove that the plane YOZ bisects all chords of the paraboloid $ax^2+by^2=2z$, parallel to OX.*

The equation of the paraboloid is $ax^2+by^2=2z$ ...(1)

Let the equations of *any* chord ∥ to OX$(y=0, z=0)$ be $y=\mu$, $z=\nu$.

It meets the paraboloid (1) where [putting $y=\mu$, $z=\nu$ in (1)],

$$ax^2+b\mu^2=2\nu, \text{ or } x^2=\frac{2\nu-b\mu^2}{a}, \text{ or } x=\pm\sqrt{\frac{2\nu-b\mu^2}{a}},$$

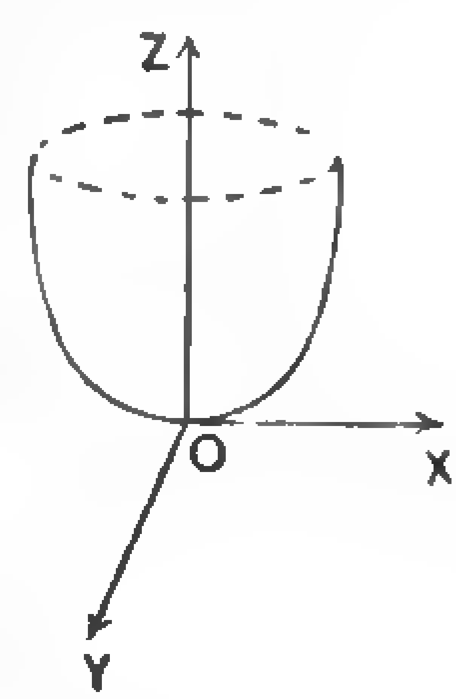
i.e., in the pts. $P\left(\sqrt{\frac{2\nu-b\mu^2}{a}}, \mu, \nu\right)$, $P'\left(-\sqrt{\frac{2\nu-b\mu^2}{a}}, \mu, \nu\right)$.

The mid-pt. of PP′ is $(0, \mu, \nu)$, [Art. 5, Cor.]
which lies on the plane YOZ $(x=0)$.

∴ the plane YOZ bisects all chords of the paraboloid, ∥ to OX.

Similarly the plane ZOX bisects all chords of the paraboloid, ∥ to OY.

142. (*a*) **Diametral plane.** *The plane which bisects a system of parallel chords of a paraboloid is called a* **diametral plane.**



Thus the plane YOZ is the **diametral plane** of the paraboloid $ax^2+by^2=2z$, which bisects chords parallel to OX (Art. 141) or, more shortly, the plane YOZ is the **diametral plane of OX.**

Similarly the plane ZOX is the diametral plane of OY.

(*b*) **Conjugate diametral planes. Def.** Two planes, which are such that each is parallel to the chords which the other bisects, are called **conjugate diametral planes** of the paraboloid.

Thus the planes YOZ and ZOX are **conjugate diametral planes**

of the paraboloid $ax^2+by^2=2z$.

For the plane YOZ passes thro' OY, and OY is ∥ to the chords which the plane ZOX bisects (Art. 141).

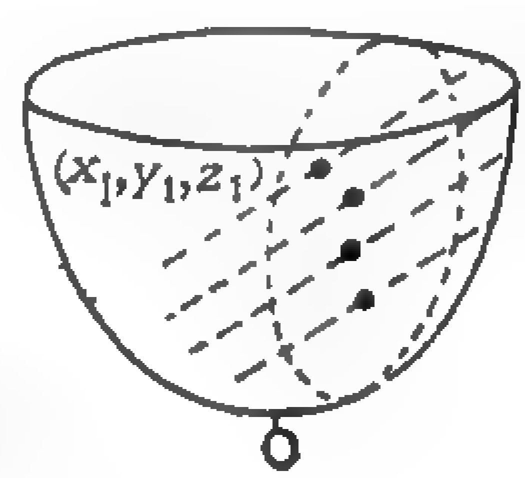
∴ the plane YOZ is ∥ to the chords which the plane ZOX bisects.

(c) **Principal planes. Def.** Diametral planes, which are perpendicular to the chords which they bisect, are called **principal planes**.

Thus the planes YOZ and ZOX are the **principal planes** of the paraboloid $ax^2+by^2=2z$.

Cor. (The axes being rectangular) *The equation* $ax^2+by^2=2z$ *represents a paraboloid referred to a tangent plane XOY (See Art. 111, (a), (iii)), and two principal planes YOZ and ZOX as the co-ordinate planes.*

143. Equation of the diametral plane. *To find the equation of the diametral plane of the paraboloid* $ax^2+by^2=2z$, *which bisects chords parallel to the line* $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.



The equation of the paraboloid is

$$ax^2+by^2=2z \quad \dots (1)$$

and the equations of the line are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \dots (2)$$

[To find the locus of the mid-pts. of chords of the paraboloid (1), ∥ to the line (2).

(Def. Art. 142, (a))]

Let (x_1, y_1, z_1) be the mid-pt. of *any* chord ∥ to the line (2).

Then the equations of the chord are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the paraboloid (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2=2(z_1+nr)$$

or $$a(x_1+lr)^2+b(y_1+mr)^2-2(z_1+nr)=0$$

or $$r^2(al^2+bm^2)+2r(alx_1+bmy_1-n)+(ax_1^2+by_1^2-2z_1)=0 \quad \dots (3)$$

which is a quadratic in r.

∵ (x_1, y_1, z_1) is the mid-pt. of the chord, the quadratic (3) has equal and opposite roots, ∴ sum of the roots = 0, ∴ coeff. of $r=0$, *i.e.*, $alx_1+bmy_1-n=0$.

∴ the locus of (x_1, y_1, z_1) is [changing (x_1, y_1, z_1) to (x, y, z)],

$$\mathbf{alx + bmy - n = 0} \quad \ldots (4)$$

which is the required equation of the diametral plane.

Abridged notation. If $F(x, y, z) = ax^2+by^2-2z$, so that $F(x, y, z) = 0$ is the equation of the paraboloid, then

$$\frac{\partial F}{\partial x} = a.2x, \frac{\partial F}{\partial y} = b.2y, \frac{\partial F}{\partial z} = -2$$

$\therefore$ from (4), the equation of the diametral plane is

$$l.\tfrac{1}{2}\frac{\partial F}{\partial x} + m.\tfrac{1}{2}\frac{\partial F}{\partial y} + n.\tfrac{1}{2}\frac{\partial F}{\partial z} = 0$$

or $$\mathbf{l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0}.$$ (See Note, Art. 129.)

Cor. 1. *Any diametral plane of a paraboloid is parallel to the axis of the paraboloid.*

For, from (4) (Art. 143), the diametral plane is $\parallel$ to the axis of the paraboloid, *i.e.*, $\parallel$ to the z-axis, if its normal (direction-cosines proportional to $al, bm, 0$) is $\perp$ to the z-axis (direction-cosines $0, 0, 1$), *i.e.*, if $al(0)+bm(0)+0(1)=0$, or $0=0$, which is true.

$\therefore$ the diametral plane is $\parallel$ to the axis of the paraboloid.

Cor. 2. Equation of any diametral plane. *The equation of any diametral plane of the paraboloid $ax^2+by^2=2z$, is* $\mathbf{alx+bmy-n=0}$.

EXAMPLES

1. Converse of Cor. 1. *Any plane parallel to the axis of a paraboloid is a diametral plane.*

2. *If the diametral plane of a line OP with respect to a paraboloid is parallel to a line OQ, then the diametral plane of OQ is parallel to OP.*

Let the equation of the paraboloid be $ax^2+by^2=2z$ ...(1)

Let the equations of OP, OQ be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}, \quad \frac{x}{l'}=\frac{y}{m'}=\frac{z}{n'}.$$

The equation of the diametral plane of OP w.r.t. the paraboloid (1) is $alx+bmy-n=0$. [Art. 143]

$$\left[\, l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0, \text{ here } F(x, y, z) = ax^2+by^2-2z \,\right]$$

If it is $\parallel$ to OQ, then its normal (direction-cosines proportional to $al, bm, 0$) is $\perp$ to OQ (direction-cosines proportional to l', m', n')

$\therefore$ $all'+bmm'+0(n')=0$, or $all'+bmm'=0$.

The *symmetry* of this result shows that it is also the condition that the diametral plane of OQ should be $\parallel$ to OP.

Cor. Condition of conjugacy of two diametral planes. *The condition, that the diametral planes $alx+bmy-n=0$, $al'x+bm'y-n'=0$, of the paraboloid $ax^2+by^2=2z$, should be conjugate, is $all'+bmm'=0$.*

The equations of the diametral planes are

$$alx+bmy-n=0 \quad ...(1)$$
$$al'x+bm'y-n'=0 \quad ...(2)$$

If they are conjugate, the plane (1) is $\parallel$ to the chords which the plane (2) bisects, *i.e.*, $\parallel$ to the line $\dfrac{x}{l'}=\dfrac{y}{m'}=\dfrac{z}{n'}$ ...(3)

$\therefore$ its normal (direction-cosines proportional to $al, bm, 0$) is $\perp$ to the line (3) (direction-cosines proportional to l', m', n')

$\therefore$ $all'+bmm'+0(n')=0$, or $all'+bmm'=0$,
which is the required condition.

3. The plane $3x+4y=1$ is a diametral plane of the paraboloid $5x^2+6y^2=2z$. Find the equations to the chord through (3, 4, 5) which it bisects. [*P. U. H. 1954*]

MISCELLANEOUS EXAMPLES ON CHAPTER XI

1. Any three mutually orthogonal lines drawn through a fixed point C meet the quadric $ax^2+by^2+cz^2=1$ in P_1, P_2 ; Q_1, Q_2; R_1, R_2 respectively ; prove that

$$\frac{P_1P_2^2}{CP_1^2 . CP_2^2}+\frac{Q_1Q_2^2}{CQ_1^2 . CQ_2^2}+\frac{R_1R_2^2}{CR_1^2 . CR_2^2}$$

and
$$\frac{1}{CP_1 . CP_2}+\frac{1}{CQ_1 . CQ_2}+\frac{1}{CR_1 . CR_2}$$

are constants. [*P. U. H. 1937*]

**2. If P, Q are any two points on an ellipsoid, and planes through the centre parallel to the tangent planes at P, Q meet PQ in P′, Q′, show that PP′=QQ′.

3. Prove that the locus of the foot of the perpendicular drawn from the centre of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ to any of its tangent planes is $a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2$. [*Ag. U. 1940*]

4. The normal at a variable point P of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ meets the plane XOY in A, and AQ is drawn parallel to OZ and equal to AP. Prove that the locus of Q is given by

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1. \qquad [\ P(P).\ U.\ 1956\ ]$$

Find the locus of R if OR is drawn from the centre parallel and equal to AP.

5. P, Q are points on an ellipsoid. The normal at P meets the tangent plane at Q in R, and the normal at Q meets the tangent plane at P in S. If p, q are the perpendiculars from the centre on the tangent planes at P, Q, show that PR : QS$=q : p$.

6. Prove that two normals to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ lie in the plane $lx+my+nz=0$ and the line joining their feet has direction-cosines proportional to

$$a^2(b^2-c^2)mn,\ b^2(c^2-a^2)nl,\ c^2(a^2-b^2)lm.$$

Also obtain the co-ordinates of these points. [*P.U.H. 1957*]

[The equations of the normal at (x_1, y_1, z_1) are

$$\frac{x-x_1}{\dfrac{x_1}{a^2}}=\frac{y-y_1}{\dfrac{y_1}{b^2}}=\frac{z-z_1}{\dfrac{z_1}{c^2}}.$$

If it lies in the plane $lx+my+nz=0$, then

$$l\frac{x_1}{a^2}+m\frac{y_1}{b^2}+n\frac{z_1}{c^2}=0,$$

and $$lx_1+my_1+nz_1=0.$$ [Art. 43, (*c*)]

Solving for x_1, y_1, z_1 by cross-multiplication, it will be found that

$$\frac{x_1}{a^2mn(b^2-c^2)}=\frac{y_1}{b^2nl(c^2-a^2)}=\frac{z_1}{c^2lm(a^2-b^2)}$$

or $$\frac{\dfrac{x_1}{a}}{amn(b^2-c^2)}=\frac{\dfrac{y_1}{b}}{bnl(c^2-a^2)}=\frac{\dfrac{z_1}{c}}{clm(a^2-b^2)}$$

$$=\frac{\sqrt{\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}}}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}=\frac{\pm 1}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}} \quad \dots (1)$$

Taking $+$ve sign in the last member of (1), the pt. is [(x_1, y_1, z_1), *i.e.*,]

$$\left[\frac{a^2mn(b^2-c^2)}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}, \frac{b^2nl(c^2-a^2)}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}, \frac{c^2lm(a^2-b^2)}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}\right],$$

and taking $-$ve sign in the last member of (1), the pt. is

$$\left[-\frac{a^2mn(b^2-c^2)}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}, -\frac{b^2nl(c^2-a^2)}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}, -\frac{c^2lm(a^2-b^2)}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}\right].$$

These are the required co-ordinates of the two pts.

The direction-cosines of the join of the above two pts. are proportional to $a^2mn(b^2-c^2)$, $b^2nl(c^2-a^2)$, $c^2lm(a^2-b^2)$.]

7. Show that the points on a conicoid the normals at which intersect the normal at a fixed point all lie on a cone of the second degree whose vertex is the fixed point. [*B. U. 1941*]

[Find the equation of the surface on which the pt. (x_1, y_1, z_1) lies, and then change the origin to the fixed pt. (α, β, γ).]

8. Show that the normals to $x^2/a^2+y^2/b^2+z^2/c^2=1$ at all points of its intersection with $lyz+mzx+nxy=0$ intersect the line

$$\frac{a^2x}{l(a^2-b^2)(c^2-a^2)}=\frac{b^2y}{m(b^2-c^2)(a^2-b^2)}=\frac{c^2z}{n(c^2-a^2)(b^2-c^2)}.$$ [*Ag. U. 1945*]

9. Prove that the points on an ellipsoid the normals at which intersect a given line lie on the curve of intersection of the ellipsoid and a conicoid. [*P. U. H. 1958*]

10. Prove that of the six normals from a point to an ellipsoid at least two are real. [*B. U. 1941*]

[Proceed as in Art. 120. It will be found that

$$(\lambda+a^2)^2(\lambda+b^2)^2(\lambda+c^2)^2$$
$$-a^2\alpha^2(\lambda+b^2)^2(\lambda+c^2)^2-b^2\beta^2(\lambda+c^2)^2(\lambda+a^2)^2-c^2\gamma^2(\lambda+a^2)^2(\lambda+b^2)^2=0 \quad ...(1)$$

Let $f(\lambda)=(\lambda+a^2)^2(\lambda+b^2)^2(\lambda+c^2)^2-a^2\alpha^2(\lambda+b^2)^2(\lambda+c^2)^2$
$$-b^2\beta^2(\lambda+c^2)^2(\lambda+a^2)^2-c^2\gamma^2(\lambda+a^2)^2(\lambda+b^2)^2.$$

[Assume that $a^2>b^2>c^2$, $\therefore$ $-a^2<-b^2<-c^2$.]

Now $f(-\infty)=+\infty=+\text{ve}$,

$$f(-a^2)=-a^2\alpha^2(-a^2+b^2)^2(-a^2+c^2)^2=-\text{ve},$$
$$f(-b^2)=-b^2\beta^2(-b^2+c^2)^2(-b^2+a^2)^2=-\text{ve},$$
$$f(-c^2)=-c^2\gamma^2(-c^2+a^2)^2(-c^2+b^2)^2=-\text{ve},$$
$$f(+\infty)=+\infty=+\text{ve}.$$

$\therefore$ at least one root of (1) lies between $-\infty$ and $-a^2$, and at least one between $-c^2$ and $+\infty$. (From Theory of Equations)]

11. Prove that four normals to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, pass through any point of the curve of intersection of the ellipsoid and the conicoid $x^2(b^2+c^2)+y^2(c^2+a^2)+z^2(a^2+b^2)=b^2c^2+c^2a^2+a^2b^2$.

12. If the feet of the six normals from (α, β, γ) to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ are (x_r, y_r, z_r) $(r=1, 2, ..., 6)$, prove that

$$a^2\alpha\Sigma\left(\frac{1}{x_r}\right)+b^2\beta\Sigma\left(\frac{1}{y_r}\right)+c^2\gamma\Sigma\left(\frac{1}{z_r}\right)=0.$$ [*P.U.H. 1952*]

13. Prove that six normals can be drawn from any point P to a central quadric surface and that these six normals are generators of a quadric cone with vertex P. [*P. U. H. 1957*]

**Prove that the conic in which the cone meets any one of the principal planes of the quadric surface remains fixed when P moves along a straight line perpendicular to that plane. [*Birmingham U.*]

14. Show that, if the feet of three of the six normals drawn from any point to a central conicoid $ax^2+by^2+cz^2=1$, lie on the plane $lx+my+nz+p=0$, the feet of the other three will lie on the plane

$$\frac{ax}{l}+\frac{by}{m}+\frac{cz}{n}-\frac{1}{p}=0. \qquad [\text{ P. U. H. }]$$

15. If the feet of three of the normals from P to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, the feet of the other three lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$, and P lies on the line $a(b^2-c^2)x=b(c^2-a^2)y=c(a^2-b^2)z$. [*Ag. U. 1944*]

16. A is a fixed point and P a variable point such that its polar plane with respect to an ellipsoid is perpendicular to AP. Prove that the locus of P is the cubic curve through the feet of the normals from A.

17. Pairs of planes are drawn which are conjugate with respect to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, the first member of each pair passing through the line $y=mx$, $z=k$ and the second member of each pair passing through the line $y=-mx$, $z=-k$; prove that the line of intersection of the two members of any pair lies on the surface

$$(b^2-a^2m^2)(z^2-k^2)+(y^2-m^2x^2)(c^2+k^2)=0. \qquad [\text{ B. U. }]$$

[**Note. Conjugate planes. Def.** Two planes, which are such that the pole of each with respect to a conicoid lies on the other, are called **conjugate planes** with respect to the conicoid.]

18. Show that the polar of a given line with respect to a central conicoid is the chord of contact of the two tangent planes through the line.

19. If the normals at P and Q, points on an ellipsoid, intersect, prove that PQ is perpendicular to its polar with respect to the ellipsoid.

20. Show that the generators of the cone which passes through the normals from a given point to an ellipsoid are perpendicular to their polars with respect to the ellipsoid.

21. Show that any normal to the conicoid

$$\frac{x^2}{pa+q}+\frac{y^2}{pb+q}+\frac{z^2}{pc+q}=1,$$

is perpendicular to its polar line with respect to the conicoid

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=1. \qquad [\text{ B. U. }]$$

22. Prove that the line joining P to the centre of a conicoid passes through the centre of the section of the conicoid by the polar plane of P with respect to the conicoid.

23. Prove that the locus of the mid-points of chords of the

conicoid $ax^2+by^2+cz^2=1$ which are parallel to the plane $x=0$ and touch the sphere $x^2+y^2+z^2=r^2$, is the surface

$$by^2(bx^2+by^2+cz^2-br^2)+cz^2(cx^2+by^2+cz^2-cr^2)=0.$$

24. If three of the feet of the normals from a point to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ lie on the plane $lx+my+nz=p$, show that the equation of the plane through the other three is

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$$

Also, show that if one of the planes contains the extremities of three conjugate semi-diameters, the other plane cuts the co-ordinate planes in a triangle whose centroid lies on a coaxal ellipsoid. [*B. U.*]

[**Note 1. Coaxal ellipsoid.** The **ellipsoid** $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=k^2$ is said to be **coaxal** with the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

For they have the same axes, *viz.*, the co-ordinate axes.

Note 2. *Note that, if* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *are the extremities of three conjugate semi-diameters, then*

$$\left(\frac{x_1+x_2+x_3}{a}\right)^2+\left(\frac{y_1+y_2+y_3}{b}\right)^2+\left(\frac{z_1+z_2+z_3}{c}\right)^2=3.\ ]$$

25. If (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) be the extremities of three conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

show that the equation of the plane through the three points (x_1, x_2, x_3), (y_1, y_2, y_3), (z_1, z_2, z_3) is

$$\left(\frac{x_1}{a^2}+\frac{y_1}{b^2}+\frac{z_1}{c^2}\right)x+\left(\frac{x_2}{a^2}+\frac{y_2}{b^2}+\frac{z_2}{c^2}\right)y+\left(\frac{x_3}{a^2}+\frac{y_3}{b^2}+\frac{z_3}{c^2}\right)z=1.$$

Show also that it touches the sphere

$$(x^2+y^2+z^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=1. \qquad [\textit{P. U. H.}]$$

26. If P, (x_1, y_1, z_1), Q, (x_2, y_2, z_2), R, (x_3, y_3, z_3) are the extremities of three conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and $OP=r_1$, $OQ=r_2$, $OR=r_3$, prove that the equation to the sphere OPQR can be written

$$x^2+y^2+z^2-r_1^2\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}\right)-r_2^2\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}\right)$$
$$-r_3^2\left(\frac{xx_3}{a^2}+\frac{yy_3}{b^2}+\frac{zz_3}{c^2}\right)=0,$$

and prove that the locus of the centres of spheres through the

origin and the extremities of three equal conjugate semi-diameters is $12(a^2x^2+b^2y^2+c^2z^2)=(a^2+b^2+c^2)^2$. [*P.U.H. 1957*]

27. OP, OQ, OR are conjugate diameters of an ellipsoid, and chords are drawn through a given point parallel to OP, OQ, OR. Prove that the sum of the squares of the ratios of the respective chords to OP, OQ, OR is constant.

28. Prove that the sum of the products of the perpendiculars from the two extremities of each of three conjugate diameters on any tangent plane to an ellipsoid is equal to twice the square on the perpendicular from the centre on that tangent plane. [*Bar. U.*]

29. Show that any one of three equal conjugates of an ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is on the cone whose equation is

$$(a^2+b^2+c^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=3(x^2+y^2+z^2). \qquad [\ \textit{Ag. U. 1955}\ ]$$

****30. Show that any two sets of three conjugate diameters of an ellipsoid lie on a cone of the second degree.**

31. Conjugate diameters of $a_1x^2+b_1y^2+c_1z^2=1$ meet $a_2x^2+b_2y^2+c_2z^2=1$ in P, Q, R. Prove that the plane PQR touches the conicoid $a_3x^2+b_3y^2+c_3z^2=1$,

where $\frac{a_3}{a_1}=\frac{b_3}{b_1}=\frac{c_3}{c_1}=\frac{a_2}{a_1}+\frac{b_2}{b_1}+\frac{c_2}{c_1}$.

[The equation of the first conicoid is $a_1x^2+b_1y^2+c_1z^2=1$ ...(1)

and that of the second is $a_2x^2+b_2y^2+c_2z^2=1$ ...(2)

and that of the third is $a_3x^2+b_3y^2+c_3z^2=1$ ...(3)

Let the equation of the plane PQR be $lx+my+nz=p$ ...(4)

The equation of the cone whose vertex is the origin and which passes thro' the pts. of intersection of the conicoid (2) and the plane (4) is $a_2x^2+b_2y^2+c_2z^2=\left(\frac{lx+my+nz}{p}\right)^2$.

$\because$ it passes thro' three conjugate semi-diameters of the conicoid (1), $\therefore$ (it will be found that)

$$a_2\left(\frac{1}{a_1}\right)+b_2\left(\frac{1}{b_1}\right)+c_2\left(\frac{1}{c_1}\right)=\frac{1}{p^2}\left[l^2\left(\frac{1}{a_1}\right)+m^2\left(\frac{1}{b_1}\right)+n^2\left(\frac{1}{c_1}\right)\right] \ldots(A)$$

The plane (4) touches the conicoid (3), if

$$\frac{l^2}{a_3}+\frac{m^2}{b_3}+\frac{n^2}{c_3}=p^2 \ldots (B)$$

From (A),

$$\frac{l^2}{a_1}+\frac{m^2}{b_1}+\frac{n^2}{c_1}=p^2\left(\frac{a_2}{a_1}+\frac{b_2}{b_1}+\frac{c_2}{c_1}\right) \ldots (C)$$

Compare coeffs. of l^2, m^2, n^2, p^2 in (C) and (B).]

**32. If through any given point (α, β, γ) perpendiculars are drawn to any three conjugate diameters of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$, prove that the plane through the feet of the perpendiculars passes through the fixed point $\left(\frac{a^2\alpha}{a^2+b^2+c^2}, \frac{b^2\beta}{a^2+b^2+c^2}, \frac{c^2\gamma}{a^2+b^2+c^2}\right)$.

[*P. U. 1958*]

33. The enveloping cone from a point P to an ellipsoid has three generating lines parallel to conjugate diameters of the ellipsoid ; find the locus of P. [*P. U. H. 1961*]

34. A line OP, drawn through the vertex O of the cone $ax^2+by^2+cz^2=0$, is such that the two planes through OP, each of which cuts the cone in perpendicular generators, are perpendicular, prove that the locus of OP is the cone

$$(2a+b+c)x^2+(2b+c+a)y^2+(2c+a+b)z^2=0.$$ [*P.U.H. 1961*]

35. Show that two asymptotes can be drawn from a point P to the conicoid $ax^2+by^2+cz^2=1$, and that they are perpendicular if P lies on the cone $a^2(b+c)x^2+b^2(c+a)y^2+c^2(a+b)z^2=0$.

**36. Prove that the bisectors of the angles between the lines in which the plane $ux+vy+wz=0$ cuts the cone $ax^2+by^2+cz^2=0$ lie on the cone $\frac{u(b-c)}{x}+\frac{v(c-a)}{y}+\frac{w(a-b)}{z}=0$.

[The equation of the plane is $ux+vy+wz=0$ ... (1)

and that of the given cone is $ax^2+by^2+cz^2=0$ ... (2)

A cone of the second degree can be found to pass thro' any five concurrent lines (Ex. 1, Art. 88).

(As suggested by the form of the result to be proved, *viz.*, $u(b-c)yz+v(c-a)zx+w(a-b)xy=0$) let us find the equation of the cone thro' the three axes and the two bisectors of the angles between the lines in which the plane (1) meets the cone (2).

Let the required equation of the cone be

$$fyz+gzx+hxy=0 \;...\; (3)$$ [Art. 89]

The bisectors of the angles between two lines are

(*i*) $\perp$, and (*ii*) harmonically conjugate w. r. t. the lines *.

From (*i*), the lines of section of the plane (1) and the cone (3) are $\perp$

$\therefore$ (it will be found that) $fvw+gwu+huv=0$

[As in Ex. 2, Art. 102]

or $$\frac{f}{u}+\frac{g}{v}+\frac{h}{w}=0 \;...\; (4)$$

* See the author's *New Modern Pure Geometry* for B. A. Students (Fourth Edition), Prop. 17.

Eliminating z from (1) and (2), it will be found that the equations of the projections of the lines of section of the plane (1) and the cone (2) on the xy-plane are

$$(aw^2+cu^2)\,x^2+2cuvxy+(bw^2+cv^2)\,y^2=0,\ z=0 \ \dots (5)$$

Similarly, eliminating z from (1) and (3), it will be found that the equations of the projections of the lines of section of the plane (1) and the cone (3) on the xy-plane are

$$gux^2+(fu+gv-hw)\,xy+fvy^2=0,\ z=0 \ \dots (6)$$

From (ii) above, the lines (5) and (6) are harmonically conjugate

$\therefore$ (it will be found that)

$$afvw+bgwu+chuv=0 \qquad [\ ab'+a'b=2hh'\ *\ ]$$

or
$$\frac{af}{u}+\frac{bg}{v}+\frac{ch}{w}=0 \ \dots (7)$$

Solve (4) and (7) for $\frac{f}{u}, \frac{g}{v}, \frac{h}{w}$ by cross-multiplication, and substitute these values of f, g, h in (3). Divide thro' out by xyz.]

37. If the lines in which the plane $lx+my+nz=0$ meets the cone $\alpha x^2+\beta y^2+\gamma z^2=0$ are conjugate diameters of the ellipse in which it meets the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, prove that the locus of the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is the cone

$$a^2(\beta b^2+\gamma c^2)x^2+b^2(\gamma c^2+\alpha a^2)\,y^2+c^2(\alpha a^2+\beta b^2)z^2=0.$$

[By Analytical Plane Geometry, if the lines $a'x^2+2h'xy+b'y^2=0$ are conjugate diameters of the conic $ax^2+2hxy+by^2=1$, then $\mathbf{ab'+a'b=2hh'}$.]

38. Two perpendicular tangent planes to the paraboloid

$$\frac{x^2}{a}+\frac{y^2}{b}=2z$$

intersect in a straight line lying in the plane $x=0$. Show that the line touches the parabola $x=0,\ y^2=(a+b)(2z+a)$. [*Ag. U. 1930*]

[The equation of the paraboloid is $\frac{x^2}{a}+\frac{y^2}{b}=2z \ \dots (1)$

Let the equations of the line lying in the plane $x=0$ be

$$lx+my+nz=p,\ x=0.$$

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* By Analytical Plane Geometry, the condition, that the lines

$$ax^2+2hxy+by^2=0,$$

and
$$a'x^2+2h'xy+b'y^2=0$$

may be harmonically conjugate, is

$$\mathbf{ab'+a'b=2hh'}.$$

i.e., $$my+nz=p,\ x=0 \quad ...(2)$$

From the given condition it will be found that

$$m^2(a+b)+n^2a+2np=0 \quad ...(3)$$

In order to prove that the line (2) touches the given parabola

$$y^2=(a+b)(2z+a),\ x=0$$

let us *find the envelope of the line* (2) *subject to the condition* (3).

Eliminating p from (2) and (3), the equations of the line (2) become $$m^2(a+b)+n^2a+2n\,(my+nz)=0,\ x=0$$

or $$m^2(a+b)+2mny+n^2\,(2z+a)=0,\ x=0$$

or, dividing the first equation thro' out by n^2,

$$\frac{m^2}{n^2}(a+b)+2\frac{m}{n}y+(2z+a)=0,\ x=0.$$

The first equation is a quadratic in the parameter $\dfrac{m}{n}$,

$\therefore$ the envelope is $4y^2=4(a+b)(2z+a),\ x=0.$

($B^2=4\,AC$ (From Differential Calculus))]

39. Show that the feet of the normals from the point (α, β, γ) to the paraboloid $x^2+y^2=2az$ lie on the sphere

$$x^2+y^2+z^2-z(a+\gamma)-\frac{y}{2\beta}(\alpha^2+\beta^2)=0. \qquad [\text{ B. U. 1948 }]$$

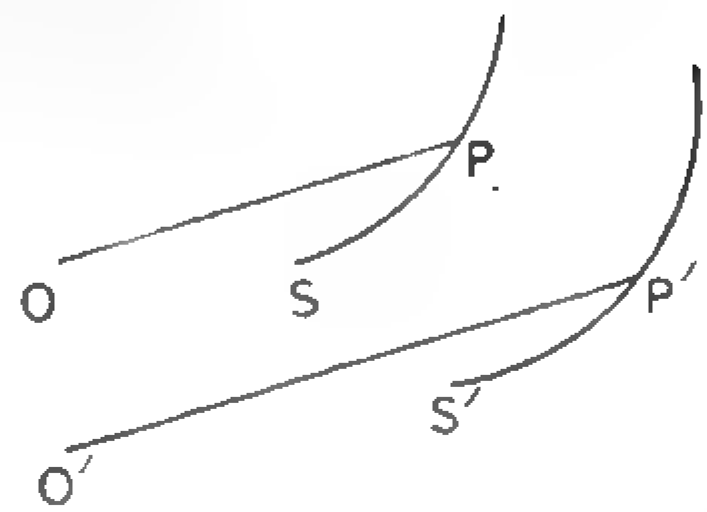
CHAPTER XII

PLANE SECTIONS OF A CONICOID

SECTION I

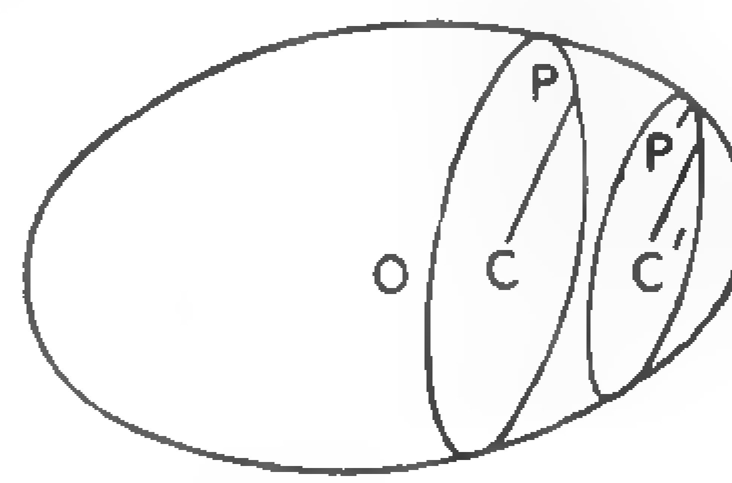
NATURE OF A PLANE SECTION OF A CONICOID

144. Similar and similarly situated curves. Def.



Two plane curves S and S′ are said to be **similar and similarly situated** if the radii vectores drawn to S from a point O in its plane are in a constant ratio to parallel radii vectores drawn to S′ from a point O′ in its plane, *i.e.*, if OP is ∥ to O′P′, and $\frac{OP}{O'P'}$=constant.

145. *All parallel plane sections of a central conicoid are similar and similarly situated conics.*



[****Proof.** Let the equation of the central conicoid be

$$ax^2+by^2+cz^2=1 \quad ...(1)$$

Let $C(x_1, y_1, z_1)$, $C'(x_2, y_2, z_2)$ be the centres of two ∥ plane sections of the conicoid (1).

Then the equations of the planes of the sections are

$$axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1 \quad [\text{ T}=\text{S}_1 \text{ (Art. 126) }]$$

or $$axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2 \quad ...(2)$$

and $$axx_2+byy_2+czz_2=ax_2^2+by_2^2+cz_2^2 \quad ...(3)$$

∵ these planes are ∥

$$\therefore \frac{ax_1}{ax_2}=\frac{by_1}{by_2}=\frac{cz_1}{cz_2}, \quad \text{or} \quad \frac{x_1}{x_2}=\frac{y_1}{y_2}=\frac{z_1}{z_2}.$$

Let CP, C′P′ be two ∥ radii vectores of the conics, the sections of the conicoid (1) by the planes (2) and (3), [Art. 114, Cor. 2] and let their direction-cosines be l, m, n.

Then the equations of CP [thro' $C(x_1, y_1, z_1)$] are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the conicoid (1), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1$$

or $$r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)$$
$$+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad ...(4)$$
which is a quadratic in r.

$\because$ (x_1, y_1, z_1) is the centre of the section, the quadratic (4) has equal and opposite roots

$\therefore$ sum of the roots$=0$, $\therefore$ coeff. of $r=0$,

i.e., $alx_1+bmy_1+cnz_1=0$, and then from (4),
$$r^2(al^2+bm^2+cn^2)+(ax_1^2+by_1^2+cz_1^2-1)=0$$
or $$r^2=\frac{1-ax_1^2-by_1^2-cz_1^2}{al^2+bm^2+cn^2},$$
i.e., $$CP^2=\frac{1-ax_1^2-by_1^2-cz_1^2}{al^2+bm^2+cn^2}.$$
Similarly $$C'P'^2=\frac{1-ax_2^2-by_2^2-cz_2^2}{al^2+bm^2+cn^2}.$$
$$\therefore \quad \frac{CP^2}{C'P'^2}=\frac{1-ax_1^2-by_1^2-cz_1^2}{1-ax_2^2-by_2^2-cz_2^2},$$
which being independent of l, m, n, is constant
$$\therefore \quad \frac{CP}{C'P'}=\text{constant}$$
$\therefore$ $\parallel$ plane sections of the central conicoid (1) by the planes (2) and (3) are similar and similarly situated conics. [Def. (Art. 144)]

Similarly parallel plane sections of a paraboloid are similar and similarly situated conics.]

EXAMPLES

1. *Nature of a plane section of a central conicoid.* **Find the conditions that the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ should be**

(*i*) **an ellipse,** (*ii*) **a parabola,** (*iii*) **a hyperbola.** [*P.U.H. 1955*]

The equation of the conicoid is $ax^2+by^2+cz^2=1$ ...(1)

and that of the plane is $lx+my+nz=p$ ...(2)

and that of the plane $\parallel$ to the plane (2) and passing thro' the origin (0, 0, 0) is $lx+my+nz=0$ ...(3)

The section of the conicoid (1) by the plane (2) is an ellipse, parabola, or hyperbola according as the section of the conicoid (1) by the plane (3) is an ellipse, parabola, or hyperbola

[$\because$ $\parallel$ plane sections of a conicoid are similar and similarly situated conics (Art. 145)]

Eliminating z from (1) and (3) [by substituting its value $\left(z=-\dfrac{lx+my}{n}\right)$ from (3) in (1)], [*Assume that $n\neq 0$*]
$$ax^2+by^2+c\left(\frac{lx+my}{n}\right)^2=1$$

or $\quad n^2(ax^2+by^2)+c(lx+my)^2=n^2$

or $\quad x^2\ (an^2+cl^2)+2\ clmxy+y^2\ (bn^2+cm^2)-n^2=0.$

$\therefore$ the equations of the projection of the section of the conicoid (1) by the plane (3) on the xy-plane are

$$(an^2+cl^2)\ x^2+2\ clmxy+(bn^2+cm^2)\ y^2-n^2=0,\ z=0.$$

[Compare the first equation with $ax^2+2hxy+by^2+2gx+2fy+c=0$, here "$a$"$=an^2+cl^2$, "$b$"$=bn^2+cm^2$, "$c$"$=-n^2$, $f\ =0\quad ,\ g\ =0\quad ,\ h\ =clm.$]

It is an ellipse, parabola, or hyperbola according as

$$(an^2+cl^2)(bn^2+cm^2)-c^2l^2m^2 \text{ is +ve, 0, or -ve} \quad ...(4)$$

["$ab-h^2$" is +ve, 0, or −ve. (From Analytical Plane Geometry)]

Now $\quad (an^2+cl^2)(bn^2+cm^2)-c^2l^2m^2$
$=abn^4+cam^2n^2+bcn^2l^2+c^2l^2m^2-c^2l^2m^2$
$=n^2\ (bcl^2+cam^2+abn^2).$

$\therefore$ from (4), the projection of the section of the conicoid (1) by the plane (3) is an ellipse, parabola, or hyperbola according as

$$n^2\ (bcl^2+cam^2+abn^2) \text{ is +ve, 0, or -ve}$$

i.e., dividing by n^2, (which $\neq 0$)
according as $bcl^2+cam^2+abn^2$ is +ve, 0, or −ve.

$\therefore$ the section of the conicoid (1) by the plane (3) is an ellipse, parabola, or hyperbola according as

$$bcl^2+cam^2+abn^2 \text{ is +ve, 0, or -ve}$$

[$\because$ the (orthogonal) projection of an ellipse, parabola, or hyperbola is respectively an ellipse, parabola, or hyperbola (From Modern Pure Geometry)]

$\therefore$ the section of the conicoid (1) by the plane (2) is also an ellipse, parabola, or hyperbola according as

$$bcl^2+cam^2+abn^2 \text{ is +ve, 0, or -ve.}$$

2. Prove that the section of a hyperboloid by a plane parallel to a tangent plane of the asymptotic cone is a parabola.

3. *Nature of a plane section of a paraboloid.* **Find the nature of the section of the paraboloid $ax^2+by^2=2z$ by the plane $lx+my+nz=p$.** [*P. U. H. 1957*]

[The equation of the paraboloid is $ax^2+by^2=2z$ ...(1)
and that of the plane is $\quad lx+my+nz=p$ ...(2)
and that of the plane ∥ to the plane (2) and passing thro' the origin (0, 0, 0) is $\quad lx+my+nz=0$ ...(3)

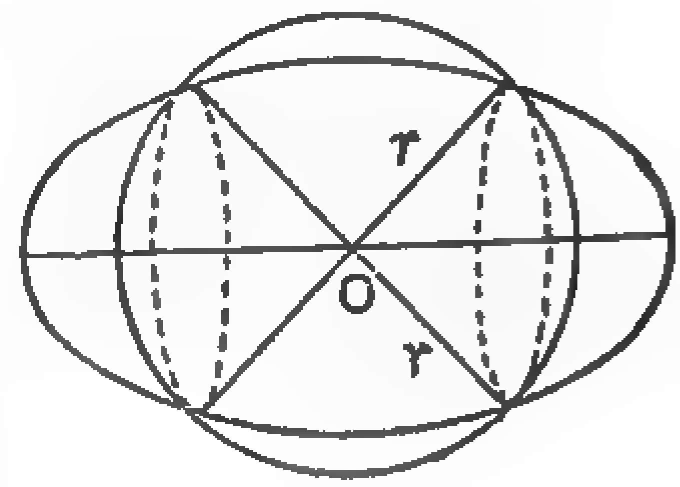
Find the equations of the projection of the section of the paraboloid (1) by the plane (3) on the *yz-plane* (*by eliminating x from (1) and (3) etc.*)]

Cor. All sections of a paraboloid which are parallel to the axis of the paraboloid are parabolas; all other sections of an elliptic paraboloid are ellipses, and of a hyperbolic paraboloid are hyperbolas.

SECTION II
AXES OF A PLANE SECTION OF A CONICOID

Axes of a central section* of a central conicoid.

146. To find the lengths and direction-ratios of the axes of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$.



The equation of the conicoid is

$$ax^2+by^2+cz^2=1 \quad ...(1)$$

and that of the plane is

$$lx+my+nz=0 \quad ...(2)$$

(*a*) *To find the lengths of the semi-axes.*

The equation of a sphere whose centre is the centre of the conicoid, *i.e.*, origin and radius r, is $x^2+y^2+z^2=r^2$

or $$\frac{x^2+y^2+z^2}{r^2}=1 \quad ...(3)$$

The equation of the cone whose vertex is the origin, and which passes thro' the pts. of intersection of the conicoid (1) and the sphere (3), is [making (1) homogeneous by means of (3)],

$$ax^2+by^2+cz^2=\frac{x^2+y^2+z^2}{r^2}$$

or $$\left(a-\frac{1}{r^2}\right)x^2+\left(b-\frac{1}{r^2}\right)y^2+\left(c-\frac{1}{r^2}\right)z^2=0 \quad ...(4)$$

The plane (2) meets the cone (4) in two diameters each of semi-length r.

If r = length of either semi-axis, the two diameters coincide.

$\therefore$ the plane (2) touches the cone (4)

$$\therefore \quad \frac{\mathbf{l}^2}{\mathbf{a}-\frac{1}{\mathbf{r}^2}}+\frac{\mathbf{m}^2}{\mathbf{b}-\frac{1}{\mathbf{r}^2}}+\frac{\mathbf{n}^2}{\mathbf{c}-\frac{1}{\mathbf{r}^2}}=\mathbf{0} \quad ...(5)$$

$$\left[\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0 \text{ (Art. 135, (2))}\right]$$

or $$l^2\left(b-\frac{1}{r^2}\right)\left(c-\frac{1}{r^2}\right)+m^2\left(c-\frac{1}{r^2}\right)\left(a-\frac{1}{r^2}\right)+n^2\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=0$$

*Central section. Def. A section of a central conicoid whose plane passes through the *centre* is called a central section.

or $$l^2\left(\frac{1}{r^2}-b\right)\left(\frac{1}{r^2}-c\right)+m^2\left(\frac{1}{r^2}-c\right)\left(\frac{1}{r^2}-a\right)$$
$$+n^2\left(\frac{1}{r^2}-a\right)\left(\frac{1}{r^2}-b\right)=0$$

or $$l^2\left[\frac{1}{r^4}-\frac{1}{r^2}(b+c)+bc\right]+m^2\left[\frac{1}{r^4}-\frac{1}{r^2}(c+a)+ca\right]$$
$$+n^2\left[\frac{1}{r^4}-\frac{1}{r^2}(a+b)+ab\right]=0$$

or $$\frac{\mathbf{1}}{\mathbf{r^4}}(\mathbf{l^2+m^2+n^2})-\frac{\mathbf{1}}{\mathbf{r^2}}\left[\mathbf{l^2(b+c)+m^2(c+a)+n^2(a+b)}\right]$$
$$+(\mathbf{l^2.bc+m^2.ca+n^2.ab})=\mathbf{0} \quad \ldots(6)$$

which is a quadratic in $\frac{1}{r^2}$, giving two values of $\frac{1}{r^2}$, say, $\frac{1}{r_1^2}$, $\frac{1}{r_2^2}$, and hence r_1, r_2, the lengths of the semi-axes.

(b) *To find the direction-ratios of the axes.*

Let λ, μ, ν be the direction-ratios of an axis.

Then the equations of the axis [thro' the centre (0, 0, 0)] are
$$\frac{x-0}{\lambda}=\frac{y-0}{\mu}=\frac{z-0}{\nu} \quad \text{(Art. 37, Cor. 1)}$$

or $$\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu} \quad \ldots(7)$$

The plane (2) touches the cone (4) along the line (7).

Any pt. on the line (7) is $(\lambda\rho, \mu\rho, \nu\rho)$.

The equation of the tangent plane to the cone (4) at $(\lambda\rho, \mu\rho, \nu\rho)$ is
$$\left(a-\frac{1}{r^2}\right)x\lambda\rho+\left(b-\frac{1}{r^2}\right)y\mu\rho+\left(c-\frac{1}{r^2}\right)z\nu\rho=0,$$

or, dividing thro' out by ρ,
$$\left(a-\frac{1}{r^2}\right)\lambda x+\left(b-\frac{1}{r^2}\right)\mu y+\left(c-\frac{1}{r^2}\right)\nu z=0 \quad \ldots(8)$$

$\because$ it is the same as the equation of the plane (2)

$\therefore$ comparing coeffs. in (8) and (2),
$$\frac{\lambda\left(\mathbf{a}-\frac{\mathbf{1}}{\mathbf{r^2}}\right)}{\mathbf{l}}=\frac{\mu\left(\mathbf{b}-\frac{\mathbf{1}}{\mathbf{r^2}}\right)}{\mathbf{m}}=\frac{\nu\left(\mathbf{c}-\frac{\mathbf{1}}{\mathbf{r^2}}\right)}{\mathbf{n}} \quad \ldots(9)$$

giving the direction-ratios (λ, μ, ν) of the axis of semi-length r.

Substituting the two values of $\frac{1}{r^2}$ found from (6), one by one, in (9), we get the direction-ratios of the axes.

Cor. 1. *Area of a central section of an ellipsoid.* **The area of the section of the ellipsoid $\frac{\mathbf{x^2}}{\mathbf{a^2}}+\frac{\mathbf{y^2}}{\mathbf{b^2}}+\frac{\mathbf{z^2}}{\mathbf{c^2}}=\mathbf{1}$ by the plane**

$$\mathbf{lx + my + nz = 0,}$$

is $$\pi \mathbf{abc} \frac{\sqrt{\mathbf{l^2 + m^2 + n^2}}}{\sqrt{\mathbf{a^2l^2 + b^2m^2 + c^2n^2}}}.$$

The equation of the ellipsoid is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ ... (1)

and that of the plane is $lx + my + nz = 0$ ... (2)

[Compare (1) with $ax^2 + by^2 + cz^2 = 1$,

here "a" $= \dfrac{1}{a^2}$, "b" $= \dfrac{1}{b^2}$, "c" $= \dfrac{1}{c^2}$.]

The equation giving the lengths (r) of the semi-axes of the section of the ellipsoid (1) by the plane (2), is

$$\frac{1}{r^4}(l^2+m^2+n^2) - \frac{1}{r^2}\left[l^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+m^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+n^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right]$$
$$+\left(l^2.\frac{1}{b^2c^2}+m^2.\frac{1}{c^2a^2}+n^2.\frac{1}{a^2b^2}\right)=0 \quad ...(3)$$

$$\Big[\frac{1}{r^4}(l^2+m^2+n^2) - \frac{1}{r^2}[l^2(b+c)+m^2(c+a)+n^2(a+b)]$$
$$+(l^2.bc+m^2.ca+n^2.ab) = 0 \text{ (Art. 146, (6)) }\Big]$$

which is a quadratic in $\dfrac{1}{r^2}$, giving two values of $\dfrac{1}{r^2}$, say, $\dfrac{1}{r_1^2}$, $\dfrac{1}{r_2^2}$.

Now area of the section (ellipse) is

$$A = \pi r_1 r_2 \quad ... (4)$$

But from (3), (product of the roots)

$$\frac{1}{r_1^2}\cdot\frac{1}{r_2^2} = \frac{\dfrac{l^2}{b^2c^2}+\dfrac{m^2}{c^2a^2}+\dfrac{n^2}{a^2b^2}}{l^2+m^2+n^2} = \frac{a^2l^2+b^2m^2+c^2n^2}{a^2b^2c^2(l^2+m^2+n^2)}.$$

$$\therefore \quad r_1^2 r_2^2 = \frac{a^2b^2c^2(l^2+m^2+n^2)}{a^2l^2+b^2m^2+c^2n^2}, \text{ or } r_1 r_2 = abc\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}.$$

$$\therefore \text{ from (4), } \mathbf{A} = \pi\mathbf{abc}\frac{\sqrt{\mathbf{l^2+m^2+n^2}}}{\sqrt{\mathbf{a^2l^2+b^2m^2+c^2n^2}}}.$$

Cor. 2. *Condition for a central section to be a rectangular hyperbola.* **The condition, that the section of the conicoid $\mathbf{ax^2+by^2+cz^2=1}$ by the plane $\mathbf{lx+my+nz=0}$, should be a rectangular hyperbola, is $\mathbf{l^2(b+c)+m^2(c+a)+n^2(a+b)=0}$.**

The equation of the conicoid is $ax^2+by^2+cz^2=1$ ...(1)

and that of the plane is $lx+my+nz=0$ ...(2)

The equation giving the lengths (r) of the semi-axes of the section of the conicoid (1) by the plane (2), is

$$\frac{1}{r^4}(l^2+m^2+n^2) - \frac{1}{r^2}[l^2(b+c)+m^2(c+a)+n^2(a+b)]$$
$$+(l^2.bc+m^2.ca+n^2.ab)=0 \quad ...(3)$$

which is a quadratic in $\frac{1}{r^2}$, giving two values of $\frac{1}{r^2}$, say, $\frac{1}{r_1^2}, \frac{1}{r_2^2}$.

If the section is a rectangular hyperbola,

$r_2^2=-r_1^2$ [$b^2=-a^2$ (From Analytical Plane Geometry]

$\therefore$ $r_2^2+r_1^2=0$, or dividing thro' out by $r_1^2 r_2^2$, $\frac{1}{r_1^2}+\frac{1}{r_2^2}=0$

$\therefore$ sum of the roots of the quadratic (3)$=0$, $\therefore$ coeff. of $\frac{1}{r^2}=0$,

i.e., $$\mathbf{l^2(b+c)+m^2(c+a)+n^2(a+b)=0,}$$

which is the required condition.

EXAMPLES

1. Find the axes of any central plane section of an ellipsoid. [*Ag. U. 1954*]

If $2\alpha, 2\beta$ are the lengths of the axes of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $lx+my+nz=0$, prove that

$$\alpha^2+\beta^2=\frac{a^2(b^2+c^2)\,l^2+b^2(c^2+a^2)m^2+c^2(a^2+b^2)n^2}{a^2l^2+b^2m^2+c^2n^2},$$

$$\alpha^2\beta^2=\frac{a^2b^2c^2\ (l^2+m^2+n^2)}{a^2l^2+b^2m^2+c^2n^2}.$$

2. Find the lengths of the semi-axes of the conic given by

$3x^2+2y^2+6z^2=1,\ x+y+z=0.$ [*P. U. H. 1938*]

[The equation of the conicoid is $3x^2+2y^2+6z^2=1$ ...(1)

and that of the plane is $x+y+z=0$ ...(2)

[Compare (1) with $ax^2+by^2+cz^2=1$, and (2) with $lx+my+nz=0$,

here $a=3,\ b=2,\ c=6$;

$l=1,\ m=1,\ n=1$]

$\therefore$ the equation giving the lengths (r) of the semi-axes of the section, is

$$\frac{1}{r^4}\ (l^2+m^2+n^2)-\frac{1}{r^2}\ [\ l^2(b+c)+m^2(c+a)+n^2(a+b)\ ]$$
$$+(l^2.bc+m^2.ca+n^2.ab)=0, \text{ (Art. 146, (6))}$$

i.e., (substitute the values of a, b, c ; l, m, n).]

3. Prove that the equation of the conic $x^2+7y^2-10z^2=1$, $x+2y+3z=0$, referred to its principal axes, is $47x^2-14y^2=14$.

4. Prove that the axes of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ lie on the cone

$$(b-c)\ \frac{l}{x}+(c-a)\ \frac{m}{y}+(a-b)\ \frac{n}{z}=0.$$

[The equations giving the direction-ratios (λ,μ,ν) of the axes are

$$\frac{\lambda\left(a-\frac{1}{r^2}\right)}{l}=\frac{\mu\left(b-\frac{1}{r^2}\right)}{m}=\frac{\nu\left(c-\frac{1}{r^2}\right)}{n}=k \text{ (say) (Art. 146,(9))}$$

$$\therefore\quad a-\frac{1}{r^2}=\frac{lk}{\lambda},\quad \Big|\ b-c$$

$$b-\frac{1}{r^2}=\frac{mk}{\mu},\quad \Big|\ c-a$$

$$c-\frac{1}{r^2}=\frac{nk}{\nu}.\quad \Big|\ a-b$$

Multiply these equations by $b-c, c-a, a-b$ respectively, and add vertically.]

5. Prove that the axes of sections of the conicoid

$$\mathbf{ax^2+by^2+cz^2=1}$$

which pass through the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ lie on the cone

$$\frac{b-c}{x}(mz-ny)+\frac{c-a}{y}(nx-lz)+\frac{a-b}{z}(ly-mx)=0. \quad [\textit{P.U.H. 1959}]$$

6. *Area of a central section.* **Prove that the area of the section of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ by the plane $lx+my+nz=0$ is $\frac{\pi abc}{p}$, where p is the perpendicular from the centre to the tangent plane which is parallel to the given plane.** [*D. U. H. 1954*]

7. If A_1, A_2, A_3 are the areas of three mutually perpendicular central sections of an ellipsoid, show that $A_1^{-2}+A_2^{-2}+A_3^{-2}$ is constant.

8. Find the axes and area of any central plane section of an ellipsoid. [*Ag. U. 1954*]

If central plane sections of an ellipsoid be of constant area, prove that their planes touch a cone of the second degree. [*P. U. H. 1934*]

[(*b*) $\pi abc\dfrac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}=\text{constant}=\dfrac{\pi abc}{k}$ (say).

It will be found that

$$l^2(a^2-k^2)+m^2(b^2-k^2)+n^2(c^2-k^2)=0.$$

$\therefore$ the plane $lx+my+nz=0$ touches the cone

$$\frac{x^2}{a^2-k^2}+\frac{y^2}{b^2-k^2}+\frac{z^2}{c^2-k^2}=0.$$

[Art. 135, (2) $\left(\because \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0\right)$].]

9. Prove that the section of the conicoid $ax^2+by^2+cz^2=1$

by a tangent plane to the cone $\dfrac{x^2}{b+c}+\dfrac{y^2}{c+a}+\dfrac{z^2}{a+b}=0$, is a rectangular hyperbola. [*P. U. H. 1957*]

Axes of any section of a central conicoid.

147. To find the lengths and direction-ratios of the axes of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane

$$lx+my+nz=p.$$

[Method of centre.]

The equation of the conicoid is $ax^2+by^2+cz^2=1$ ... (1)
and that of the plane is $lx+my+nz=p$ ... (2)

[To find the centre of the section of the conicoid (1) by the plane (2).]

Let (x_1, y_1, z_1) be the centre of the section of the conicoid (1) by the plane (2).

Then the equation of the plane of the section is

$$axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1 \qquad | \; ax^2+by^2+cz^2-1=0$$

[$T=S_1$ (Art. 126)]

or $\quad axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2$ ... (3)

$\because$ it is the same as the equation of the given plane (2)

$\therefore$ comparing coeffs. in (3) and (2),

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{cz_1}{n}=\frac{ax_1^2+by_1^2+cz_1^2}{p}=\rho \text{ (say)} \quad \ldots (4)$$

or $\quad x_1=\dfrac{l\rho}{a},\; y_1=\dfrac{m\rho}{b},\; z_1=\dfrac{n\rho}{c}$ ... (5)

Substituting these values of x_1, y_1, z_1 in the last two members of (4),

$$\frac{1}{p}\left[a\frac{l^2\rho^2}{a^2}+b\frac{m^2\rho^2}{b^2}+c\frac{n^2\rho^2}{c^2}\right]=\rho \qquad [\text{Cancel } \rho]$$

or $\quad \rho=\dfrac{p}{\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}}$ $\qquad \left[\text{Put } \dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}=p_0^2\right]$

$$=\frac{p}{p_0^2}.$$

Substituting this value of ρ in (5),

$$x_1=\frac{l}{a}\frac{p}{p_0^2},\; y_1=\frac{m}{b}\frac{p}{p_0^2},\; z_1=\frac{n}{c}\frac{p}{p_0^2} \quad \ldots (6)$$

Changing the origin to (x_1, y_1, z_1), the equation of the conicoid (1) becomes

$$a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2=1$$

or $\quad ax^2+by^2+cz^2+2(axx_1+byy_1+czz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(7)

Substituting the values of x_1, y_1, z_1 from (6) in (7),

$$ax^2+by^2+cz^2+2\left(ax\frac{l}{a}\cdot\frac{p}{p_0^2}+by\frac{m}{b}\frac{p}{p_0^2}+cz\frac{n}{c}\frac{p}{p_0^2}\right)$$
$$+\left(a\frac{l^2}{a^2}\frac{p^2}{p_0^4}+b\frac{m^2}{b^2}\frac{p^2}{p_0^4}+c\frac{n^2}{c^2}\frac{p^2}{p_0^4}-1\right)=0$$

or $$ax^2+by^2+cz^2+2\frac{p}{p_0^2}(lx+my+nz)+\frac{p^2}{p_0^4}\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)-1=0$$

$$\left[\text{But } \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p_0^2\right]$$

or $$ax^2+by^2+cz^2+2\frac{p}{p_0^2}(lx+my+nz)+\frac{p^2}{p_0^2}-1=0$$

or $$ax^2+by^2+cz^2+2\frac{p}{p_0^2}(lx+my+nz)=1-\frac{p^2}{p_0^2}=k^2 \text{ (say)} \quad ...(8)$$

The equation of the plane (2) becomes

$$l(x+x_1)+m(y+y_1)+n(z+z_1)=p$$

or $$lx+my+nz+(lx_1+my_1+nz_1-p)=0$$

[But (x_1, y_1, z_1), the centre of the section, lies on the plane (2),

$\therefore\ lx_1+my_1+nz_1=p$]

or $lx+my+nz=0$ ...(9)

Now the section of the conicoid (8) by the plane (9) is the same as [substituting from (9) in (8)], the section of the conicoid

$$ax^2+by^2+cz^2=k^2, \text{ i.e., } \frac{a}{k^2}x^2+\frac{b}{k^2}y^2+\frac{c}{k^2}z^2=1 \quad ...(10)$$

by the plane $lx+my+nz=0$.

[Compare (10) with $ax^2+by^2+cz^2=1$, here "a" $=\frac{a}{k^2}$, "b" $=\frac{b}{k^2}$, "c" $=\frac{c}{k^2}$]

$\therefore$ the equation giving the lengths (r) of the semi-axes of the section is

$$\frac{l^2}{\frac{a}{k^2}-\frac{1}{r^2}}+\frac{m^2}{\frac{b}{k^2}-\frac{1}{r^2}}+\frac{n^2}{\frac{c}{k^2}-\frac{1}{r^2}}=0 \quad ...(11)$$

$$\left[\frac{l^2}{a-\frac{1}{r^2}}+\frac{m^2}{b-\frac{1}{r^2}}+\frac{n^2}{c-\frac{1}{r^2}}=0 \text{ (Art. 146, (5))}\right]$$

or $$l^2\left(\frac{b}{k^2}-\frac{1}{r^2}\right)\left(\frac{c}{k^2}-\frac{1}{r^2}\right)+m^2\left(\frac{c}{k^2}-\frac{1}{r^2}\right)\left(\frac{a}{k^2}-\frac{1}{r^2}\right)$$
$$+n^2\left(\frac{a}{k^2}-\frac{1}{r^2}\right)\left(\frac{b}{k^2}-\frac{1}{r^2}\right)=0$$

or $$l^2\left(\frac{1}{r^2}-\frac{b}{k^2}\right)\left(\frac{1}{r^2}-\frac{c}{k^2}\right)+m^2\left(\frac{1}{r^2}-\frac{c}{k^2}\right)\left(\frac{1}{r^2}-\frac{a}{k^2}\right)$$
$$+n^2\left(\frac{1}{r^2}-\frac{a}{k^2}\right)\left(\frac{1}{r^2}-\frac{b}{k^2}\right)=0$$

or $l^2\left[\frac{1}{r^4}-\frac{1}{r^2k^2}(b+c)+\frac{bc}{k^4}\right]+m^2\left[\frac{1}{r^4}-\frac{1}{r^2k^2}(c+a)+\frac{ca}{k^4}\right]$
$$+n^2\left[\frac{1}{r^4}-\frac{1}{r^2k^2}(a+b)+\frac{ab}{k^4}\right]=0$$

or $\frac{\mathbf{1}}{\mathbf{r}^4}(\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)-\frac{\mathbf{1}}{\mathbf{r}^2\mathbf{k}^2}[\mathbf{l}^2(\mathbf{b}+\mathbf{c})+\mathbf{m}^2(\mathbf{c}+\mathbf{a})+\mathbf{n}^2(\mathbf{a}+\mathbf{b})]$
$$+\frac{\mathbf{1}}{\mathbf{k}^4}(\mathbf{l}^2.\mathbf{bc}+\mathbf{m}^2.\mathbf{ca}+\mathbf{n}^2.\mathbf{ab})=\mathbf{0} \quad \ldots (12)$$

which is a quadratic in $\frac{1}{r^2}$, giving two values of $\frac{1}{r^2}$, say, $\frac{1}{r_1^2}$, $\frac{1}{r_2^2}$, and hence r_1, r_2, the lengths of the semi-axes.

The equations giving the direction-ratios (λ, μ, ν) of the axis of semi-length r, are

$$\frac{\lambda\left(\frac{\mathbf{a}}{\mathbf{k}^2}-\frac{\mathbf{1}}{\mathbf{r}^2}\right)}{\mathbf{l}}=\frac{\mu\left(\frac{\mathbf{b}}{\mathbf{k}^2}-\frac{\mathbf{1}}{\mathbf{r}^2}\right)}{\mathbf{m}}=\frac{\nu\left(\frac{\mathbf{c}}{\mathbf{k}^2}-\frac{\mathbf{1}}{\mathbf{r}^2}\right)}{\mathbf{n}} \quad \ldots (13)$$

$$\left[\frac{\lambda\left(a-\frac{1}{r^2}\right)}{l}=\frac{\mu\left(b-\frac{1}{r^2}\right)}{m}=\frac{\nu\left(c-\frac{1}{r^2}\right)}{n} \text{ (Art. 146, (9))}\right]$$

where $\mathbf{k}^2=\mathbf{1}-\frac{\mathbf{p}^2}{\mathbf{p}_0^2}$, and $\mathbf{p}_0^2=\frac{\mathbf{l}^2}{\mathbf{a}}+\frac{\mathbf{m}^2}{\mathbf{b}}+\frac{\mathbf{n}^2}{\mathbf{c}}$.

Substituting the two values of $\frac{1}{r^2}$ found from (12), one by one, in (13), we get the direction-ratios of the axes.

Note 1. Method of centre. Since in this method we first find the *centre* of the section of the conicoid by the given plane, it may be called the **method of centre.**

Note 2. Rule to write down the equation giving the lengths (r) of the semi-axes of the section of the conicoid $\mathbf{ax^2+by^2+cz^2=1}$ by the plane $\mathbf{lx+my+nz=p}$ from the corresponding equation for the section of the conicoid by the parallel central plane

$$\mathbf{lx+my+nz=0}.$$

In the corresponding equation for the section of the conicoid by the parallel central plane $lx+my+nz=0$, change **a** *to* $\frac{\mathbf{a}}{\mathbf{k}^2}$, **b** *to* $\frac{\mathbf{b}}{\mathbf{k}^2}$, **c** *to* $\frac{\mathbf{c}}{\mathbf{k}^2}$, *thus getting*

$$\frac{\mathbf{l}^2}{\frac{\mathbf{a}}{\mathbf{k}^2}-\frac{\mathbf{1}}{\mathbf{r}^2}}+\frac{\mathbf{m}^2}{\frac{\mathbf{b}}{\mathbf{k}^2}-\frac{\mathbf{1}}{\mathbf{r}^2}}+\frac{\mathbf{n}^2}{\frac{\mathbf{c}}{\mathbf{k}^2}-\frac{\mathbf{1}}{\mathbf{r}^2}}=\mathbf{0}, \text{ or}$$

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2}\left[l^2\left(\frac{b}{k^2}+\frac{c}{k^2}\right)+m^2\left(\frac{c}{k^2}+\frac{a}{k^2}\right)+n^2\left(\frac{a}{k^2}+\frac{b}{k^2}\right)\right]$$
$$+\left[l^2.\left(\frac{b}{k^2}\right)\left(\frac{c}{k^2}\right)+m^2.\left(\frac{c}{k^2}\right)\left(\frac{a}{k^2}\right)+n^2.\left(\frac{a}{k^2}\right)\left(\frac{b}{k^2}\right)\right]=0$$

or $\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2k^2}[\,l^2(b+c)+m^2(c+a)+n^2(a+b)\,]$

$$+\frac{1}{k^4}(l^2.bc+m^2.ca+n^2.ab)=0,$$

where $k^2=1-\frac{p^2}{p_0^2}$, and $p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$.*

Similarly for the rule to write down the equations giving the direction-ratios (λ, μ, ν) of the axis of semi-length r.

Cor. 1. *If r_1, r_2 are the lengths of the semi-axes of the section of the conicoid $ax^2+by^2+cz^2=1$ by a central plane $lx+my+nz=0$, and r_1', r_2' the lengths of the corresponding semi-axes of the section of the conicoid by a parallel plane $lx+my+nz=p$, then* [$\mathbf{r'=kr}$, *i.e.*]

$$\mathbf{r_1'=kr_1,\ r_2'=kr_2,}$$

where $\mathbf{k^2=1-\frac{p^2}{p_0^2}}$, *and* $\mathbf{p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$,

and the corresponding axes of the two sections are parallel.

The equation of the conicoid is $ax^2+by^2+cz^2=1$ ... (1)
and that of the central plane is $lx+my+nz=0$ ... (2)
and that of the || plane is $lx+my+nz=p$ ... (3)

The equation giving the lengths (r) of the semi-axes of the section of the conicoid (1) by the plane (2), is

$$\frac{l^2}{a-\frac{1}{r^2}}+\frac{m^2}{b-\frac{1}{r^2}}+\frac{n^2}{c-\frac{1}{r^2}}=0 \quad \text{... (4)}$$

and the equations giving the direction-ratios (λ, μ, ν) of the axis of semi-length r, are

$$\frac{\lambda\left(a-\frac{1}{r^2}\right)}{l}=\frac{\mu\left(b-\frac{1}{r^2}\right)}{m}=\frac{\nu\left(c-\frac{1}{r^2}\right)}{n} \quad \text{... (5)}$$

[Art. 146]

Now the equation giving the lengths (r') of the semi-axes of the section of the conicoid (1) by the plane (3), is

[From (4), changing a to $\frac{a}{k^2}$, b to $\frac{b}{k^2}$, c to $\frac{c}{k^2}$, and r to r']

$$\frac{l^2}{\frac{a}{k^2}-\frac{1}{r'^2}}+\frac{m^2}{\frac{b}{k^2}-\frac{1}{r'^2}}+\frac{n^2}{\frac{c}{k^2}-\frac{1}{r'^2}}=0 \qquad \text{[Art. 147]}$$

* The expression $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$ is already in the student's memory. For the condition, that the plane $lx+my+nz=p$ may touch the conicoid $ax^2+by^2+cz^2=1$, is $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$ (Art. 116).

or, multiplying the denominators by k^2,

$$\frac{l^2}{a-\dfrac{k^2}{r'^2}}+\frac{m^2}{b-\dfrac{k^2}{r'^2}}+\frac{n^2}{c-\dfrac{k^2}{r'^2}}=0 \;\ldots (6)$$

and the equations giving the direction-ratios (λ', μ', ν') of the axis of semi-length r', are

$$\frac{\lambda'\left(\dfrac{a}{k^2}-\dfrac{1}{r'^2}\right)}{l}=\frac{\mu'\left(\dfrac{b}{k^2}-\dfrac{1}{r'^2}\right)}{m}=\frac{\nu'\left(\dfrac{c}{k^2}-\dfrac{1}{r'^2}\right)}{n}$$

[From (5), changing a to $\dfrac{a}{k^2}$, b to $\dfrac{b}{k^2}$, c to $\dfrac{c}{k^2}$, and r to r', and λ, μ, ν to λ', μ', ν'] [Art. 147]

or, multiplying the numerators by k^2,

$$\frac{\lambda'\left(a-\dfrac{k^2}{r'^2}\right)}{l}=\frac{\mu'\left(b-\dfrac{k^2}{r'^2}\right)}{m}=\frac{\nu'\left(c-\dfrac{k^2}{r'^2}\right)}{n} \;\ldots (7)$$

The equation (6) is the same as (4),

if $\dfrac{k^2}{r'^2}=\dfrac{1}{r^2}$, or $r'^2=k^2r^2$, or $r'=kr$.

$\therefore \quad r_1'=kr_1, r_2'=kr_2.$

Again, if $\dfrac{k^2}{r'^2}=\dfrac{1}{r^2}$, then (7) becomes

$$\frac{\lambda'\left(a-\dfrac{1}{r^2}\right)}{l}=\frac{\mu'\left(b-\dfrac{1}{r^2}\right)}{m}=\frac{\nu'\left(c-\dfrac{1}{r^2}\right)}{n} \;\ldots (8)$$

Dividing (8) by (5),

$$\frac{\lambda'}{\lambda}=\frac{\mu'}{\mu}=\frac{\nu'}{\nu}.$$

$\therefore$ the corresponding axes of the two sections are $\parallel$.

[Art. 13, (*b*), Cor. 4]

Cor. 2. *If A is the area of the section of the conicoid* $ax^2+by^2+cz^2=1$ *by a central plane* $lx+my+nz=0$, *and A′ the area of the section of the conicoid by a parallel plane* $lx+my+nz=p$, *then*

$$\mathbf{A'=k^2A},$$

where $\mathbf{k^2=1-\dfrac{p^2}{p_0^2}}$, *and* $\mathbf{p_0^2=\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}}$.

The equation of the conicoid is $ax^2+by^2+cz^2=1 \;\ldots (1)$
and that of the central plane is $lx+my+nz=0 \;\ldots (2)$
and that of the $\parallel$ plane is $lx+my+nz=p \;\ldots (3)$

Let r_1, r_2 be the lengths of the semi-axes of the section of the conicoid (1) by the plane (2), and r_1', r_2' the lengths of the corresponding semi-axes of the section of the conicoid (1) by the plane (3).

Then $r_1' = kr_1, r_2' = kr_2$,

where $k^2 = 1 - \dfrac{p^2}{p_0^2}$, and $p_0^2 = \dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c}$. [Cor. 1]

$\therefore$ area of the section of the conicoid (1) by the plane (3), is

$$\mathbf{A}' = \pi r_1' r_2' = \pi(kr_1)(kr_2) = k^2(\pi r_1 r_2) = \mathbf{k^2A},$$

where $\mathbf{k^2 = 1 - \dfrac{p^2}{p_0^2}}$, and $\mathbf{p_0^2 = \dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c}}$.

Cor. 3. Area of any section of an ellipsoid. *The area of the section of the ellipsoid* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ *by the plane* $lx + my + nz = p$,

is $$\pi\mathbf{abc}\,\frac{\sqrt{\mathbf{l^2+m^2+n^2}}}{\sqrt{\mathbf{a^2l^2+b^2m^2+c^2n^2}}}\left[\mathbf{1} - \frac{\mathbf{p^2}}{\mathbf{a^2l^2+b^2m^2+c^2n^2}}\right].$$

The equation of the ellipsoid is $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ ... (1)

and that of the plane is $lx + my + nz = p$ ... (2)

and that of the central plane [*i.e.*, ∥ to the plane (2) and passing thro' the centre (0, 0, 0)] is

$$lx + my + nz = 0 \text{ ... (3)}$$

$\therefore$ area of the section of the ellipsoid (1) by the central plane (3) is

$$\mathrm{A} = \pi abc\,\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2l^2+b^2m^2+c^2n^2}} \text{ ... (4)} \qquad \text{[Art. 146, Cor. 1]}$$

Now area of the section of the ellipsoid (1) by the ∥ plane (2), is

$$\mathrm{A}' = k^2\mathrm{A} \text{ ... (5)} \qquad \text{[Cor. 2]}$$

where $k^2 = 1 - \dfrac{p^2}{p_0^2}$,

and $p_0^2 = \dfrac{l^2}{\frac{1}{a^2}} + \dfrac{m^2}{\frac{1}{b^2}} + \dfrac{n^2}{\frac{1}{c^2}}$ $\left[\dfrac{l^2}{a} + \dfrac{m^2}{b} + \dfrac{n^2}{c} \text{ (Art. 147)}\right]$

$= a^2l^2 + b^2m^2 + c^2n^2$.

$\therefore$ $k^2 = 1 - \dfrac{p^2}{a^2l^2 + b^2m^2 + c^2n^2}$.

$\therefore$ from (5),

$$\mathrm{A}' = \left[1 - \frac{p^2}{a^2l^2+b^2m^2+c^2n^2}\right]\pi abc\,\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}$$

$$= \pi\mathbf{abc}\,\frac{\sqrt{\mathbf{l^2+m^2+n^2}}}{\sqrt{\mathbf{a^2l^2+b^2m^2+c^2n^2}}}\left[\mathbf{1} - \frac{\mathbf{p^2}}{\mathbf{a^2l^2+b^2m^2+c^2n^2}}\right].$$

EXAMPLES

1. (*a*) Find the relation between the axes of a central section and a parallel section of a central conicoid. [*Ag. U. 1938*]

(*b*) Find the co-ordinates of the centre and the lengths of the semi-axes of the section of the ellipsoid $6x^2+6y^2+3z^2=5$ by the plane

$$2x+2y+z=1.$$

[(*b*) (*ii*) The equation of the ellipsoid is

$$\tfrac{6}{5}x^2+\tfrac{6}{5}y^2+\tfrac{3}{5}z^2=1 \quad \dots (1)$$

[Form $ax^2+by^2+cz^2=1$ (*Note this step*)]

and that of the plane is $2x+2y+z=1$ ... (2)

[Compare (1) with $ax^2+by^2+cz^2=1$, and (2) with $lx+my+nz=p$,

here $a=\frac{6}{5},\ b=\frac{6}{5},\ c=\frac{3}{5}$;

$l=2,\ m=2,\ n=1,\ p=1$] ... (3)

The equation giving the lengths (r) of the semi-axes of the section is

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2k^2}\,[l^2(b+c)+m^2(c+a)+n^2(a+b)]$$

$$+\frac{1}{k^4}[\,l^2.bc+m^2.ca+n^2.ab\,]=0 \quad \text{[Art. 147, (12)]}$$

[Substitute the values of a, b, c; l, m, n]

i.e., (it will be found that)

$$\frac{3}{r^4}-\frac{1}{r^2k^2}\left(\frac{28}{5}\right)+\frac{1}{k^4}\left(\frac{12}{5}\right)=0$$

$$\therefore \quad \frac{1}{r^2}=\frac{6}{5k^2},\ \frac{2}{3k^2}$$

or $r=\sqrt{\tfrac{5}{6}}\,k,\ \sqrt{\tfrac{3}{2}}\,k$ ... (4)

Now $k^2=1-\dfrac{p^2}{p_0^2}$,

here $p=1$, [From (3)]

$$p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=(2)^2.\frac{5}{6}+(2)^2.\frac{5}{6}+(1)^2.\frac{5}{3}=\frac{25}{3}$$

$$\therefore \quad k^2=1-\frac{1}{\frac{25}{3}}=1-\frac{3}{25}=\frac{22}{25},\ \text{or } k=\frac{\sqrt{22}}{\sqrt{25}}=\frac{\sqrt{22}}{5}.$$

Substitute this value of k in (4).]

2. Axes of any section of a cone. Prove that the axes of the section of the cone $ax^2+by^2+cz^2=0$ by the plane $lx+my+nz=p$ are given by

$$\frac{l^2}{ap_0^2r^2+p^2}+\frac{m^2}{bp_0^2r^2+p^2}+\frac{n^2}{cp_0^2r^2+p^2}=0,$$

where $p_0^2\equiv\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}$. [*Ag. U. 1947*]

Note. *Sometimes the word 'axes' is used in the sense of* **semi-axes,** as here.

3. (*a*) **If OP, OQ, OR are conjugate semi-diameters of an ellipsoid, prove that the area of the section of the ellipsoid by the plane PQR is two-thirds the area of the parallel central section.** [*Ag. U. 1949*]

(*b*) **Show that the area of the section of an ellipsoid, by a plane which passes through the extremities of three conjugate diameters, is in a constant ratio to the area of the parallel central section.** [*P. U. H. 1956*]

4. Find the area of the section of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$. [*P. U. H. 1952*]

5. Find the locus of the centres of sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose area is constant, $(=\pi k^2)$. [*P. U. H. 1957*]

6. **Prove that tangent planes to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}+1=0$ which cut $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}-1=0$ in ellipses of constant area πk^2 have their points of contact on the surface $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{k^4}{4a^2b^2c^2}$.**

7. *Relations between the direction-cosines of the axes of any section.* **Prove that if λ_1, μ_1, ν_1 ; λ_2, μ_2, ν_2 are the direction-cosines of the axes of any plane section of the ellipsoid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

$$\frac{\lambda_1\lambda_2}{a^2(b^2-c^2)}=\frac{\mu_1\mu_2}{b^2(c^2-a^2)}=\frac{\nu_1\nu_2}{c^2(a^2-b^2)}.$$

[The equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

$\because$ the corresponding axes of any section of a central conicoid and the $\parallel$ central section are $\parallel$ (Art. 147, Cor. 1)

$\therefore$ the direction-cosines of the axes of the $\parallel$ central section are also λ_1, μ_1, ν_1 ; λ_2, μ_2, ν_2.

Let the lengths of these semi-axes be r_1, r_2.

Then the extremities of these semi-axes are $P(\lambda_1 r_1, \mu_1 r_1, \nu_1 r_1)$, $Q(\lambda_2 r_2, \mu_2 r_2, \nu_2 r_2)$ (Art. 8).

Now the semi-axes OP, OQ are conjugate semi-diameters of the section (ellipse),

$\therefore$ if OR is the semi-diameter whose diametral plane is the plane POQ, then

OP, OQ, OR are conjugate semi-diameters of the ellipsoid (1). [Art. 133]

$$\therefore \quad \frac{\lambda_1 r_1.\lambda_2 r_2}{a^2}+\frac{\mu_1 r_1.\mu_2 r_2}{b^2}+\frac{\nu_1 r_1.\nu_2 r_2}{c^2}=0$$

$$\left[\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0 \text{ (Art. 131 (B))}\right]$$

or $$\frac{\lambda_1\lambda_2}{a^2}+\frac{\mu_1\mu_2}{b^2}+\frac{\nu_1\nu_2}{c^2}=0 \quad ...(2)$$

Also $\because$ the semi-axes OP, OQ are $\perp$

$\therefore\ \lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2=0 \quad ...(3)$

Solve (2) and (3) for $\lambda_1\lambda_2$, $\mu_1\mu_2$, $\nu_1\nu_2$ by cross-multiplication.]

8. Show that the condition, that the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ should be a rectangular hyperbola, is $\mathbf{l^2(b+c)+m^2(c+a)+n^2(a+b)=0}$.

9. Show that the plane $x+2y+3z=1$ cuts the hyperboloid $2x^2+y^2-2z^2=1$ in a parabola, the direction-ratios of whose axis are $(1, 4, -3)$. [*P. U. B.Sc. H. 1955*]

Axes of any section of a paraboloid.

148. To find the lengths and direction-ratios of the axes of the section of the paraboloid $\mathbf{ax^2-by^2=2z}$ by the plane

$$\mathbf{lx+my+nz=p.}$$

[**Method of centre.**]

The equation of the paraboloid is $ax^2+by^2=2z \quad ...(1)$

and that of the plane is $lx+my+nz=p \quad ...(2)$

[To find the centre of the section of the paraboloid (1) by the plane (2).]

Let (x_1, y_1, z_1) be the centre of the section of the paraboloid (1) by the plane (2).

Then the equation of the plane of the section is

$$axx_1+byy_1-(z+z_1)=ax_1^2+by_1^2-2z_1 \qquad | \ ax^2+by^2-2z=0$$

[T=S$_1$ (Art. 137, (5)]

or $$axx_1+byy_1-z=ax_1^2+by_1^2-z_1 \quad ...(3)$$

$\because$ it is the same as the equation of the given plane (2),

$\therefore$ comparing coeffs. in (3) and (2),

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{-1}{n}=\frac{ax_1^2+by_1^2-z_1}{p}$$

or $$x_1=-\frac{l}{an},\ y_1=-\frac{m}{bn}.$$

$$ax_1^2+by_1^2-z_1=-\frac{p}{n}.$$

$\therefore\ z_1=ax_1^2+by_1^2+\frac{p}{n}$ [Substitute the values of x_1, y_1 found above]

$$=a\frac{l^2}{a^2n^2}+b\frac{m^2}{b^2n^2}+\frac{p}{n}=\frac{1}{n^2}\left[\frac{l^2}{a}+\frac{m^2}{b}+np\right] \quad ...(4)$$

Changing the origin to (x_1, y_1, z_1), the equation of the paraboloid (1) becomes

$$a(x+x_1)^2+b(y+y_1)^2=2(z+z_1)$$

or $\quad a(x+x_1)^2+b(y+y_1)^2-2(z+z_1)=0$

or $\quad ax^2+by^2+2(axx_1+byy_1-z)+(ax_1^2+by_1^2-2z_1)=0 \quad ...(5)$

Substituting the values of x_1, y_1, z_1 from (4) in (5),

$$ax^2+by^2+2\left[ax\left(-\frac{l}{an}\right)+by\left(-\frac{m}{bn}\right)-z\right]$$
$$+\left[a\frac{l^2}{a^2n^2}+b\frac{m^2}{b^2n^2}-2\frac{1}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}+np\right)\right]=0$$

or $\quad ax^2+by^2-\dfrac{2}{n}(lx+my+nz)-\dfrac{l^2}{an^2}-\dfrac{m^2}{bn^2}-\dfrac{2}{n}p=0$

or $\quad ax^2+by^2-\dfrac{2}{n}(lx+my+nz)-\dfrac{1}{n^2}\left(\dfrac{l^2}{a}+\dfrac{m^2}{b}+2np\right)=0$

$$\left[\text{Put } \frac{l^2}{a}+\frac{m^2}{b}+2np=p_0^2\right]$$

or $\quad ax^2+by^2-\dfrac{2}{n}(lx+my+nz)-\dfrac{p_0^2}{n^2}=0$

or $\quad ax^2+by^2-\dfrac{2}{n}(lx+my+nz)=\dfrac{p_0^2}{n^2}=k^2$ (say) ...(6)

The equation of the plane (2) becomes

$$l(x+x_1)+m(y+y_1)+n(z+z_1)=p$$

or $\quad lx+my+nz+(lx_1+my_1+nz_1-p)=0$

[But (x_1, y_1, z_1), the centre of the section, lies on the plane (2)

$\therefore\ lx_1+my_1+nz_1=p$]

or $\quad lx+my+nz=0 \quad ...(7)$

Now the section of the paraboloid (6) by the plane (7) is the same as [substituting from (7) in (6)] the section of the cylinder

$$ax^2+by^2=k^2, \text{ i.e., } \frac{a}{k^2}x^2+\frac{b}{k^2}y^2=1 \quad ...(8)$$

by the plane $lx+my+nz=0$.

$\left[\text{Compare (8) with } ax^2+by^2+cz^2=1, \text{ here } “a”=\dfrac{a}{k^2}, “b”=\dfrac{b}{k^2}, “c”=0\right]$

$\therefore$ the equation giving the lengths (r) of the semi-axes of the section is

$$\frac{l^2}{\frac{a}{k^2}-\frac{1}{r^2}}+\frac{m^2}{\frac{b}{k^2}-\frac{1}{r^2}}+\frac{n^2}{-\frac{1}{r^2}}=0 \quad ...(9)$$

$$\left[\frac{l^2}{a-\frac{1}{r^2}}+\frac{m^2}{b-\frac{1}{r^2}}+\frac{n^2}{c-\frac{1}{r^2}}=0 \text{ (Art. 146, (5))}\right]$$

or $\quad l^2\left(\dfrac{b}{k^2}-\dfrac{1}{r^2}\right)\left(-\dfrac{1}{r^2}\right)+m^2\left(-\dfrac{1}{r^2}\right)\left(\dfrac{a}{k^2}-\dfrac{1}{r^2}\right)$

$$+n^2\left(\frac{a}{k^2}-\frac{1}{r^2}\right)\left(\frac{b}{k^2}-\frac{1}{r^2}\right)=0$$

or $l^2\left(\frac{1}{r^2}-\frac{b}{k^2}\right)\left(\frac{1}{r^2}\right)+m^2\left(\frac{1}{r^2}\right)\left(\frac{1}{r^2}-\frac{a}{k^2}\right)$

$$+n^2\left(\frac{1}{r^2}-\frac{a}{k^2}\right)\left(\frac{1}{r^2}-\frac{b}{k^2}\right)=0$$

or $l^2\left[\frac{1}{r^4}-\frac{b}{r^2k^2}\right]+m^2\left[\frac{1}{r^4}-\frac{a}{r^2k^2}\right]+n^2\left[\frac{1}{r^4}-\frac{1}{r^2k^2}(a+b)+\frac{ab}{k^4}\right]=0$

or $\frac{1}{\mathbf{r}^4}(\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)-\frac{1}{\mathbf{r}^2\mathbf{k}^2}[\mathbf{l}^2(\mathbf{b})+\mathbf{m}^2(\mathbf{a})+\mathbf{n}^2(\mathbf{a}+\mathbf{b})]$

$$+\frac{1}{\mathbf{k}^4}(\mathbf{n}^2.\mathbf{ab})=0 \quad \ldots(10)$$

which is a quadratic in $\frac{1}{r^2}$, giving two values of $\frac{1}{r^2}$, say, $\frac{1}{r_1^2}$, $\frac{1}{r_2^2}$, and hence r_1, r_2, the lengths of the semi-axes.

The equations giving the direction-ratios (λ, μ, ν) of the axis of semi-length r, are

$$\frac{\lambda\left(\frac{\mathbf{a}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{l}}=\frac{\mu\left(\frac{\mathbf{b}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{m}}=\frac{\nu\left(-\frac{1}{\mathbf{r}^2}\right)}{\mathbf{n}} \quad \ldots(11)$$

$$\left[\frac{\lambda\left(a-\frac{1}{r^2}\right)}{l}=\frac{\mu\left(b-\frac{1}{r^2}\right)}{m}=\frac{\nu\left(c-\frac{1}{r^2}\right)}{n} \text{(Art. 146, (9))}\right]$$

where $\mathbf{k}^2=\frac{\mathbf{p}_0^2}{\mathbf{n}^2}$, and $\mathbf{p}_0^2=\frac{\mathbf{l}^2}{\mathbf{a}}+\frac{\mathbf{m}^2}{\mathbf{b}}+2\mathbf{np}$.

Substituting the two values of $\frac{1}{r^2}$ found from (10), one by one, in (11), we get the direction-ratios of the axes.

Note 1. Method of centre. Since in this method we first find the *centre* of the section of the paraboloid by the given plane, it may be called the **method of centre.**

Note 2. Rule to write down the equation giving the lengths (r) of the semi-axes of the section of the paraboloid $\mathbf{ax^2+by^2=2z}$ by the plane $\mathbf{lx+my+nz=p}$ from the corresponding equation for the section of the central conicoid $\mathbf{ax^2+by^2+cz^2=1}$ by the plane $\mathbf{lx+my+nz=p}$.

In the corresponding equation for the section of the central conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$, **put $\mathbf{c=0}$,** *thus getting*

$$\frac{\mathbf{l}^2}{\frac{\mathbf{a}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}}+\frac{\mathbf{m}^2}{\frac{\mathbf{b}}{\mathbf{k}^2}-\frac{1}{\mathbf{r}^2}}+\frac{\mathbf{n}^2}{-\frac{1}{\mathbf{r}^2}}=0$$

or $\frac{1}{\mathbf{r}^4}(\mathbf{l}^2+\mathbf{m}^2+\mathbf{n}^2)-\frac{1}{\mathbf{r}^2\mathbf{k}^2}[\mathbf{l}^2(\mathbf{b})+\mathbf{m}^2(\mathbf{a})+\mathbf{n}^2(\mathbf{a}+\mathbf{b})]+\frac{1}{\mathbf{k}^4}(\mathbf{n}^2.\mathbf{ab})=0,$

where $\mathbf{k^2=\frac{p_0^2}{n^2}}$, *and* $\mathbf{p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+2np}$.*

[*Note the new values of* k^2 *and* p_0^2]

Similarly for the rule to write down the equations giving the direction-ratios (λ, μ, ν) **of the axis of semi-length r.**

Cor. 1. Area of any section of a paraboloid. *The area of the section of the paraboloid* $ax^2+by^2=2z$ *by the plane* $lx+my+nz=p$, *is*

$$\frac{\pi}{\mathbf{n^3}}\ \frac{\sqrt{\mathbf{l^2+m^2+n^2}}}{\sqrt{\mathbf{ab}}}\left[\frac{\mathbf{l^2}}{\mathbf{a}}+\frac{\mathbf{m^2}}{\mathbf{b}}+\mathbf{2np}\right].$$

The equation of the paraboloid is $ax^2+by^2=2z$ ...(1)
and that of the plane is $lx+my+nz=p$ ...(2)

The equation giving the lengths (r) of the semi-axes of the section of the paraboloid (1) by the plane (2), is

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2k^2}[\,l^2(b)+m^2(a)+n^2(a+b)]+\frac{1}{k^4}(n^2.ab)=0 \quad ...(3)$$

[Art. 148, (10)]

which is a quadratic in $\frac{1}{r^2}$, giving two values of $\frac{1}{r^2}$, say, $\frac{1}{r_1^2}$, $\frac{1}{r_2^2}$.

Now area of the section (ellipse) $=\pi r_1 r_2$ ... (4)

But from (3), (product of the roots)

$$\frac{1}{r_1^2}\cdot\frac{1}{r_2^2}=\frac{\frac{n^2ab}{k^4}}{l^2+m^2+n^2}=\frac{n^2ab}{k^4(l^2+m^2+n^2)},\ \text{or}\ r_1^2\,r_2^2=\frac{k^4(l^2+m^2+n^2)}{n^2ab}.$$

$$\therefore\ r_1\,r_2=\frac{k^2}{n}\ \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{ab}}.$$

$\therefore$ from (4),

$$\text{area of the section}=\pi\frac{k^2}{n}\ \frac{\sqrt{l^2+m^2+n^2}}{\sqrt{ab}}$$

$$\left[\text{But } k^2=\frac{p_0^2}{n^2}\ \left(\text{where } p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+2np\right)=\frac{1}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}+2np\right)\right]$$

$$=\pi\frac{1}{n^2}\left(\frac{l^2}{a}+\frac{m^2}{b}+2np\right)\cdot\frac{1}{n}\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{ab}}$$

$$=\frac{\pi}{\mathbf{n^3}}\frac{\sqrt{\mathbf{l^2+m^2+n^2}}}{\sqrt{\mathbf{ab}}}\left[\frac{\mathbf{l^2}}{\mathbf{a}}+\frac{\mathbf{m^2}}{\mathbf{b}}+\mathbf{2np}\right].$$

---

* The expression $\frac{l^2}{a}+\frac{m^2}{b}+2np$ is already in the student's memory. For the condition, that the plane $lx+my+nz=p$ may touch the paraboloid $ax^2+by^2=2z$, is $\frac{l^2}{a}+\frac{m^2}{b}=-2np$, or $\frac{l^2}{a}+\frac{m^2}{b}+2np=0$.

Cor. 2. *Condition for any section to be a rectangular hyperbola.* **The condition, that the section of the paraboloid $ax^2+by^2=2z$ by the plane $lx+my+nz=p$ should be a rectangular hyperbola, is**

$$l^2(b)+m^2(a)+n^2(a+b)=0.$$

The equation of the paraboloid is $ax^2+by^2=2z$ ... (1)
and that of the plane is $lx+my+nz=p$ ... (2)

The equation giving the lengths (r) of the semi-axes of the section of the paraboloid (1) by the plane (2), is

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2k^2}[\,l^2(b)+m^2(a)+n^2(a+b)\,]$$
$$+\frac{1}{k^4}(\,n^2.ab\,)=0 \;...\;(3)\;\text{[Art. 148, (10)]}$$
$$\left(\text{where } k^2=\frac{p_0^2}{n^2}, \text{ and } p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+2np\right)$$

which is a quadratic in $\frac{1}{r^2}$, giving two values of $\frac{1}{r^2}$, say, $\frac{1}{r_1^2}, \frac{1}{r_2^2}$.

If the section is a rectangular hyperbola,

$r_2^2=-r_1^2$ [$b^2=-a^2$ (From Analytical Plane Geometry)]

or $r_2^2+r_1^2=0$ or, dividing thro' out by $r_1^2r_2^2$, $\frac{1}{r_1^2}+\frac{1}{r_2^2}=0$.

$\therefore$ sum of the roots of the quadratic (3) $=0$, $\therefore$ coeff. of $\frac{1}{r^2}=0$,

i.e., $\frac{1}{k^2}[\,l^2(b)+m^2(a)+n^2(a+b)\,]=0$,

or $$\mathbf{l^2(b)+m^2(a)+n^2(a+b)=0,}$$

which is the required condition.

[**Aid to memory.** In the condition for the section of the *central conicoid* $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$ to be a rectangular hyperbola, *viz.*, $l^2(b+c)+m^2(c+a)+n^2(a+b)=0$, [Ex. 8, Art. 147]
put $c=0$, thus getting,

$$l^2(b)+m^2(a)+n^2(a+b)=0,$$

which is the condition for the section of the paraboloid $ax^2+by^2=2z$ by the plane $lx+my+nz=p$ to be a rectangular hyperbola (Cor. 2). Compare Note 2, Art. 148.]

EXAMPLES

1. Find the lengths of the semi-axes of the section of the paraboloid $x^2+2y^2=z$ by the plane $x+y+z=1$.

[The equation of the paraboloid is $x^2+2y^2=z$
or, *multiplying thro' out by 2,* [*Note this step*]

$2x^2+4y^2=2z$ ... (1) (Form $ax^2+by^2=2z$)

The equation of the plane is $x+y+z=1$ ... (2)

[Compare (1) with $ax^2+by^2=2z$, and (2) with $lx+my+nz=p$,
here $a=2,\; b=4$;
$l=1,\; m=1,\; n=1,\; p=1$] ... (3)

The equation giving the lengths (r) of the semi-axes of the section, is

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2k^2}[l^2(b)+m^2(a)+n^2(a+b)]+\frac{1}{k^4}(n^2.ab)=0$$

(Art. 148, (10))

[Substitute the values of a, b ; l, m, n]

i.e., it will be found that $\frac{3}{r^4}-\frac{12}{r^2k^2}+\frac{8}{k^4}=0.$

$\therefore \quad \frac{1}{r^2}=\frac{2}{3k^2}(3\pm\sqrt{3})$, or $r^2=\frac{3k^2}{2(3\pm\sqrt{3})}$ [Rationalize the denom.]

or $r^2=\frac{3k^2}{2}\cdot\frac{3\mp\sqrt{3}}{6}=\frac{k^2}{4}(3\pm\sqrt{3}).$

$\therefore \quad r=\frac{k}{2}\sqrt{3\pm\sqrt{3}}$ ...(4)

Now $k^2=\frac{p_0^2}{n^2}$,

here $p_0^2=\frac{l^2}{a}+\frac{m^2}{b}+2np$, which from (3),

$=\frac{(1)^2}{2}+\frac{(1)^2}{4}+2(1)(1)=\frac{11}{4}$,

$n=1$

$\therefore \quad k^2=\frac{11}{4(1)^2}$, or $k=\frac{\sqrt{11}}{2}$.

Substitute this value of k in (4).]

**2. A plane section through the vertex of the paraboloid of revolution $x^2+y^2=2az$ makes an angle θ with the axis of the surface. Prove that its principal semi-axes are $a \cot\theta \operatorname{cosec}\theta$, $a\cot\theta$. [*P.U.*]

3. Find the locus of the centres of sections of the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ which are of constant area πk^3. [*P. U. H. 1961*]

4. **Planes are drawn through a fixed point** (α, β, γ) **so that their sections of the paraboloid $ax^2+by^2=2z$ are rectangular hyperbolas. Prove that they touch the cone**

$$\frac{(x-\alpha)^2}{b}+\frac{(y-\beta)^2}{a}+\frac{(z-\gamma)^2}{a+b}=0. \quad [\textit{P. U. H. 1961}]$$

SECTION III

CIRCULAR SECTIONS OF A CONICOID

Circular sections of a central conicoid.

149. *Circular sections of an ellipsoid.* **To find the circular**

sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

[Method of sphere.]

The equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad ...(1)$$

Consider the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1-k^*\left(x^2+y^2+z^2-\frac{1}{k}\right)=0^\dagger \quad ...(2)$$

It is a homogeneous equation of the second degree in x, y, z.

If it represents a pair of planes, they meet the ellipsoid (1) where [substituting from (1) in (2)] they meet the sphere

$$x^2+y^2+z^2-\frac{1}{k}=0 \quad ...(3)$$

i.e., in two circles. [$\because$ a plane meets a sphere in a circle]

Now (2) is $\left(\frac{1}{a^2}-k\right)x^2+\left(\frac{1}{b^2}-k\right)y^2+\left(\frac{1}{c^2}-k\right)z^2=0 \quad ...(4)$

[Compare this with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$,

here "a"$=\frac{1}{a^2}-k$, "b"$=\frac{1}{b^2}-k$, "c"$=\frac{1}{c^2}-k$; $f=0$, $g=0$, $h=0$.]

It represents a pair of planes if

$$\left(\frac{1}{a^2}-k\right)\left(\frac{1}{b^2}-k\right)\left(\frac{1}{c^2}-k\right)=0,$$

[$abc+2fgh-af^2-bg^2-ch^2=0$ (Art. 35)]

i.e., if $k=\frac{1}{a^2}, \frac{1}{b^2}, \frac{1}{c^2}$.

* and *not* $+k$ to avoid negative values of k in what follows.

†**How to write this equation.** Write the equation **ellipsoid $-$k (sphere) $=0$**, *where 'ellipsoid' stands for the 'L. H. S. of the equation of the ellipsoid (R. H. S. being zero)', and so for the 'sphere', the equation of the sphere being taken so that the equation ellipsoid $-k$ (sphere) $=0$, is a homogeneous equation of the second degree in x, y, z.*

Thus here we have taken the equation of the sphere to be $x^2+y^2+z^2-\frac{1}{k}=0$, because, by doing so, the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1-k\left(x^2+y^2+z^2-\frac{1}{k}\right)=0$ is a homogeneous equation of the second degree in x, y, z, the constant term (-1) in the 'ellipsoid' cancelling the constant term $-k\left(-\frac{1}{k}\right)$.

A similar trick will be useful in the case of any central conicoid or a paraboloid

Substituting these values of k, one by one, in (4),

$$\left(\frac{1}{b^2}-\frac{1}{a^2}\right)y^2+\left(\frac{1}{c^2}-\frac{1}{a^2}\right)z^2=0 \quad ...(5)$$

$$\left(\frac{1}{a^2}-\frac{1}{b^2}\right)x^2+\left(\frac{1}{c^2}-\frac{1}{b^2}\right)z^2=0 \quad ...(6)$$

$$\left(\frac{1}{a^2}-\frac{1}{c^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{c^2}\right)y^2=0 \quad ...(7)$$

Let a^2 *be* $> b^2 > c^2$. Then $\frac{1}{a^2}<\frac{1}{b^2}<\frac{1}{c^2}$.

$\therefore$ the planes (5) are imaginary,

[$\because$ coeff. of y^2 is +ve, and of z^2 is also +ve*]

the planes (6) are real, [$\because$ coeff. of x^2 is −ve, and of z^2 is +ve]

and the planes (7) are imaginary,

[$\because$ coeff. of x^2 is −ve, and of y^2 is also −ve]

$\therefore$ the equation of the planes of real central circular sections is (6),

i.e., $\left(\frac{b^2-a^2}{a^2b^2}\right)x^2+\left(\frac{b^2-c^2}{b^2c^2}\right)z^2=0$ [Cancel b^2]

or, transposing,

$$\left(\frac{a^2-b^2}{a^2}\right)x^2=\left(\frac{b^2-c^2}{c^2}\right)z^2, \text{ or } \frac{\sqrt{a^2-b^2}}{a}x=\pm\frac{\sqrt{b^2-c^2}}{c}z,$$

or $$\frac{\mathbf{x}}{\mathbf{a}}\sqrt{\mathbf{a^2-b^2}}\pm\frac{\mathbf{z}}{\mathbf{c}}\sqrt{\mathbf{b^2-c^2}}=\mathbf{0},$$

which pass thro' $z=0, x=0$, *i.e.*, thro' the y-axis, *i.e.*, thro' the mean axis.

$\therefore$ the equations of the planes of the circular sections are

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda,$$

$$\frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}=\mu,$$

for all values of λ and μ.

[$\because$ $\parallel$ plane sections of a conicoid are similar and similarly situated conics (Art. 145)]

Cor. 1. Important. *The value of k which gives real central circular sections is* $\frac{1}{b^2}$, *i.e., which lies between the other two values of k, viz.,* $\frac{1}{a^2}$, $\frac{1}{c^2}$. *This value of* $k=\frac{1}{b^2}$ *is called the* **mean value** *of k. Hence we have a* **short cut.**

*so that the equation of the planes is of the form $By^2+Cz^2=0$, where B, C are +ve, *i.e.*, $By^2=-Cz^2$, or $\sqrt{B}y=\pm i\sqrt{C}z$, which are imaginary.

Short cut for numerical examples. *In a numerical example, after finding the three values of k, arrange them in ascending* order of magnitude, take the mean value of k, i.e., which lies between the other two values of k, and substitute it in the equation of the planes, and find their separate equations.* [Rule (Ex. 3, Art. 35)]

Cor. 2. *The radius of the central circular sections of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *is b.*

Substituting the value of $k=\frac{1}{b^2}$ (which gives *real* central circular sections (6)) in (3), the equation of the sphere becomes

$$x^2+y^2+z^2-b^2=0, \text{ or } x^2+y^2+z^2=b^2,$$

whose radius $= b$

$\therefore$ the radius of the central circular sections is also b.

EXAMPLES

1. Find the real central circular sections of the ellipsoid

$$x^2+2y^2+3z^2=4.$$

2. Find the radius of the circle in which the plane

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda$$

cuts the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. [*P. U. H. 1955*]

The equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

and that of the plane of the circular section is

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda \quad ...(2)$$

and that of the $\parallel$ central plane is

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=0 \quad ...(3)$$

The radius of the circular section of the ellipsoid (1) by the central plane (3) $= b$. [Art. 149, Cor. 2]

$\therefore$ radius of the circular section of the ellipsoid (1) by the plane (2) $= kb$ ...(4) [$r'=kr$ (Art. 147, Cor. 1)]

[Compare (1) with $ax^2+by^2+cz^2=1$, and (2) with $lx+my+nz=p$,

here "a" $=\frac{1}{a^2}$, "b" $=\frac{1}{b^2}$, "c" $=\frac{1}{c^2}$;

$l=\frac{1}{a}\sqrt{a^2-b^2}$, $m=0$, $n=\frac{1}{c}\sqrt{b^2-c^2}$, $p=\lambda$] ...(5)

* or descending

Now $k^2=1-\frac{p^2}{p_0^2}$,

here, from (5), $p=\lambda$

$$p_0^2=\text{“}\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\text{”}=\frac{1}{a^2}(a^2-b^2)\,a^2+0+\frac{1}{c^2}\,(b^2-c^2)c^2$$
$$=a^2-b^2+b^2-c^2=a^2-c^2$$

$\therefore \quad k^2=1-\frac{\lambda^2}{a^2-c^2}$, or $k=\sqrt{1-\frac{\lambda^2}{a^2-c^2}}$,

$\therefore$ from (4), radius $=\sqrt{1-\frac{\lambda^2}{a^2-c^2}}\cdot b$.

3. Central circular sections of a hyperboloid of one sheet. *Prove that the central circular sections of the hyperboloid of one sheet* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, $(a^2>b^2)$, *are* $\frac{y}{b}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{a^2+c^2}=0$.

Cor. *The radius of the central circular sections of the hyperboloid of one sheet* $\frac{x^2}{a^2}+\frac{y^3}{b^2}-\frac{z^2}{c^2}=1$, $(a^2>b^2)$ *is* a.

4. Circular sections of a hyperboloid of two sheets. *Prove that the circular sections of the hyperboloid of two sheets*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ (b^3>c^2),$$

are
$$\frac{x}{a}\sqrt{a^2+b^2}+\frac{z}{c}\sqrt{b^2-c^2}=\lambda,$$
$$\frac{x}{a}\sqrt{a^2+b^3}-\frac{z}{c}\sqrt{b^2-c^2}=\mu,$$

for all values of λ and μ.

Note. It will be found that the equation of the sphere is $x^2+y^2+z^2+b^2=0$, or $x^2+y^2+z^3=-b^2$, $\therefore$ radius of the sphere$=ib$.

$\therefore$ radius of the central circular sections is also $=ib$, which is imaginary.

$\therefore$ the central planes $\frac{x}{a}\sqrt{a^2+b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0$, do not meet the hyperboloid of two sheets in real pts. They are the planes thro' the centre to the systems of planes which meet the hyperboloid of two sheets in real circles.

5. Show that the section of the hyperboloid of two sheets $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ by the plane $\frac{x}{a}\sqrt{a^2+b^2}+\frac{z}{c}\sqrt{b^2-c^3}=\lambda$ is real if $\lambda^2>a^2+c^2$.

150. Any two circular sections of an ellipsoid which are not parallel lie on a sphere.

Let the equation of the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad \dots (1)$$

Let the equations of the planes of *any* two circular sections which are not $\parallel$, be

$$\left.\begin{aligned} \frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}&=\lambda, \\ \frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}&=\mu, \end{aligned}\right\} \dots (2) \qquad [\text{Art. } 149]$$

so that their combined equation is

$$\left(\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}-\lambda\right)\left(\frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}-\mu\right)=0 \dots (3)$$

The equation of *any* conicoid thro' the pts. of intersection of the ellipsoid (1) and the planes of the circular sections (3), is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$
$$+k\left[\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}-\lambda\right]\left[\frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}-\mu\right]=0 \quad \dots (4)$$

It is a sphere if

coeff. of x^2 = coeff. of y^2 = coeff. of z^2 ;

and coeff. of $yz=0$, coeff. of $zx=0$, coeff. of $xy=0$,

i.e., if $\frac{1}{a^2}+\frac{k}{a^2}(a^2-b^2)=\frac{1}{b^2}=\frac{1}{c^2}-\frac{k}{c^2}(b^2-c^2)$ ...(5)

[$\because$ coeffs. of yz, zx, xy are each $=0$]

From the first two members,

$$\frac{k}{a^2}(a^2-b^2)=\frac{1}{b^2}-\frac{1}{a^2}=\frac{a^2-b^2}{a^2b^2} \qquad \left[\text{Cancel } \frac{a^2-b^2}{a^2}\right]$$

or $$k=\frac{1}{b^2}.$$

Substituting this value of $k\left(=\frac{1}{b^2}\right)$ in (5), we get

$$\frac{1}{a^2}+\frac{1}{a^2b^2}(a^2-b^2)=\frac{1}{b^2}=\frac{1}{c^2}-\frac{1}{b^2c^2}(b^2-c^2)$$

or $\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{a^2}=\frac{1}{b^2}=\frac{1}{c^2}-\frac{1}{c^2}+\frac{1}{b^2}$, or $\frac{1}{b^2}=\frac{1}{b^2}=\frac{1}{b^2}$,

which is true,

$\therefore$ the circular sections (2) lie on a sphere.

Cor. *To find the equation of the sphere on which the two circular sections lie.*

Substituting the value of $k\left(=\frac{1}{b^2}\right)$ in (4),

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1$$
$$+\frac{1}{b^2}\left[\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}-\lambda\right]\left[\frac{x}{a}\sqrt{a^2-b^2}-\frac{z}{c}\sqrt{b^2-c^2}-\mu\right]=0$$

or $$x^2\left[\frac{1}{a^2}+\frac{1}{a^2b^2}(a^2-b^2)\right]+y^2\left[\frac{1}{b^2}\right]+z^2\left[\frac{1}{c^2}-\frac{1}{b^2c^2}(b^2-c^2)\right]-1$$
$$+\frac{1}{b^2}\cdot\frac{x}{a}\sqrt{a^2-b^2}\,(-\lambda-\mu)+\frac{1}{b^2}\,\frac{z}{c}\sqrt{b^2-c^2}\,(\lambda-\mu)+\frac{1}{b^2}\,\lambda\mu=0$$

or $$\frac{x^2}{b^2}+\frac{y^2}{b^2}+\frac{z^2}{b^2}$$
$$-\frac{1}{b^2}(\lambda+\mu)\,\frac{x}{a}\sqrt{a^2-b^3}+\frac{1}{b^3}(\lambda-\mu)\,\frac{z}{c}\sqrt{b^2-c^2}+\frac{1}{b^2}\,\lambda\mu-1=0$$

or, multiplying thro' out by b^2,

$$x^2+y^2+z^2-(\lambda+\mu)\,\frac{x}{a}\sqrt{a^2-b^2}+(\lambda-\mu)\,\frac{z}{c}\sqrt{b^2-c^2}+(\lambda\mu-b^2)=0,$$

which is the required equation of the sphere.

EXAMPLES

1. Prove that any two circular sections of opposite systems, of an ellipsoid, lie on a sphere. [*P. U. H. 1952*]

Prove that the planes $2x+3z-4=0$, $2x-3z+6=0$ meet the hyperboloid $-x^2+3y^2+12z^2=16$ in circles which lie on the sphere

$$3x^2+3y^2+3z^2+4x+30z-40=0.$$

2. Find the locus of the centres of spheres which pass through the origin and cut the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^3}+\frac{z^2}{c^2}=1$ in a pair of circles.

3. Find the locus of the centres of spheres of constant radius k which cut the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ in a pair of circles.

[*P. U. M. 1943*]

151. Circular sections of any central conicoid. *To find the central circular sections of the central conicoid*

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1.$$

[**Method of sphere.**]

The equation of the central conicoid is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

or $$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-1=0 \quad ...(1)$$

Consider the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-1-k\left(x^2+y^2+z^2-\frac{1}{k}\right)=0^* \quad ...(2)$$

It is a homogeneous equation of the second degree in x, y, z.

If it represents a pair of planes, they meet the conicoid (1) where [substituting from (1) in (2)] they meet the sphere $x^2+y^2+z^2-\frac{1}{k}=0$, *i.e.*, in two circles. [$\because$ a plane meets a sphere in a circle]

Now (2) is

$$(a-k)x^2+(b-k)y^2+(c-k)z^2+2fyz+2gzx+2hxy=0 \quad ...(3)$$

[Compare this with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$, here "$a$"$=a-k$, "$b$"$=b-k$, "$c$"$=c-k$]

It represents a pair of planes if

$$(a-k)(b-k)(c-k)+2fgh-(a-k)f^2-(b-k)g^2-(c-k)h^2=0$$

[$abc+2fgh-af^2-bg^2-ch^2=0$ (Art. 35)]

or, multiplying thro' out by -1,

$$(k-a)(k-b)(k-c)-2fgh-(k-a)f^2-(k-b)g^2-(k-c)h^2=0,$$

which is a cubic in k, giving three real values of k.

[From Theory of Equations]

Arranging them in ascending order of magnitude, *taking the mean value of k, i.e., which lies between the other two values of k, and substituting it in (3), we get the equation of the planes of real central circular sections.*

The separate equations of these planes are the required equations.

[Rule (Ex. 3, Art. 35)]

EXAMPLES

1. Find the real central circular sections of the conicoid $\mathbf{3x^2+5y^2+3z^2+2xz = 4.}$ [*Ag. U. 1953*]

The equation of the conicoid is $3x^2+5y^2+3z^2+2xz=4$

or $$3x^2+5y^2+3z^2+2zx-4=0 \quad ...(1)$$

Consider the equation

$$3x^2+5y^2+3z^2+2zx-4-k\left(x^2+y^2+z^2-\frac{4}{k}\right)=0 \quad ...(2)$$

It is a homogeneous equation of the second degree in x, y, z.

If it represents a pair of planes, they meet the conicoid (1) where [substituting from (1) in (2)] they meet the sphere $x^2+y^2+z^2-\frac{4}{k}=0$ *i.e.*, in two circles. [$\because$ a plane meets a sphere in a circle]

Now (2) is

$$(3-k)x^2+(5-k)y^2+(3-k)z^2+2zx=0 \quad ...(3)$$

*How to write this equation. See the footnote† on page 307.

[Compare this with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$,

here $a=3-k,\quad b=5-k,\quad c=3-k$;

$f=0,\quad g=1,\quad h=0$]

It represents a pair of planes if

$(3-k)(5-k)(3-k)+2(0)-(3-k)0-(5-k)(1)^2-(3-k)0=0$

[$abc+2fgh-af^2-bg^2-ch^2=0$ (Art. 35)]

or, multiplying thro' out by -1,

$$(k-3)(k-5)(k-3)-(k-5)=0$$

$\therefore$ *either* $k-5=0$, *i.e.*, $k=5$,

or, dividing by $k-5$,

$$(k-3)^2-1=0, \text{ or } (k-3)^2=1, \text{ or } k-3=\pm 1$$

or $\quad k=4, 2.$

$\therefore\quad k=5, 4, 2,$

or $\quad k=2, 4, 5.\qquad$ [*Arranging in ascending order of magnitude*]

The mean value of $k=4$.

Substituting this value of $k\ (=4)$ in (3),

$$-x^2+y^2-z^2+2zx=0, \text{ or } y^2-(x^2-2xz+z^2)=0$$

or $$y^2-(x-z)^2=0, \text{ or } (y+x-z)(y-x+z)=0$$

or $$x+y-z=0,\quad x-y-z=0,$$

which are the required equations of the planes of the central circular sections.

2. Find the equations of the real central circular sections of the conicoid $5x^2-8y^2-14yz-10zx+18xy+27=0$.

3. Find the real circular sections of the surface

$$\mathbf{4x^2+2y^2+z^2+3yz+zx=1.}\qquad \text{[\textit{P.U.H. 1952}]}$$

4. Find the conditions that the equations

$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$, $lx+my+nz=0$ should determine a circle. [*P. U. H. 1951*]

The equation of the conicoid is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

or $$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-1=0 \quad\ldots(1)$$

and the equation of the plane is $\quad lx+my+nz=0 \quad\ldots(2)$

[To find the central circular sections of the conicoid (1).]

Consider the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-1-k\left(x^2+y^2+z^2-\frac{1}{k}\right)=0 \quad\ldots(3)$$

It is a homogeneous equation of the second degree in x, y, z.

If it represents a pair of planes, they meet the conicoid (1) where [substituting from (1) in (3)] they meet the sphere $x^2+y^2+z^2-\dfrac{1}{k}=0$, *i.e.*, in two circles. [$\because$ a plane meets a sphere in a circle]

Now (3) is

$$(a-k)x^2+(b-k)y^2+(c-k)z^2+2fyz+2gzx+2hxy=0 \quad \ldots (4)$$

[Compare this with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$.

here "a" $= a-k$, "b" $= b-k$, "c" $= c-k$.]

It represents a pair of planes if

$$(a-k)(b-k)(c-k)+2fgh-(a-k)f^2-(b-k)g^2-(c-k)h^2=0,$$

[$abc+2fgh-af^2-bg^2-ch^2=0$ (Art. 35)]

or, multiplying thro' out by (-1),

$$(k-a)(k-b)(k-c)-2fgh-(k-a)f^2-(k-b)g^2-(k-c)h^2=0,$$

which is a cubic in k, giving three real values of k.

[From Theory of Equations]

If the section of the conicoid (1) by the plane (2) is a circle, then the plane (2) is one of the planes (4).

$\therefore$ let $(a-k)x^2+(b-k)y^2+(c-k)z^2+2fyz+2gzx+2hxy$

$$=(lx+my+nz)\left[\frac{a-k}{l}x+\frac{b-k}{m}y+\frac{c-k}{n}z\right]$$

[Adjusting coeffs. of x^2, y^2, z^2]

[*Assume that none of* $l, m, n = 0$]

Equating coeffs. of yz, zx, xy on both sides,

$$2f=\frac{m(c-k)}{n}+\frac{n(b-k)}{m} \quad \ldots (5)$$

$$2g=\frac{l(c-k)}{n}+\frac{n(a-k)}{l} \quad \ldots (6)$$

$$2h=\frac{l(b-k)}{m}+\frac{m(a-k)}{l} \quad \ldots (7)$$

From (5), $2fmn=m^2(c-k)+n^2(b-k)$, $\therefore$ $k=\dfrac{bn^2+cm^2-2fmn}{m^2+n^2}$.

From (6), $2gnl=l^2(c-k)+n^2(a-k)$, $\therefore$ $k=\dfrac{cl^2+an^2-2gnl}{n^2+l^2}$.

From (7), $2hlm=l^2(b-k)+m^2(a-k)$, $\therefore$ $k=\dfrac{am^2+bl^2-2hlm}{l^2+m^2}$.

$\therefore$ equating the values of k,

$$\frac{bn^2+cm^2-2fmn}{m^2+n^2}=\frac{cl^2+an^2-2gnl}{n^2+l^2}=\frac{am^2+bl^2-2hlm}{l^2+m^2},$$

which are the required conditions.

5. If $lx+my=0$ is a circular section of $ax^2+by^2+cz^2+2hxy=1$, prove that $(b-c)l^2-2hlm+(a-c)m^2=0$.

6. Find the conditions that the section of $ax^2+by^2+cz^2=1$ by $lx+my+nz=p$ should be a circle.

Circular sections of a paraboloid.

152. *To find the circular sections of the paraboloid* $ax^2+by^2=2z$.

The equation of the paraboloid is $ax^2+by^2=2z$

or $$ax^2+by^2-2z=0 \quad ...(1)$$

Consider the equation

$$ax^2+by^2-2z-k\left(x^2+y^2+z^2-\frac{2z}{k}\right)=0 \quad ...(2)$$

It is a homogeneous equation of the second degree in x, y, z.

If it represents a pair of planes, they meet the paraboloid (1) where [substituting from (1) in (2)] they meet the sphere

$$x^2+y^2+z^2-\frac{2z}{k}=0 \quad ...(3)$$

i.e., in two circles. [$\because$ a plane meets a sphere in a circle]

Now (2) is $(a-k)x^2+(b-k)y^2-kz^2=0 \quad ...(4)$

[Compare this with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$,
here "a"$=a-k$, "b"$=b-k$, "c"$=-k$;
$f=0$, $g=0$, $h=0$.]

It represents a pair of planes if

$$(a-k)(b-k)(-k)=0$$

[$abc+2fgh-af^2-bg^2-ch^2=0$ (Art. 35)]

i.e., if $k=a, b, 0$.

Substituting these values of k, one by one, in (4),

$$(b-a)y^2-az^2=0 \quad ...(5)$$
$$(a-b)x^2-bz^2=0 \quad ...(6)$$
$$ax^2+by^2=0 \quad ...(7)$$

Case I. Let a, b be both +ve, and $a>b$.

Then the planes (5) are imaginary,

[$\because$ coeff. of y^2 is $-$ve, and of z^2 is also $-$ve]

the planes (6) are real, [$\because$ coeff. of x^2 is +ve, and of z^2 is $-$ve]
and the planes (7) are imaginary.

[$\because$ coeff. of x^2 is +ve, and of y^2 is also +ve]

$\therefore$ the equation of the planes of real circular sections thro' the vertex is (6)

i.e., $(a-b)x^2=bz^2$, or $\sqrt{a-b}\,x=\pm\sqrt{b}.z$

or $$\sqrt{a-b}\,x\pm\sqrt{b}.z=0.$$

$\therefore$ the equations of the planes of circular sections are

$$\sqrt{a-b}\,x+\sqrt{b}.z=\lambda, \quad \sqrt{a-b}\,x-\sqrt{b}.z=\mu,$$

for all values of λ and μ.

[$\because$ $\parallel$ plane sections of a conicoid are similar and similarly situated conics (Art. 145)]

Case II. Let one of a and b, say, a be +ve and (the other) b be $-$ve.

Then the planes (5) are imaginary,

[$\because$ coeff. of y^2 is $-$ve, and of z^2 is also $-$ve]

the planes (6) are imaginary,

[$\because$ coeff. of x^2 is $+$ve, and of z^2 is also $+$ve]

and the planes (7) are real.

[$\because$ coeff. of x^2 is $+$ve, and of y^2 is $-$ve]

In this case, $\because$ $k=0$,

$\therefore$ radius of the sphere (3)$=\sqrt{\left(-\dfrac{1}{k}\right)^2}=\dfrac{1}{k}=\infty$ [$\because k=0$]

$\therefore$ the circular sections (7) are of infinite radius, *i.e.*, st. lines.

They are the st. lines in which the plane $z=0$ meets the paraboloid (1).

Cor. Important. *The value of k which gives real circular sections through the vertex is b, i.e., which lies between the other two values of k, viz., 0, a.* [$\because a>b>0$, $\therefore 0<b<a$]

This value of k is called the **mean value** *of k. Hence we have a* **short cut.**

Short cut for numerical examples. *See* **short cut for numerical examples** *in Cor. I, Art. 149.*

EXAMPLES

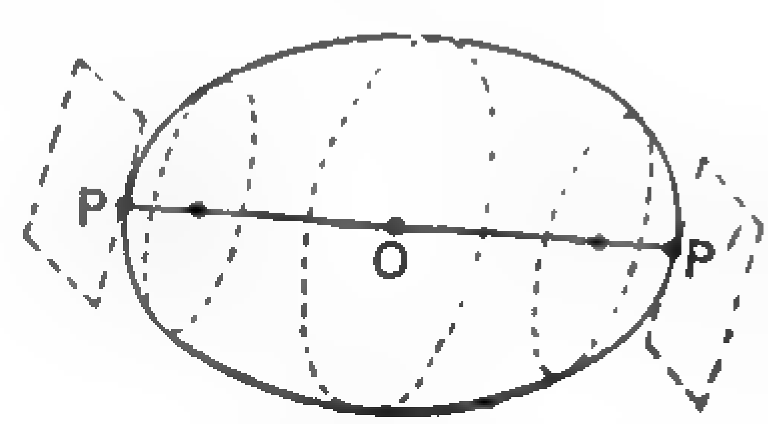
1. Find the circular sections of the paraboloid $x^2+5z^2=2y$.

[*P. U. H. 1960*]

2. Find the radius of the circle in which the plane $x+2z=3$ cuts the paraboloid $5x^2+4y^2=8z$.

Umbilics.

153. Umbilics. The centres of parallel plane sections of a conicoid lie on a diameter of the conicoid [Ex. 3, Art. 126 and Ex. 7, Art. 137], and the tangent plane at an extremity of the diameter is parallel to the plane sections.



$\therefore$ if P, P′ are the extremities of the diameter on which lie the centres of a system of parallel circular sections of an ellipsoid, the sections of the ellipsoid by the tangent planes at P, P′, being similar and similarly situated to the parallel circular sections (Art. 145), are circles of zero radius.

Umbilic. Def. A circular section of zero radius, of a conicoid is called an **umbilic.**

Cor. Important. *The tangent plane at an umbilic of a conicoid is parallel to the plane of a central circular section of the conicoid.*

[Art. 153]

154. Umbilics of an ellipsoid. *To find the umbilics of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

The equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

Let (x_1, y_1, z_1) be an umbilic.

The equation of the tangent plane at (x_1, y_1, z_1) to the ellipsoid (1) is
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1.$$

$\because$ it is $\parallel$ to the plane of a central circular section
$$\frac{x}{a}\sqrt{a^2-b^2}\pm\frac{z}{c}\sqrt{b^2-c^2}=0 \qquad \text{[Art. 153, Cor.]}$$

$$\therefore \quad \frac{\frac{x_1}{a^2}}{\frac{1}{a}\sqrt{a^2-b^2}}=\frac{\frac{y_1}{b^2}}{0}=\frac{\frac{z_1}{c^2}}{\pm\frac{1}{c}\sqrt{b^2-c^2}}$$
$$\left[\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'} \text{ (Art. 27, Cor. 2)}\right]$$

or $$\frac{x_1}{a\sqrt{a^2-b^2}}=\frac{y_1}{0}=\frac{z_1}{\pm c\sqrt{b^2-c^2}}$$

or $$\frac{\frac{x_1}{a}}{\sqrt{a^2-b^2}}=\frac{\frac{y_1}{b}}{0}=\frac{\frac{z_1}{c}}{\pm\sqrt{b^2-c^2}}=\frac{\sqrt{\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}}}{\sqrt{a^2-b^2+(0)^2+b^2-c^2}}$$
$$=\frac{\pm 1}{\sqrt{a^2-c^2}}$$

$\left[\because \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1, \text{ since } (x_1, y_1, z_1) \text{ lies on the ellipsoid (1)}\right]$

or $$\frac{x_1}{a\sqrt{a^2-b^2}}=\frac{y_1}{0}=\frac{z_1}{\pm c\sqrt{b^2-c^2}}=\frac{\pm 1}{\sqrt{a^2-c^2}}$$

$$\therefore \quad x_1=\pm\frac{a\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}},\ y_1=0,\ z_1=\pm\frac{c\sqrt{b^2-c^2}}{\sqrt{a^2-c^2}},$$

which are the required co-ordinates of the umbilics.

N. B. The **co-ordinates of the umbilics** of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ are given by

$$\mathbf{\frac{x}{a}=\pm\frac{\sqrt{a^2-b^2}}{\sqrt{a^2-c^2}},\ y=0,\ \frac{z}{c}=\pm\frac{\sqrt{b^2-c^2}}{\sqrt{a^2-c^2}},\ (a^2>b^2>c^2)}$$

EXAMPLES

1. Show that the umbilics of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

lie on the sphere $x^2+y^2+z^2=a^2-b^2+c^2$.

2. Prove that the perpendicular distance from the centre to the tangent plane at an umbilic of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is ac/b. [*P. U. H. 1958*]

3. Prove that the central circular sections of the conicoid $(a-b)x^2+ay^2+(a+b)z^2=1$ are at right angles and that the umbilics are given by $x=\pm\sqrt{\frac{a-b}{2a(a-b)}}, y=0, z=\pm\sqrt{\frac{a+b}{2a(a+b)}}$. [*Ag.U.1956*]

4. Prove that the umbilics of the conicoid

$$\frac{x^2}{a+b}+\frac{y^2}{a}+\frac{z^2}{a-b}=1$$

are the extremities of the equal conjugate diameters of the ellipse

$$y=0, \quad \frac{x^2}{a+b}+\frac{z^2}{a-b}=1. \qquad [\text{ Ag. U. 1950 }]$$

5. Umbilics of a hyperboloid of one sheet. *Show that the hyperboloid of one sheet* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ *has no real umbilics.*

6. Umbilics of a hyperboloid of two sheets. *Prove that the umbilics of the hyperboloid of two sheets* $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ *are given by*

$$x=\pm\frac{a\sqrt{a^2+b^2}}{\sqrt{a^2+c^2}},\ y=0,\ z=\pm\frac{c\sqrt{b^2-c^2}}{\sqrt{a^2+c^2}}, \quad (b^2>c^2) \quad [\text{P.U.H. 1960}]$$

7. Umbilics of an elliptic paraboloid. *Find the umbilics of the elliptic paraboloid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z, \quad (a^2>b^2)$.

MISCELLANEOUS EXAMPLES ON CHAPTER XII

1. Show that the section of a conicoid by a tangent plane to the asymptotic cone is two parallel straight lines.

2. *Find the angle between the asymptotes of the section of the conicoid* $ax^2+by^2+cz^2=1$ *by the plane* $lx+my+nz=0$.

The equation of the conicoid is $ax^2+by^2+cz^2=1$ ... (1)

and that of the plane is $lx+my+nz=0$ ... (2)

The equation giving the lengths (r) of the semi-axes of the section of the conicoid (1) by the plane (2), is

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2}[\,l^2(b+c)+m^2(c+a)+n^2(a+b)\,]$$
$$+(l^2.bc+m^2.ca+n^2.ab)=0 \quad ...(3)$$

[Art. 146, (6)]

which is a quadratic in $\frac{1}{r^2}$, giving two values of $\frac{1}{r^2}$, say, $\frac{1}{r_1^2}$, $\frac{1}{r_2^2}$.

The equation of the section (hyperbola) referred to its axes as co-ordinate axes, *in their plane*, is $\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1$.

$$\left[\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ (From Analytical Plane Geometry)}\right]$$

and the equation of its asymptotes *in that plane* is

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 0 \quad \dots (4)$$

$$\left[\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \text{ (From Analytical Plane Geometry)}\right]$$

[Compare (4) with $ax^2 + 2hxy + by^2 = 0$,

here $a = \frac{1}{r_1^2}$, $b = \frac{1}{r_2^2}$, $h = 0$]

$\therefore$ if θ is the angle between the asymptotes (4),

then $$\tan\theta = \frac{2\sqrt{0 - \frac{1}{r_1^2} \cdot \frac{1}{r_2^2}}}{\frac{1}{r_1^2} + \frac{1}{r_2^2}}$$

$$\left[2\frac{\sqrt{h^2 - ab}}{a+b} \text{ (From Analytical Plane Geometry}\right]$$

or, squaring, $$\tan^2\theta = \frac{-4\frac{1}{r_1^2} \cdot \frac{1}{r_2^2}}{\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right)^2} \quad \dots (5)$$

Now from the quadratic (3),

(sum of the roots, *i.e.*,) $\frac{1}{r_1^2} + \frac{1}{r_2^2} = \frac{l^2(b+c) + m^2(c+a) + n^2(a+b)}{l^2 + m^2 + n^2}$,

and (product of the roots, *i.e.*,) $\frac{1}{r_1^2} \cdot \frac{1}{r_2^2} = \frac{l^2.bc + m^2.ca + n^2.ab}{l^2 + m^2 + n^2}$.

Substituting these values in (5),

$$\tan^2\theta = \frac{-4\,\frac{l^2.bc + m^2.ca + n^2.ab}{l^2 + m^2 + n^2}}{\left[\frac{l^2(b+c) + m^2(c+a) + n^2(a+b)}{l^2 + m^2 + n^2}\right]^2}$$

$$= \frac{-4\,(l^2.bc + m^2.ca + n^2.ab)\,(l^2 + m^2 + n^2)}{[\,l^2(b+c) + m^2(c+a) + n^2(a+b)\,]^2}.$$

3. If through the centre of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ a perpendicular is drawn to any central section and lengths equal to the axes of the section are marked off along the perpendicular, the

locus of their extremities is given by $\dfrac{a^2x^2}{r^2-a^2}+\dfrac{b^2y^2}{r^2-b^2}+\dfrac{c^2z^2}{r^2-c^2}=0,$

where $r^2 \equiv x^2+y^2+z^2$. [*P. U. H. 1952*]

4. One axis of a central section of the conicoid

$$\mathbf{ax^2+by^2+cz^2=1}$$

lies in the plane ux+vy+wz=0. Show that the other lies on the cone (b−c)uyz+(c−a)vzx+(a−b)wxy=0. [*P. U. H. 1958*]

[The equation of the conicoid is $ax^2+by^2+cz^2=1$ ...(1)

Let λ_1, μ_1, ν_1 ; λ_2, μ_2, ν_2 be the direction-cosines of the axes of a central section of the conicoid (1), then

$$a\lambda_1\lambda_2+b\mu_1\mu_2+c\nu_1\nu_2=0, \qquad \text{[As in Ex. 7, Art. 147]}$$

and
$$\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2=0 \qquad [\because \text{ the axes are } \perp]$$

$\therefore$ by cross-multiplication,

$$\frac{\lambda_1\lambda_2}{b-c}=\frac{\mu_1\mu_2}{c-a}=\frac{\nu_1\nu_2}{a-b}. \;]$$

5. Show that, if the plane $lx+my+nz=p$ cut the surface $ax^2+by^2+cz^2=1$ in a parabola, the co-ordinates of the vertex of the parabola satisfy the equation

$$\frac{ax}{l}\left(\frac{1}{b}-\frac{1}{c}\right)+\frac{by}{m}\left(\frac{1}{c}-\frac{1}{a}\right)+\frac{cz}{n}\left(\frac{1}{a}-\frac{1}{b}\right)=0. \qquad [\text{ } A. U.]$$

[The equations giving the direction-ratios (λ, μ, ν) of the axis of semi-length r of the section are

$$\frac{\lambda\left(\dfrac{a}{k^2}-\dfrac{1}{r^2}\right)}{l}=\frac{\mu\left(\dfrac{b}{k^2}-\dfrac{1}{r^2}\right)}{m}=\frac{\nu\left(\dfrac{c}{k^2}-\dfrac{1}{r^2}\right)}{n}.$$

If the section is a parabola, $r\to\infty$, $\therefore \dfrac{1}{r^2}\to 0$.

$$\therefore \quad \frac{\lambda a}{l}=\frac{\mu b}{m}=\frac{\nu c}{n}, \quad \text{or} \quad \frac{\lambda}{\dfrac{l}{a}}=\frac{\mu}{\dfrac{m}{b}}=\frac{\nu}{\dfrac{n}{c}}.$$

$\therefore$ the direction-ratios of the axis are

$$\frac{l}{a}, \frac{m}{b}, \frac{n}{c}.$$

If (x_1, y_1, z_1) is the vertex of the parabolic section, then the line of intersection of the tangent plane at (x_1, y_1, z_1) to the conicoid and the plane $lx+my+nz=p$ of the section is $\perp$ to the axis of the section.]

6. If $\Delta_1, \Delta_2, \Delta_3$; $\delta_1, \delta_2, \delta_3$ are the areas of the sections of the ellipsoids $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2}=1$, by three conjugate

diametral planes of the former, prove that

$$\frac{\Delta_1^2}{\delta_1^2}+\frac{\Delta_2^2}{\delta_2^2}+\frac{\Delta_3^2}{\delta_3^2}=\frac{a^2b^2c^2}{\alpha^2\beta^2\gamma^2}\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right). \qquad [P. U. H. 1959]$$

7. If A_1, A_2, A_3 are the areas of the sections of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ by the diametral planes of three mutually perpendicular semi-diameters of lengths r_1, r_2, r_3, prove that

$$\frac{A_1^2}{r_1^2}+\frac{A_2^2}{r_2^2}+\frac{A_3^2}{r_3^2}=\pi^2\left(\frac{b^2c^2}{a^2}+\frac{c^2a^2}{b^2}+\frac{a^2b^2}{c^2}\right).$$

8. The normal section of an enveloping cylinder of the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ has a given area πk^2. Prove that the plane of contact of the cylinder and ellipsoid touches the cone

$$\frac{x^2}{a^4(b^2c^2-k^4)}+\frac{y^2}{b^4(c^2a^2-k^4)}+\frac{z^2}{c^4(a^2b^2-k^4)}=0. \qquad [Ag. U. 1944]$$

****9. Axes of a central section of any central conicoid.** *Prove that the lengths (r) of the semi-axes of the section of the conicoid*

$$f(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

by the plane $lx+my+nz=0$ are given by

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2}\left[(a+b+c)(l^2+m^2+n^2)-f(l, m, n)\right] +(Al^2+\ldots+2Fmn+\ldots)=0,$$

where $A=bc-f^2, \ldots, F=gh-af, \ldots$.

Prove also that the axes are the lines in which the plane cuts the cone $(mz-ny)\dfrac{\partial f}{\partial x}+(nx-lz)\dfrac{\partial f}{\partial y}+(ly-mx)\dfrac{\partial f}{\partial z}=0.$

[Proceed as in Art. 146. It will be found that the equation of the cone is

$$\left(a-\frac{1}{r^2}\right)x^2+\left(b-\frac{1}{r^2}\right)y^2+\left(c-\frac{1}{r^2}\right)z^2+2fyz+2gzx+2hxy=0 \quad \ldots(4)$$

∴ the plane (2) (Art. 146) touches the cone (4)

$$\therefore \left[\left(b-\frac{1}{r^2}\right)\left(c-\frac{1}{r^2}\right)-f^2\right]l^2+\ldots+2\left[gh-\left(a-\frac{1}{r^2}\right)f\right]mn+\ldots=0$$

[$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$ (Art. 98)]

Again, it will be found that

$$\frac{\left(a-\frac{1}{r^2}\right)\lambda+h\mu+g\nu}{l}=\frac{h\lambda+\left(b-\frac{1}{r^2}\right)\mu+f\nu}{m}$$

$$=\frac{g\lambda+f\mu+\left(c-\frac{1}{r^2}\right)\nu}{n}=k \text{ (say)}$$

$$\therefore \quad \frac{a\lambda+h\mu+g\nu}{l} - \frac{1}{r^2}\,\frac{\lambda}{l}=k$$

or $$\frac{1}{r^2}\,\frac{\lambda}{l}+k=\frac{a\lambda+h\mu+g\nu}{l}, \text{ etc.}$$

Multiplying these equations by $\frac{\mu}{m}-\frac{\nu}{n}$, etc., and adding vertically,

$$\Sigma\,\frac{a\lambda+h\mu+g\nu}{l}\left(\frac{\mu}{m}-\frac{\nu}{n}\right)=0, \text{ or } \Sigma(a\lambda+h\mu+g\nu)(n\mu-m\nu)=0.$$

$\therefore$ the axes $\left(\frac{x}{\lambda}=\frac{y}{\mu}=\frac{z}{\nu}\right)$ lie on the cone

$\Sigma\,(ax+hy+gz)\,(ny-mz)=0$, *i.e.*, on $\Sigma\,(mz-ny)\,\frac{\partial f}{\partial x}=0.$

Also the axes lie on the plane.]

****10.** Find the lengths of the semi-axes of the sections of the conicoid $4yz-5zx+5xy=2$ by the planes

(*i*) $x-y+z=0$, (*ii*) $2x-y+z=0$.

****11.** Find the condition that the section of the conicoid

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

by the plane $lx+my+nz=0$ may be a rectangular hyperbola.

[The equation of the conicoid is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1 \quad ...(1)$$

and that of the plane is $$lx+my+nz=0 \quad ...(2)$$

Let r_1, r_2 be the lengths of the semi-axes of the section of the conicoid (1) by the plane (2), and l_1, m_1, n_1 ; l_2, m_2, n_2 their direction-cosines. The direction-cosines of the semi-diameter of the ellipsoid $\perp$ to the plane (2) are l, m, n ; let its length be r.

$\because$ the pts. (l_1r_1, m_1r_1, n_1r_1), (l_2r_2, m_2r_2, n_2r_2), (lr, mr, nr) lie on the conicoid (1), $\therefore$ (it will be found that)

$$al_1^2+bm_1^2+cn_1^2+2fm_1n_1+2gn_1l_1+2hl_1m_1=\frac{1}{r_1^2} \quad ...(3)$$

$$al_2^2+bm_2^2+cn_2^2+2fm_2n_2+2gn_2l_2+2hl_2m_2=\frac{1}{r_2^2} \quad ...(4)$$

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=\frac{1}{r^2} \quad ...(5)$$

Adding (3), (4), (5) vertically, it will be found that

$$a+b+c=\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r^2} \quad ...(6) \qquad \text{[Art. 58, (C), (D)]}$$

If the section is a rectangular hyperbola, then $r_2^2=-r_1^2$,

$$\therefore \quad \frac{1}{r_1^2}+\frac{1}{r_2^2}=0.$$

$\therefore$ from (6), $a+b+c=\dfrac{1}{r^2}$ [which from (5)]

$$=al^2+bm^2+cn^2+2fmn+2gnl+2hlm$$

or $(a+b+c)(l^2+m^2+n^2)=al^2+bm^2+cn^2+2fmn+2gnl+2hlm$, which is the required condition.]

**12. Prove that the axes of the section, whose centre is P, of a central conicoid, are the lines in which the plane of section meets the cone through the normals from P.

**13. Show that the area of the section of the conicoid

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

by the plane which passes through the extremities of its principal axes is

$$\frac{2\pi}{3\sqrt{3}}\sqrt{\left(\frac{a+b+c}{\Delta}\right)},$$

where $\Delta=abc+2fgh-af^2-bg^2-ch^2$.

[Find the area of the section of the ellipsoid $\dfrac{x^2}{\alpha^2}+\dfrac{y^2}{\beta^2}+\dfrac{z^2}{\gamma^2}=1$ by the plane $\dfrac{x}{\alpha}+\dfrac{y}{\beta}+\dfrac{z}{\gamma}=1$, and use invariants (Art. 101, Cor.)]

14. Through a given point (α, β, γ) planes are drawn parallel to three conjugate diametral planes of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$.

Show that the sum of the ratios of the areas of the sections by these planes to the areas of the sections by the parallel diametral planes is $\qquad 3-\dfrac{\alpha^2}{a^2}-\dfrac{\beta^2}{b^2}-\dfrac{\gamma^2}{c^2}$. [*Ag. U. 1957*]

15. Find the area of the section of the cone $ax^2+by^2+cz^2=0$ by the plane $lx+my+nz=p$.

The equation of the cone is $ax^2+by^2+cz^2=0$ ...(1)
and that of the plane is $lx+my+nz=p$ ...(2)

Consider the central conicoid $ax^2+by^2+cz^2=d$ ...(3)

i.e., $$\frac{a}{d}x^2+\frac{b}{d}y^2+\frac{c}{d}z^2=1 \quad ...(4)$$

$\left[\text{Compare this with } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1,\right.$

$\left.\text{here } ``a^2"=\dfrac{d}{a},\ ``b^2"=\dfrac{d}{b},\ ``c^2"=\dfrac{d}{c}\right]$

The area of the section of the conicoid (4) by the plane (2),

$$=\pi abc\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}\left[1-\frac{p^2}{a^2l^2+b^2m^2+c^2n^2}\right]$$

$$=\pi\sqrt{\frac{d}{a}}\sqrt{\frac{d}{b}}\sqrt{\frac{d}{c}}\;\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{\frac{d}{a}l^2+\frac{d}{b}m^2+\frac{d}{c}n^2}}\times$$

$$\cdot\left[1-\frac{p^2}{\frac{d}{a}l^2+\frac{d}{b}m^2+\frac{d}{c}n^2}\right]$$

$$=\frac{\pi}{\sqrt{abc}}\;\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}}\left[d-\frac{p^2}{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}\right].$$

Taking the limits when $d\to 0$ [so that the conicoid (3) becomes the cone (1)],

the required area (*in magnitude*) $=\dfrac{\pi}{\sqrt{abc}}\;\dfrac{\sqrt{l^2+m^2+n^2}}{\left[\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}\right]^{\frac{3}{2}}}\,p^2.$

[Rule to deduce the results for the section of the cone $ax^2+by^2+cz^2=0$ by the plane $lx+my+nz=p$ from the corresponding results for the section of the central conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$.

(*i*) *Consider the central conicoid* $ax^2+by^2+cz^2=d$,

i.e., $$\frac{a}{d}x^2+\frac{b}{d}y^2+\frac{c}{d}z^2=1,$$

and write down the corresponding result for the section of this conicoid.

(*ii*) *Simplify the result as far as the powers of d are concerned.*

(*iii*) *Take the limits when* $d\to 0$.]

16. Show that the locus of the centres of sections of the cone $ax^2+by^2+cz^2=0$, such that the sum of the squares of their axes is constant, $=k^2$, is the conicoid

$$a\left(\frac{1}{b}+\frac{1}{c}\right)x^2+b\left(\frac{1}{c}+\frac{1}{a}\right)y^2+c\left(\frac{1}{a}+\frac{1}{b}\right)z^2+k^2=0.$$

[See Note in Ex. 2, Art. 147.]

17. If the area of the section of $\dfrac{y^2}{b}+\dfrac{z^2}{c}=2x$ be constant and equal to a^2, the locus of the centre is

$$a^4\left(1+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)^{-1}=\pi^2bc\left(2x-\frac{y^2}{b}-\frac{z^2}{c}\right)^2. \qquad [Ag.\ U.\ 1955\ ]$$

[Rule to write down the results for the section of the paraboloid $by^2+cz^2=2x$ by the plane $lx+my+nz=p$ from the corresponding results for the section of the paraboloid $ax^2+by^2=2z$ by the plane $lx+my+nz=p$.

In the corresponding result for the section of the paraboloid $ax^2+by^2=2z$ *by the plane* $lx+my+nz=p$, *change* a, b, c *cyclically,*

i.e., change a to b, b to c, c to a ; x, y, z cyclically, i.e., change x to y, y to z, z to x ; and l, m, n cyclically, i.e., change l to m, m to n, n to l.

Example. The area of the section of the paraboloid $ax^2+by^2=2z$ by the plane $lx+my+nz=p$,

$$=\frac{\pi}{n^3}\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{ab}}\left(\frac{l^2}{a}+\frac{m^2}{b}+2np\right). \text{ [Art. 148, Cor. 1]}$$

Changing a, b, c cyclically, x,y,z cyclically, and l,m,n cyclically, we get :

The area of the section of the paraboloid $by^2+cz^2=2x$ by the plane $my+nz+lx=p$, *i.e.*, by the plane $lx+my+nz=p$,

$$=\frac{\pi}{l^3}\frac{\sqrt{m^2+n^2+l^2}}{\sqrt{bc}}\left(\frac{m^2}{b}+\frac{n^2}{c}+2lp\right).\Big]$$

18. Show that the section of $\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=\dfrac{2x}{a}$ by the plane $lx+my+nz=0$ is a rectangular hyperbola, if $(b^2-c^2)l^2+m^2b^2-n^2c^2=0$. [*Ag. U. 1955*]

19. Show that all plane sections of $\dfrac{x^2}{a}-\dfrac{y^2}{b}=z$ which are rectangular hyperbolas, and which pass through the point (α, β, γ), touch the cone $\dfrac{(x-\alpha)^2}{a}-\dfrac{(y-\beta)^2}{b}+\dfrac{(z-\gamma)^2}{a-b}=0$. [*D.U.M. 1947*]

20. If p_1, p_2, p_3 be the lengths of the perpendiculars from the extremities of three conjugate semi-diameters on one central circular section of the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, show that

$$p_1^2+p_2^2+p_3^2=\frac{a^2c^2}{b^2}. \quad \text{[P.U.H. 1957]}$$

****21. The normals to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ at all points of a central circular section are parallel to a plane that makes an angle $\cos^{-1}\dfrac{ac}{b\sqrt{a^2-b^2+c^2}}$ with the section.** [*D. U. 1949*]

[The equation of the ellipsoid is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ ... (1)

Let the equation of the plane of a central circular section be

$$\frac{x}{a}\sqrt{a^2-b^2}+\frac{z}{c}\sqrt{b^2-c^2}=0 \text{ ... (2)}$$

Let (x_1, y_1, z_1) be *any* pt. on this circular section.

Then $\dfrac{x_1}{a}\sqrt{a^2-b^2}+\dfrac{z_1}{c}\sqrt{b^2-c^2}=0$ ... (3)

The equations of the normal at (x_1, y_1, z_1) to the ellipsoid (1) are

$$\frac{x-x_1}{\frac{x_1}{a^2}} = \frac{y-y_1}{\frac{y_1}{b^2}} = \frac{z-z_1}{\frac{z_1}{c^2}}.$$

If it is $\parallel$ to the plane $\quad lx + my + nz = 0 \quad \dots (4)$

then $$l\frac{x_1}{a^2} + m\frac{y_1}{b^2} + n\frac{z_1}{c^2} = 0 \quad \dots (5)$$

Comparing coeffs. of x_1, y_1, z_1 in (5) and (3),

$$\frac{l}{a\sqrt{a^2-b^2}} = \frac{m}{0} = \frac{n}{-c\sqrt{b^2-c^2}}.$$

Substituting these values of l, m, n in (4),

$$a\sqrt{a^2-b^2}\,x + c\sqrt{b^2-c^2}\,z = 0 \quad \dots (6)$$

which is the equation of the plane to which the normals at all pts. of the section (2) are $\parallel$.

Now find the angle between the planes (2) and (6).]

22. Prove that points on an ellipsoid, which are such that the product of their distances from the two central circular sections is constant, lie on the curve of intersection of the ellipsoid and a sphere.

23. If the radius of the sphere which passes through two circular sections of an ellipsoid is equal to its mean radius, the distances of the planes from the centre are in a constant ratio.

24. Prove that the radius of a circular section of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

at a distance p from the centre is $b\sqrt{(1-p^2b^2/a^2c^2)}$. [*P.U.H. 1957*]

25. Circular sections of a cone. Show that the circular sections of the cone $ax^2 + by^2 + cz^2 = 0$, $a > b > c$, are given by $x\sqrt{a-b} \pm z\sqrt{b-c} = \lambda$, for all values of λ, and that these also give the circular sections of the cone $(a-\mu)x^2 + (b-\mu)y^2 + (c-\mu)z^2 = 0$.

26. If the section of the cone whose vertex is $P(\alpha, \beta, \gamma)$, and base the conic $ax^2 + by^2 = 1$, $z = 0$, by the plane $x = 0$ is a circle, prove that P lies on the conic $ax^2 - bz^2 = 1$, $y = 0$, and the section of the cone by the plane $(a-b)\gamma x - 2a\alpha z = 0$ is also a circle.

[It will be found that the equation of the cone is

$$a(\alpha z - \gamma x)^2 + b(\beta z - \gamma y)^2 - (z-\gamma)^2 = 0 \quad \dots (1)$$

Further it will be found that $\beta = 0$, $a\alpha^2 - b\gamma^2 = 1 \quad \dots (2)$

(To prove that the section of the cone (1) by the plane $(a-b)\gamma x - 2a\alpha z = 0$ is also a circle.)

The equation of *any* conicoid thro' the pts. of intersection of the cone (1) and the pair of planes $x[(a-b)\gamma x-2a\alpha z]=0$, is

$$a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2-(z-\gamma)^2+kx[(a-b)\gamma x-2a\alpha z]=0.$$

It is a sphere if

$$\text{coeff. of } x^2=\text{coeff. of } y^2=\text{coeff. of } z^2\,;$$

$$\text{coeff. of } yz=0,\ \text{coeff. of } zx=0,\ \text{coeff. of } xy=0.$$

Use (2).]

27. Show that if the squares of the axes of an ellipsoid are in arithmetical progression the umbilics lie on the central circular sections ; if they are in geometrical progression the tangent planes at the umbilics touch the sphere through the central circular sections ; if they are in harmonical progression the circular sections are perpendicular.

CHAPTER XIII

GENERATING LINES OF A RULED CONICOID

SECTION I

RULED CENTRAL CONICOID

155. Ruled surface. Def. A **ruled surface** is a surface generated by a moving straight line.

The straight line in any position is called a **generating line** or a **generator.**

Thus a cone and a cylinder are *ruled surfaces.*

[Defs. (Arts. 85, 104)]

156. To prove that a hyperboloid of one sheet is a ruled surface.

Let the equation of the hyperboloid of one sheet be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Transposing, $\qquad \dfrac{x^2}{a^2}-\dfrac{z^2}{c^2}=1-\dfrac{y^2}{b^2}.$

Factorising,

$$\left(\frac{x}{a}+\frac{z}{c}\right)\left(\frac{x}{a}-\frac{z}{c}\right)=\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right)$$

$$=\lambda\left(1+\frac{y}{b}\right)\cdot\frac{1}{\lambda}\left(1-\frac{y}{b}\right)\quad\ldots(i)$$

or, interchanging the factors on the L.H.S., and changing λ to μ on the R.H.S.,

$$\left(\frac{x}{a}-\frac{z}{c}\right)\left(\frac{x}{a}+\frac{z}{c}\right)=\mu\left(1+\frac{y}{b}\right)\cdot\frac{1}{\mu}\left(1-\frac{y}{b}\right)\quad\ldots(ii)$$

From (i) and (ii), the hyperboloid is generated by the variable st. lines

(1) $\dfrac{x}{a}+\dfrac{z}{c}=\lambda\left(1+\dfrac{y}{b}\right),\ \dfrac{x}{a}-\dfrac{z}{c}=\dfrac{1}{\lambda}\left(1-\dfrac{y}{b}\right);$

and (2) $\dfrac{x}{a}-\dfrac{z}{c}=\mu\left(1+\dfrac{y}{b}\right),\ \dfrac{x}{a}+\dfrac{z}{c}=\dfrac{1}{\mu}\left(1-\dfrac{y}{b}\right);$

where λ and μ are variable parameters.

As λ varies, taking in turn all real values, the line (1) moves so as to generate the hyperboloid completely.

Similarly as μ varies, taking in turn all real values, the line (2) moves so as to generate the hyperboloid completely.

$\therefore$ the hyperboloid of one sheet is a ruled surface.

[Def. (Art. 155)]

Cor. *To prove that the two systems of generating lines are distinct.*

$\because$ it is impossible to give values to λ and μ so that the equations of the line (1) may become the same as those of the line (2)

$\therefore$ no member of the system of lines (1) coincides with any member of the system of lines (2),

i.e., the two systems of generating lines are distinct.

Note. λ- and μ- systems of generators. Def.

The system of st. lines

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\ \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right) \quad ...(1)$$

where λ is a variable parameter, is called the **λ-system,**
and the system of st. lines

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\ \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right) \quad ...(2)$$

where μ is a variable parameter, is called the **μ-system.**

[**Rule to write down the equations of a generator of the μ-system, of the hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ from the equations of a generator of the λ-system :**

In the equations of a generator of the λ-system, change c to $-c$ and λ to μ.

For, changing c to $-c$ and λ to μ in (1), we get (2).]

EXAMPLES

1. *CP, CD are any two conjugate diameters of the ellipse*

$$x^2/a^2+y^2/b^2=1,\ z=c,$$

and C'P', C'D' are the two conjugate diameters of the ellipse

$$x^2/a^2+y^2/b^2=1,\ z=-c,$$

drawn in the same directions as CP, CD. Prove that the hyperboloid $2x^2/a^2+2y^2/b^2-z^2/c^2=1$ *is generated by either PD' or P'D.*

[*Ag. U. 1950*]

2. **Show that there are two systems of generating lines on a hyperboloid of one sheet.** [*P. U. H. 1956*]

157. *If three points of a straight line lie on a conicoid, the straight line lies wholly on the conicoid.*

Let the equations of the line be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(1)

and the equation of the conicoid be $ax^2+by^2+cz^2=1$ ...(2)

Any pt. on the line (1) is $(x_1+lr,\ y_1+mr,\ z_1+nr)$.

If it lies on the conicoid (2), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1$$

or $\quad r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(3)

which is a quadratic in r.

If three pts. of the line lie on the conicoid, the quadratic (3) is satisfied by three, *i.e.*, more than two values of r,

$\therefore$ it is an identity, [From Elementary Algebra]

i.e., it is satisfied by *all* values of r.

$\therefore$ *all* pts. of the line lie on the conicoid.

158. To find the conditions that a given straight line should be a generator of a given conicoid.

Let the equations of the line be $\dfrac{x-x_1}{l}=\dfrac{y-y_1}{m}=\dfrac{z-z_1}{n}$ ...(1)

and the equation of the conicoid be $\quad ax^2+by^2+cz^2=1$ ...(2)

Any pt. on the line (1) is $(x_1+lr,\ y_1+mr,\ z_1+nr)$.

If it lies on the conicoid (2), then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2=1$$

or $\quad r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(3)

which is a quadratic in r.

If the line is a generator of the conicoid, all pts. of the line lie on the conicoid.

$\therefore$ the quadratic (3) is satisfied by *all* values of r,

$\therefore$ coeff. of $r^2=0$, coeff. of $r=0$, constant term$=0$,

[$\because$ $r^2(0)+r(0)+0=0$ for *all* values of r]

i.e.,
$$al^2+bm^2+cn^2=0 \quad ...(4)$$
$$alx_1+bmy_1+cnz_1=0 \quad ...(5)$$
$$ax_1^2+by_1^2+cz_1^2-1=0 \quad ...(6)$$

(4), (5) and (6) are the required conditions.

Cor. 1. *Any point of a generating line of a conicoid lies on the conicoid.*

[$\because$ the pt. (x_1, y_1, z_1) of the line (1) lies on the conicoid (2) (From (6))]

Cor. 2. *A generating line of a conicoid lies in the tangent plane at any point on the generating line.*

For the equation of the tangent plane at the pt. (x_1, y_1, z_1) of the generating line (1) to the conicoid (2), is

$$axx_1+byy_1+czz_1=1 \quad ...(7)$$

The generating line (1) lies in the tangent plane (7) if

$alx_1+bmy_1+cnz_1=0$, and $ax_1^2+by_1^2+cz_1^2=1$, [Art. 43, (c)]

each of which is true. [From (5) and (6)]

Cor. 3. *Through any point (x_1, y_1, z_1) of a conicoid there pass two generating lines of the conicoid.*

For, from (4) and (5) (Art. 158), we get two sets of values of $l : m : n$. Substituting these sets of values, one by one, in (1), we get the equations of the two generating lines thro' (x_1, y_1, z_1).

Cor. 4. *To find the direction-ratios of the two generating lines of a conicoid through any point (x_1, y_1, z_1) of the conicoid.*

Identically,

$$(al^2+bm^2)(ax_1^2+by_1^2)-(alx_1+bmy_1)^2$$
$$=a^2l^2x_1^2+abl^2y_1^2+abm^2x_1^2+b^2m^2y_1^2-(a^2l^2x_1^2+2ablmx_1y_1+b^2m^2y_1^2)$$
$$=ab(l^2y_1^2-2lmx_1y_1+m^2x_1^2) = ab(ly_1-mx_1)^2$$

or
$$ab(ly_1-mx_1)^2 = (al^2+bm^2)(ax_1^2+by_1^2)-(alx_1+bmy_1)^2$$
[Substitute from (4), (6) and (5)]
$$= (-cn^2)(1-cz_1^2)-c^2n^2z_1^2 = -cn^2.$$

Dividing thro' out by ab, [*Note this step*]

$$(ly_1-mx_1)^2 = -\frac{c}{ab}n^2$$

or
$$ly_1-mx_1 = \pm\sqrt{-\frac{c}{ab}}\,n$$

or
$$ly_1-mx_1 \pm\sqrt{-\frac{c}{ab}}\,n = 0 \quad \dots (8)$$

Solving (5) and (8) for l, m, n (by cross-multiplication),

$$\frac{l}{\pm by_1\sqrt{-\frac{c}{ab}}+cz_1x_1} = \frac{m}{cy_1z_1 \mp ax_1\sqrt{-\frac{c}{ab}}} = \frac{n}{-ax_1^2-by_1^2}.$$

Cor. 5. *The only ruled central conicoid is a hyperboloid of one sheet.*

For, from (8), the values of $l : m : n$ are real only if $-\frac{c}{ab}$ is +ve

i.e., if $\frac{c}{ab}$ is −ve [Multiply the numerator and denominator by c]

i.e., if $\frac{c^2}{abc}$ is −ve [But c^2 is always + ve]

i.e., if abc is −ve, *i.e.,* if two of the three quantities a, b, c are +ve and one is −ve

[∵ all the three quantities a, b, c cannot be −ve, since, if a, b, c are all −ve, then from (2), the conicoid is an imaginary ellipsoid (Note, Art. 109, (*a*))]

i.e., if from (2), the conicoid is a hyperboloid of one sheet (Art. 109, (*b*)).

Cor. 6. *The lines through the centre of a conicoid parallel to the generating lines generate the asymptotic cone.*

For, the equations of the line thro' the centre (0, 0, 0) to the generating line (1) are $\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n}$

i.e.,
$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \dots (9)$$

Eliminating l, m, n from (9) and (4) [by substituting their values from (9) in (4)], the locus of the lines thro' the centre to the generating lines is $ax^2+by^2+cz^2=0$,
which is the asymptotic cone (Ex. 10, Art. 135).

EXAMPLES

1. Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ which pass through the point (2, 3, −4). [*P. U. H. 1960*]

The equation of the hyperboloid is $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1 \quad \dots (1)$

The equations of *any* line thro' (2, 3, −4) are
$$\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n} \quad \dots (2)$$

Any pt. on this line is $(2+lr,\ 3+mr,\ -4+nr)$.

If it lies on the hyperboloid (1), then
$$\frac{1}{4}(2+lr)^2+\frac{1}{9}(3+mr)^2-\frac{1}{16}(-4+nr)^2=1$$

or $$r^2\left(\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right)+2r\left(\frac{2l}{4}+\frac{3m}{9}+\frac{4n}{16}\right)$$
$$+\frac{1}{4}(4)+\frac{1}{9}(9)-\frac{1}{16}(16)-1=0$$

or $$r^2\left(\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right)+2r\left(\frac{l}{2}+\frac{m}{3}+\frac{n}{4}\right)=0 \quad \dots (3)$$

which is a quadratic in r.

If the line is a generating line of the hyperboloid, all pts. of the line lie on the hyperboloid.

∴ the quadratic (3) is satisfied by *all* values of r

∴ coeff. of $r^2=0$, coeff. of $r=0$,

i.e.,
$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \quad \dots (4)$$
$$\frac{l}{2}+\frac{m}{3}+\frac{n}{4}=0 \quad \dots (5)$$

Substituting the value of $n \left(=-2l-\frac{4}{3}m\right)$ from (5) in (4),

$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{1}{16}(2l+\frac{4}{3}m)^2=0,$$

or $\frac{l^2}{4}+\frac{m^2}{9}-\frac{1}{16}(4l^2+\frac{16}{3}lm+\frac{16}{9}m^2)=0$

or $-\frac{1}{3}lm=0$, or $lm=0$

$\therefore$ *either* $l=0$, *or* $m=0$,

i.e., *either* $1.l+0.m+0.n=0$ ... (6)

or $0.l+1.m+0.n=0$ ... (7)

Solving (5) and (6) for l, m, n (by cross-multiplication),

$$\frac{l}{0}=\frac{m}{\frac{1}{4}}=\frac{n}{-\frac{1}{3}}.$$

or (multiplying the denoms. by 12), $\frac{l}{0}=\frac{m}{3}=\frac{n}{-4}$ ... (8)

Solving (5) and (7) for l, m, n (by cross-multiplication),

$$\frac{l}{-\frac{1}{4}}=\frac{m}{0}=\frac{n}{\frac{1}{2}}$$

or (multiplying the denoms. by 4), $\frac{l}{-1}=\frac{m}{0}=\frac{n}{2}$ ... (9)

Substituting these sets of values of l, m, n from (8) and (9), one by one, in (2),

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4},$$

$$\frac{x-2}{-1}=\frac{y-3}{0}=\frac{z+4}{2},$$

which are the required equations of the generating lines.

2. Find the equations of the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ which pass through the point $(-2, 1, -\frac{4}{3})$.

3. Find the equations of the generating lines of the hyperboloid $2yz+zx-3xy+6=0$ which pass through the point $(0, -1, 3)$.

4. Find the equations to the generators of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

which pass through the point $(a\cos\alpha, b\sin\alpha, 0)$. [*P.U.H. 1959*]

The equation of the hyperboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ ... (1)

The equations of *any* line thro' $(a\cos\alpha, b\sin\alpha, 0)$ are

$$\frac{x-a\cos\alpha}{l}=\frac{y-b\sin\alpha}{m}=\frac{z}{n} \quad ... (2)$$

Any pt. on this line is $(a\cos\alpha+lr, b\sin\alpha+mr, nr)$.

If it lies on the hyperboloid (1), then

$$\frac{1}{a^2}(a \cos \alpha+lr)^2+\frac{1}{b^2}(b \sin \alpha+mr)^2-\frac{1}{c^2}\, n^2r^2 = 1$$

or $r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)+2r\left(\frac{l \cos \alpha}{a}+\frac{m \sin \alpha}{b}\right)+\cos^2 \alpha+\sin^2 \alpha-1$
$=0$

or $r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)+2r\left(\frac{l \cos \alpha}{a}+\frac{m \sin \alpha}{b}\right)=0$ ... (3)

which is a quadratic in r.

If the line is a generator of the hyperboloid, all pts. of the line lie on the hyperboloid.

$\therefore$ the quadratic (3) is satisfied by *all* values of r

$\therefore$ coeff. of $r^2=0$, coeff. of $r=0$,

i.e., $\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0$ ...(4)

$\frac{l \cos \alpha}{a}+\frac{m \sin \alpha}{b}=0$, or $\frac{l \cos \alpha}{a}=-\frac{m \sin \alpha}{b}$

or $\dfrac{\frac{l}{a}}{\sin \alpha}=\dfrac{\frac{m}{b}}{-\cos \alpha}$ [*Note this step*]

$$=\frac{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}}{\sqrt{\sin^2 \alpha+\cos^2 \alpha}}$$ [But, from (4), $\frac{l^2}{a^2}+\frac{m^2}{b^2}=\frac{n^2}{c^2}$

$\therefore \sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}=\pm\frac{n}{c}$]

$$=\frac{\pm\frac{n}{c}}{1}$$

or $\frac{l}{a \sin \alpha}=\frac{m}{-b \cos \alpha}=\frac{n}{\pm c}$.

Substituting these sets of values of l, m, n in (2),

$$\mathbf{\frac{x-a \cos \alpha}{a \sin \alpha}=\frac{y-b \sin \alpha}{-b \cos \alpha}=\frac{z}{\pm c}},$$

which are the required equations of the generators.

[**Aid to memory.** The generator passes thro' $(a \cos \alpha, b \sin \alpha, 0)$,

$\therefore$ the numerators in its equations are

$x-a \cos \alpha,\ y-b \sin \alpha,\ z.$

The denominators of the first two members are the partial differential coefficients, w.r.t. α, of the respective numerators.

Thus $\frac{\partial}{\partial \alpha}(x-a \cos \alpha)=a \sin \alpha$, $\frac{\partial}{\partial \alpha}(y-b \sin \alpha)=-b \cos \alpha$; and the denominator of the third member is $\pm c$.

Cor. Equations of any generator. *The equations of* **any** *generator of one system, of the hyperboloid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ *are*

$$\frac{\mathbf{x-a\cos\alpha}}{\mathbf{a\sin\alpha}}=\frac{\mathbf{y-b\sin\alpha}}{\mathbf{-b\cos\alpha}}=\frac{\mathbf{z}}{\mathbf{c}}.$$

For *any* generator of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ lies wholly on the hyperboloid

$\therefore$ it meets the principal elliptic section $z=0$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, in a pt. $(a\cos\alpha, b\sin\alpha, 0)$ (say).

$\therefore$ its equations are $\frac{\mathbf{x-a\cos\alpha}}{\mathbf{a\sin\alpha}}=\frac{\mathbf{y-b\sin\alpha}}{\mathbf{-b\cos\alpha}}=\frac{\mathbf{z}}{\mathbf{c}}$.

The equations of **any** *generator of the other system are*

$$\frac{\mathbf{x-a\cos\alpha}}{\mathbf{a\sin\alpha}}=\frac{\mathbf{y-b\sin\alpha}}{\mathbf{-b\cos\alpha}}=\frac{\mathbf{z}}{\mathbf{-c}}.$$

Note 1. As α varies from 0 to 2π, each of the two above generators moves so as to generate the hyperboloid completely.

Note 2. Comparison of the different forms of the equations of generators of opposite systems of the hyperboloid

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}-\frac{\mathbf{z}^2}{\mathbf{c}^2}=\mathbf{1}.$$

The equations of generators of opposite systems, in the symmetrical form,

$$\frac{\mathbf{x-a\cos\alpha}}{\mathbf{a\sin\alpha}}=\frac{\mathbf{y-b\sin\alpha}}{\mathbf{-b\cos\alpha}}=\frac{\mathbf{z}}{\mathbf{c}};$$

and
$$\frac{\mathbf{x-a\cos\alpha}}{\mathbf{a\sin\alpha}}=\frac{\mathbf{y-b\sin\alpha}}{\mathbf{-b\cos\alpha}}=\frac{\mathbf{z}}{\mathbf{-c}}$$

are sometimes more useful than

the equations $\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right), \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right);$

and $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right).$

[Note, Art. 156]

5. Prove that the equations of a generating line through any point of the hyperboloid of one sheet $\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}-\frac{\mathbf{z}^2}{\mathbf{c}^2}=\mathbf{1}$ **can be written in the form** $\frac{\mathbf{x-a\cos\alpha}}{\mathbf{a\sin\alpha}}=\frac{\mathbf{y-b\sin\alpha}}{\mathbf{-b\cos\alpha}}=\frac{\mathbf{z}}{\mathbf{\pm c}}.$

[*Ag. U. 1949*]

Two important properties of the λ- and μ- systems of generators.

159. (*a*) **To prove that no two generators of the same system, of a hyperboloid of one sheet, intersect.**

Let the equation of the hyperboloid of one sheet be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Let the equations of *any* two generators of the λ-system be

$$\left.\begin{aligned}\frac{x}{a}+\frac{z}{c}&=\lambda\left(1+\frac{y}{b}\right) \quad \dots (i),\\ \frac{x}{a}-\frac{z}{c}&=\frac{1}{\lambda}\left(1-\frac{y}{b}\right) \quad \dots (ii)\end{aligned}\right\} \dots (1)$$

and
$$\left.\begin{aligned}\frac{x}{a}+\frac{z}{c}&=\lambda'\left(1+\frac{y}{b}\right) \quad \dots (iii),\\ \frac{x}{a}-\frac{z}{c}&=\frac{1}{\lambda'}\left(1-\frac{y}{b}\right) \quad \dots (iv)\end{aligned}\right\} \dots (2) \text{ [Note, Art. 156]}$$

[To solve (*i*), (*ii*), (*iii*) for x, y, z.]

From (*i*) and (*iii*), equating R. H. S.'s,

$$\lambda\left(1+\frac{y}{b}\right)=\lambda'\left(1+\frac{y}{b}\right)$$

$\therefore$ *either* $1+\dfrac{y}{b}=0$

or, dividing both sides by $1+\dfrac{y}{b}$,

$\lambda=\lambda'$, which is impossible.

[$\because$ if $\lambda=\lambda'$, the two generators (1) and (2) coincide]

$\therefore$ $1+\dfrac{y}{b}=0$, or $\dfrac{y}{b}=-1.$

Substituting this value of $\dfrac{y}{b}$ in (*i*) and (*ii*),

$$\frac{x}{a}+\frac{z}{c}=0$$

$$\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}.2.$$

Adding and subtracting,

$$\frac{2x}{a}=\frac{2}{\lambda},\ \frac{2z}{c}=-\frac{2}{\lambda}$$

$$\therefore \qquad \frac{x}{a}=\frac{1}{\lambda},\ \frac{z}{c}=-\frac{1}{\lambda}.$$

Substituting these values of $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ in (*iv*), we get

$$\frac{1}{\lambda}+\frac{1}{\lambda}=\frac{1}{\lambda'}(1+1), \text{ or } \frac{2}{\lambda}=\frac{2}{\lambda'}, \text{ or } \lambda=\lambda',$$

which is impossible. [Proved above]

∴ the two generators (1) and (2) do not intersect.

Similarly no two generators of the μ-system intersect.

159. (*b*) **To prove that any generator of the λ-system, of a hyperboloid of one sheet, intersects any generator of the μ-system.**

Let the equation of the hyperboloid of one sheet be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Let the equations of *any* generator of the λ-system be

$$\left.\begin{aligned} \frac{x}{a}+\frac{z}{c} &= \lambda\left(1+\frac{y}{b}\right) \;\ldots\; (i), \\ \frac{x}{a}-\frac{z}{c} &= \frac{1}{\lambda}\left(1-\frac{y}{b}\right) \;\ldots\; (ii) \end{aligned}\right\} \ldots (1)$$

and the equations of *any* generator of the μ-system be

$$\left.\begin{aligned} \frac{x}{a}-\frac{z}{c} &= \mu\left(1+\frac{y}{b}\right) \;\ldots\; (iii), \\ \frac{x}{a}+\frac{z}{c} &= \frac{1}{\mu}\left(1-\frac{y}{b}\right) \;\ldots\; (iv) \end{aligned}\right\} \ldots (2) \text{ [Note, Art. 156]}$$

[To solve (*i*), (*ii*), (*iii*) for *x, y, z.*]

From (*ii*) and (*iii*), equating R. H. S.'s,

$$\frac{1}{\lambda}\left(1-\frac{y}{b}\right)=\mu\left(1+\frac{y}{b}\right), \text{ or } \frac{y}{b}\left(\frac{1}{\lambda}+\mu\right)=\frac{1}{\lambda}-\mu$$

or $$\frac{y}{b}\left(\frac{1+\lambda\mu}{\lambda}\right)=\frac{1-\lambda\mu}{\lambda}, \quad \therefore \frac{y}{b}=\frac{1-\lambda\mu}{1+\lambda\mu}.$$

Substituting this value of $\frac{y}{b}$ in (*i*) and (*ii*),

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{1-\lambda\mu}{1+\lambda\mu}\right)=\frac{2\lambda}{1+\lambda\mu},$$

$$\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{1-\lambda\mu}{1+\lambda\mu}\right)=\frac{1}{\lambda}\left(\frac{2\lambda\mu}{1+\lambda\mu}\right)=\frac{2\mu}{1+\lambda\mu}.$$

Adding and subtracting,

$$\frac{2x}{a}=\frac{2(\lambda+\mu)}{1+\lambda\mu}, \quad \frac{2z}{c}=\frac{2(\lambda-\mu)}{1+\lambda\mu}$$

$$\therefore \quad \frac{x}{a}=\frac{\lambda+\mu}{1+\lambda\mu}, \quad \frac{z}{c}=\frac{\lambda-\mu}{1+\lambda\mu}.$$

Substituting these values of $\frac{x}{a}, \frac{y}{b}, \frac{z}{c}$ in (*iv*), we get

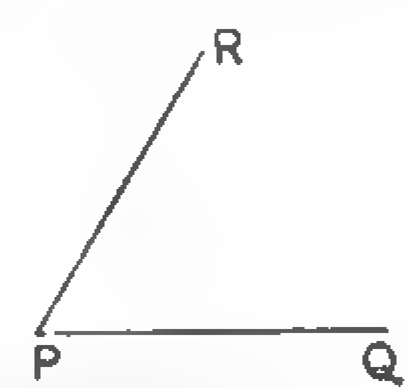
$$\frac{\lambda+\mu}{1+\lambda\mu}+\frac{\lambda-\mu}{1+\lambda\mu}=\frac{1}{\mu}\left(1-\frac{1-\lambda\mu}{1+\lambda\mu}\right)$$

or $$\frac{2\lambda}{1+\lambda\mu}=\frac{1}{\mu}\cdot\frac{2\lambda\mu}{1+\lambda\mu}, \text{ or } \frac{2\lambda}{1+\lambda\mu}=\frac{2\lambda}{1+\lambda\mu}, \text{ which is true.}$$

$\therefore$ the two generators (1) and (2) intersect.

Cor. Co-ordinates of the point of intersection. The co-ordinates of the point of intersection of a generator of the λ-system and a generator of the μ-system, of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, are given by $\frac{x}{a}=\frac{\lambda+\mu}{1+\lambda\mu}, \frac{y}{b}=\frac{1-\lambda\mu}{1+\lambda\mu}, \frac{z}{c}=\frac{\lambda-\mu}{1+\lambda\mu}$.

160. (*a*) *Section of a hyperboloid of one sheet by the tangent plane at a point.* **The section of a hyperboloid of one sheet by the tangent plane at a point is the two generating lines through the point.**



Let P be a pt. on the hyperboloid of one sheet.

$\because$ the hyperboloid is generated completely by either of the two systems of st. lines [Art. 156]

$\therefore$ thro' P there pass two generating lines PQ, PR (say), one of each system.

$\because$ PQ meets the hyperboloid at P in, at least, two coincident pts.

$\therefore$ PQ is a tangent to the hyperboloid at P.

Similarly PR is a tangent to the hyperboloid at P.

$\therefore$ PQ, PR lie in the tangent plane at P ...(1)

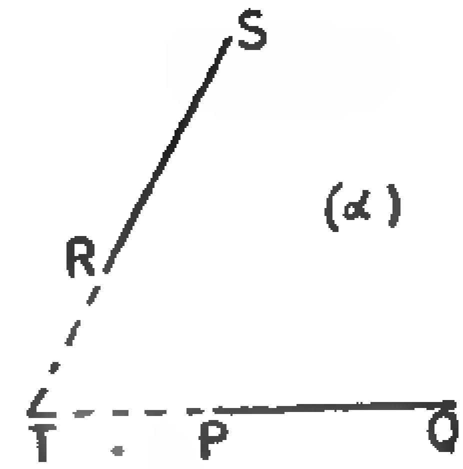
Now the section of the hyperboloid by any plane is a conic [Art. 114, Cor. 2]

$\therefore$ the section of the hyperboloid by the tangent plane at P is the two generating lines thro' P.

Note. *The plane through two intersecting generators of a hyperboloid of one sheet is the tangent plane at their point of intersection.*

For PQ, PR lie in the tangent plane at P. [Proved in (1)]

160. (*b*) **Any plane through a generating line.** *Any plane through a generating line of a hyperboloid of one sheet is a tangent plane.*



Let α be *any* plane thro' a generating line PQ of the hyperboloid of one sheet.

The section of the hyperboloid by the plane α is a conic of which the st. line PQ is a part

$\therefore$ the other part is also a st. line RS (say).

Let PQ, RS, lying in the plane α, meet in T.

$\because$ PQ meets the hyperboloid at T in, at least, two coincident pts.

$\therefore$ PQ is a tangent to the hyperboloid at T.

Similarly RS is a tangent to the hyperboloid at T.

$\therefore$ PQ, RS lie in the tangent plane at T,

i.e., the plane α is a tangent plane to the hyperboloid at T.

EXAMPLES

1. If (x_1, y_1, z_1) is a point on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,$$

interpret the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}\right)^2.$

2. Prove that the line

$$lx+my+nz+p=0,\ l'x+m'y+n'z+p'=0$$

is a generator of the hyperboloid $x^2/a+y^2/b+z^2/c=1$ if $al^2+bm^2+cn^2=p^2, al'^2+bm'^2+cn'^2=p'^2$, and $all'+bmm'+cnn'=pp'$.

[*P. U. H. 1956*]

161. *Locus of the points of intersection of perpendicular generators.* **To find the locus of the points of intersection of perpendicular generators of a hyperboloid of one sheet.**

Let the equation of the hyperboloid of one sheet be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad ...(1)$$

Let (x_1, y_1, z_1) be the pt. of intersection of two $\perp$ generators.

Let the equations of a generator thro' (x_1, y_1, z_1) be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(2)$$

Any pt. on this line is (x_1+lr, y_1+mr, z_1+nr).

$\because$ it lies on the hyperboloid (1)

$$\therefore \quad \frac{1}{a^2}(x_1+lr)^2+\frac{1}{b^2}(y_1+mr)^2-\frac{1}{c^2}(z_1+nr)^2=1$$

or $$r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)+2r\left(\frac{lx_1}{a^2}+\frac{my_1}{b^2}-\frac{nz_1}{c^2}\right)$$
$$+\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}-1\right)=0 \quad ...(3)$$

which is a quadratic in r.

$\because$ the line is a generator of the hyperboloid, all pts. of the line lie on the hyperboloid.

$\therefore$ the quadratic (3) is satisfied by *all* values of r

$\therefore$ coeff. of $r^2=0$, coeff. of $r=0$, constant term $=0$.

i.e.,
$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0 \quad ...(4)$$
$$\frac{lx_1}{a^2}+\frac{my_1}{b^2}-\frac{nz_1}{c^2}=0 \quad ...(5)$$
$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}-1=0 \quad ...(6)$$

$\because$ the two generators whose direction-ratios (l, m, n) are given by (4) and (5), are $\perp$

$$\therefore \quad \frac{x_1^2}{a^4}\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+\frac{y_1^2}{b^4}\left(-\frac{1}{c^2}+\frac{1}{a^2}\right)+\frac{z_1^2}{c^4}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=0.$$

[$u^2(b+c)+v^2(c+a)+w^2(a+b)=0$ (Note, Ex. 5, Art. 116)]

or $\quad \dfrac{x_1^2}{a^4b^2c^2}(c^2-b^2)+\dfrac{y_1^2}{b^4c^2a^2}(c^2-a^2)+\dfrac{z_1^2}{c^4a^2b^2}(a^2+b^2)=0$

Multiplying thro' out by $a^2b^2c^2$*, [*Note this step*]

$$\frac{x_1^2}{a^2}(c^2-b^2)+\frac{y_1^2}{b^2}(c^2-a^2)+\frac{z_1^2}{c^2}(a^2+b^2)=0$$

or, *multiplying thro' out by* -1†, [*Note this step*]

$$\frac{x_1^2}{a^2}(b^2-c^2)+\frac{y_1^2}{b^2}(a^2-c^2)-\frac{z_1^2}{c^2}(a^2+b^2)=0$$

or $\quad \dfrac{x_1^2}{a^2}(a^2+b^2-c^2)-x_1^2+\dfrac{y_1^2}{b^2}(a^2+b^2-c^2)-y_1^2-\dfrac{z_1^2}{c^2}(a^2+b^2-c^2)-z_1^2=0$

or $\quad (a^2+b^2-c^2)\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-\dfrac{z_1^2}{c^2}\right)-x_1^2-y_1^2-z_1^2=0$

$\left[\text{But } \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-\dfrac{z_1^2}{c^2}=1 \text{ (From (6))}\right]$

$\therefore$ $x_1^2+y_1^2+z_1^2=a^2+b^2-c^2$.

$\therefore$ (x_1, y_1, z_1) lies on the director sphere $x^2+y^2+z^2=a^2+b^2-c^2$ of the hyperboloid (1). [As in Art. 117]

But (x_1, y_1, z_1) also lies on the hyperboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$

[From (6)]

*Why this step. To get $\dfrac{x_1^2}{a^2}, \dfrac{y_1^2}{b^2}, \dfrac{z_1^2}{c^2}$ instead of $\dfrac{x_1^2}{a^4}, \dfrac{y_1^2}{b^4}, \dfrac{z_1^2}{c^4}$, and to clear of other fractions.

†Why this step. To get $-c^2$ (as it occurs in the equation of the hyperboloid) instead of c^2 in the terms which follow.

$\therefore$ the locus of (x_1, y_1, z_1) is the curve of intersection of the hyperboloid and its director sphere.

162. Co-ordinates of any point on a hyperboloid of one sheet in terms of two variables. *The co-ordinates of any point on the hyperboloid of one sheet* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ *are*

$$(\mathbf{a \cos \theta \sec \phi,\ b \sin \theta \sec \phi,\ c \tan \phi}).$$

The equation of the hyperboloid of one sheet is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad ...(1)$$

Substituting the co-ordinates of the pt.

$$(a \cos \theta \sec \phi, b \sin \theta \sec \phi, c \tan \phi)$$

in (1), we get

$$\frac{a^2 \cos^2\theta \sec^2\phi}{a^2}+\frac{b^2 \sin^2\theta \sec^2\phi}{b^2}-\frac{c^2 \tan^2\phi}{c^2}=1$$

or $\quad \sec^2\phi\ (\cos^2\theta+\sin^2\theta)-\tan^2\phi=1$

or $\quad \sec^2\phi-\tan^2\phi=1 \qquad [\because \cos^2\theta+\sin^2\theta=1]$

which is true.

$\therefore$ the co-ordinates of any pt. on the hyperboloid (1) are

$$(\mathbf{a \cos \theta \sec \phi,\ b \sin \theta \sec \phi,\ c \tan \phi}).$$

Note. Point "θ, ϕ". Def. *The point whose co-ordinates are* (**a cos θ sec ϕ, b sin θ sec ϕ, c tan ϕ**) *is, for shortness, called the* **point "θ, ϕ".**

Cor. *For all points "θ, ϕ" on a generator of the λ-system, of the hyperboloid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, $\theta-\phi$ *is constant, and for all points on a generator of the μ-system, $\theta+\phi$ is constant.*

The equation of the hyperboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

Let the equations of a given generator of the λ-system be

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right), \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right). \text{ [Note, Art. 156]}$$

$\because$ it passes thro' the pt. "θ, ϕ",

i.e., thro' $(a \cos \theta \sec \phi, b \sin \theta \sec \phi, c \tan \phi)$, [Note, Art. 162]

$\therefore$ substituting in the first equation,

$$\frac{a \cos \theta \sec \phi}{a}+\frac{c \tan \phi}{c}=\lambda\left(1+\frac{b \sin \theta \sec \phi}{b}\right)$$

or $$\frac{\cos \theta}{\cos \phi}+\frac{\sin \phi}{\cos \phi}=\lambda\left(1+\frac{\sin \theta}{\cos \phi}\right)$$

or $$\cos \theta+\sin \phi=\lambda(\cos \phi+\sin \theta)$$

$$\therefore \quad \frac{\lambda}{1}=\frac{\cos\theta+\sin\phi}{\cos\phi+\sin\theta}$$

$\therefore$ by componendo and dividendo,

$$\frac{1+\lambda}{1-\lambda}=\frac{\cos\phi+\sin\theta+\cos\theta+\sin\phi}{\cos\phi+\sin\theta-\cos\theta-\sin\phi}=\frac{(\cos\phi+\cos\theta)+(\sin\theta+\sin\phi)}{(\cos\phi-\cos\theta)+(\sin\theta-\sin\phi)}$$

$$=\frac{2\cos\frac{\phi+\theta}{2}\cos\frac{\phi-\theta}{2}+2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}}{2\sin\frac{\phi+\theta}{2}\sin\frac{\theta-\phi}{2}+2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}} \qquad [\text{ Cancel } 2]$$

$$=\frac{\cos\frac{\theta-\phi}{2}\left(\cos\frac{\phi+\theta}{2}+\sin\frac{\theta+\phi}{2}\right)}{\sin\frac{\theta-\phi}{2}\left(\sin\frac{\phi+\theta}{2}+\cos\frac{\theta+\phi}{2}\right)}=\cot\frac{\theta-\phi}{2}$$

$\therefore \quad \cot\frac{\theta-\phi}{2}=\frac{1+\lambda}{1-\lambda}$, which is constant

$\therefore \quad \frac{\theta-\phi}{2}$ is constant,

$\therefore$ **$\theta-\phi$ is constant.**

Similarly for all pts. "θ, ϕ" on a generator of the μ-system, **$\theta+\phi$ is constant.**

EXAMPLES

****1. Find the equations to the generating lines through "θ, ϕ" of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.** [*Ag. U. 1951*]

The equation of the hyperboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

The equations of *any* generator of one system [thro' $(a\cos\alpha, b\sin\alpha, 0)$] are $\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}$. ...(1)

[Ex. 4, Cor., Art. 158]

If it passes thro' the pt. "θ, ϕ",

i.e., thro' $(a\cos\theta\sec\phi, b\sin\theta\sec\phi, c\tan\phi)$, then

$$\frac{a\cos\theta\sec\phi-a\cos\alpha}{a\sin\alpha}=\frac{b\sin\theta\sec\phi-b\sin\alpha}{-b\cos\alpha}=\frac{c\tan\phi}{c} \quad ...(2)$$

[To find the values of $\sin\alpha$, $\cos\alpha$ for the direction-ratios of the generator (1).]

From the first and third members of (2),

$$\frac{\cos\theta}{\cos\phi}-\cos\alpha=\sin\alpha\frac{\sin\phi}{\cos\phi} \quad ...(3)$$

and from the second and third members of (2),

$$\frac{\sin\theta}{\cos\phi}-\sin\alpha=-\cos\alpha\frac{\sin\phi}{\cos\phi} \quad ...(4)$$

From (3) and (4),

$\cos\theta-\cos\alpha\cos\phi=\sin\alpha\sin\phi,\ \sin\theta-\sin\alpha\cos\phi=-\cos\alpha\sin\phi$

or $\cos\alpha\cos\phi+\sin\alpha\sin\phi=\cos\theta,\ \sin\alpha\cos\phi-\cos\alpha\sin\phi=\sin\theta$

or $\cos(\alpha-\phi)=\cos\theta,\ \sin(\alpha-\phi)=\sin\theta$

or $\alpha-\phi=\theta,$

$\therefore\quad \alpha=\theta+\phi.$

Now from (1), the direction-ratios of the generator are

$$a\sin\alpha,\ -b\cos\alpha,\ c,$$

i.e., $a\sin(\theta+\phi),\ -b\cos(\theta+\phi),\ c.$

$\therefore$ the equations of the generator thro'

$$(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$$

are $$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta+\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta+\phi)}=\frac{z-c\tan\phi}{c}.$$

Similarly [considering the equations of *any* generator of the other system [thro'$(a\cos\alpha,\ b\sin\alpha,\ 0)$], *viz.*, $\left.\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{-c}\right]$, the equations of the other generator thro'

$$(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$$

are $$\frac{x-a\cos\theta\sec\phi}{a\sin(\theta-\phi)}=\frac{y-b\sin\theta\sec\phi}{-b\cos(\theta-\phi)}=\frac{z-c\tan\phi}{-c}.$$

2. If four generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ form a skew quadrilateral whose vertices are "θ_r, ϕ_r," $r=1,2,3,4$, prove that

$$\theta_1+\theta_3=\theta_2+\theta_4,\ \phi_1+\phi_3=\phi_2+\phi_4. \qquad \text{[Ag. U. 1956]}$$

163. (*a*) **Principal elliptic section. Def.** The plane $z=0$ meets the hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is an *ellipse in that plane*. It is called the **principal elliptic section** of the hyperboloid, and its equations are

$$\frac{\mathbf{x}^2}{\mathbf{a}^2}+\frac{\mathbf{y}^2}{\mathbf{b}^2}=\mathbf{1},\ \mathbf{z}=\mathbf{0}.$$

EXAMPLE

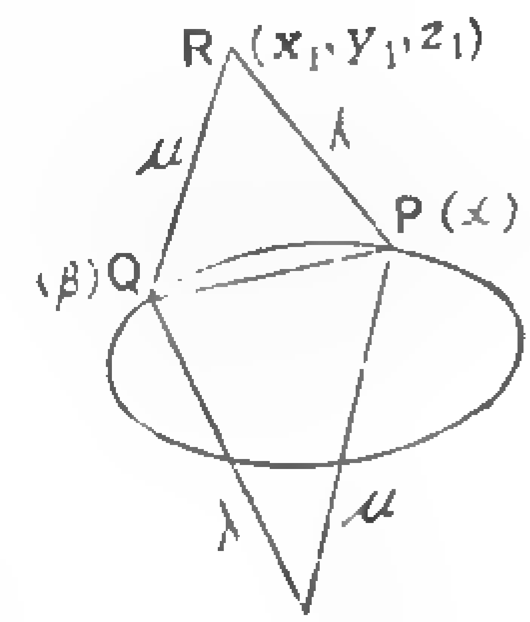
Prove that the generators of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ given by $\lambda=t,\ \mu=-\frac{1}{t}$ are parallel, and that they meet the principal elliptic section in the extremities of a diameter.

163. (*b*) **Point of intersection of two generators through two points on the principal elliptic section.** *If* α, β *are the eccentric angles of two points P, Q on the principal elliptic section of the*

hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, *then the co-ordinates of R, the point of intersection of two generators of opposite systems, one drawn through P and the other through Q, are*

$$\left[a\cos\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2},\ b\sin\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2},\ \pm c\tan\frac{\beta-\alpha}{2}\right].\ (\beta>\alpha)$$

The equation of the hyperboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ ...(1)



Let (x_1, y_1, z_1) be the co-ordinates of R, the pt. of intersection of two generators of opposite systems, one drawn thro' P and the other thro' Q.

The plane PRQ is the tangent plane at R to the hyperboloid (1). [Note, Art. 160, (*a*)]

∴ its equation is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1\ \text{...(2)}$$

It meets the plane $z=0$, where [putting $z=0$ in (2)]

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1.$$

∴ the equations of PQ are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1,\ z=0\ \text{...(3)}$$

Also, α, β being the eccentric angles of P,Q, the equations of PQ are $\frac{x}{a}\cos\frac{\beta+\alpha}{2}+\frac{y}{b}\sin\frac{\beta+\alpha}{2}=\cos\frac{\beta-\alpha}{2}$, $(\beta>\alpha)$, $z=0$

[From Analytical Plane Geometry]

or, dividing the first equation thro' out by $\cos\frac{\beta-\alpha}{2}$,

[To get 1 on the R.H.S.]

$$\frac{x}{a}\cos\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2}+\frac{y}{b}\sin\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2}=1,\ z=0\ \text{...(4)}$$

∴ comparing coeffs. in (3) and (4),

$$\frac{x_1}{a^2}=\frac{1}{a}\cos\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2},\ \frac{y_1}{b^2}=\frac{1}{b}\sin\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2}$$

[∵ constant terms in the first equations of (3) and (4) are equal]

or $x_1=a\cos\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2}$, $y_1=b\sin\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2}$ ...(5)

∵ (x_1, y_1, z_1) lies on the hyperboloid (1)

$$\therefore\ \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-\frac{z_1^2}{c^2}=1\ \text{...(6)}$$

Substituting the values of x_1, y_1 from (5) in (6),

$$\frac{1}{a^2}a^2\cos^2\frac{\beta+\alpha}{2}\sec^2\frac{\beta-\alpha}{2}+\frac{1}{b^2}b^2\sin^2\frac{\beta+\alpha}{2}\sec^2\frac{\beta-\alpha}{2}-\frac{z_1^2}{c^2}=1$$

or $\sec^2\frac{\beta-\alpha}{2}\left(\cos^2\frac{\beta+\alpha}{2}+\sin^2\frac{\beta+\alpha}{2}\right)-\frac{z_1^2}{c^2}=1$

or $\sec^2\frac{\beta-\alpha}{2}-\frac{z_1^2}{c^2}=1$ $\left[\because \cos^2\frac{\beta+\alpha}{2}+\sin^2\frac{\beta+\alpha}{2}=1\right]$

or $\frac{z_1^2}{c^2}=\sec^2\frac{\beta-\alpha}{2}-1=\tan^2\frac{\beta-\alpha}{2}$

or $\frac{z_1}{c}=\pm\tan\frac{\beta-\alpha}{2}$, or $z_1=\pm c\tan\frac{\beta-\alpha}{2}$.

$\therefore$ the co-ordinates of R are

$$\left[a\cos\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2},\ b\sin\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2},\ \pm c\tan\frac{\beta-\alpha}{2}\right].$$

[(x_1, y_1, z_1)]

Note. Important. *For problems relating to the point of intersection of two generators of opposite systems drawn through P, Q, points on the principal elliptic section of the hyperboloid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, *it is useful to remember that the co-ordinates of the point of intersection are*

$$\mathbf{x}=\mathbf{a}\cos\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2},\ \mathbf{y}=\mathbf{b}\sin\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2},\ \mathbf{z}=\pm\mathbf{c}\tan\frac{\beta-\alpha}{2},$$

$(\beta>\alpha)$

where α, β are the eccentric angles of P, Q.

EXAMPLES

1. P, D are the extremities of conjugate diameters of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, and generators of opposite systems through P, D intersect in Q. Find the locus of Q.

Prove also that $QP^2+QD^2=a^2+b^2+2c^2$.

2. Show that the generators through points on the principal elliptic section of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, such that the eccentric angle of the one is double the eccentric angle of the other, intersect on the curves given by

$$\mathbf{x}=\frac{a(1-3t^2)}{1+t^2},\ \mathbf{y}=\frac{bt(3-t^2)}{1+t^2},\ \mathbf{z}=\pm ct.$$ [*Ag. U. 1957*]

3. If α, β are the eccentric angles of two points P, Q on the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$,

and two generators of opposite systems, one drawn through P and the other through Q, intersect at R, the point "θ, ϕ", then

$\theta+\phi=\beta,\ \theta-\phi=\alpha.$ [*Ag. U. 1956*]

164. Projections of the generators on a principal plane. *To prove that the projections of the generators of a hyperboloid of one sheet on a principal plane are tangents to the section of the hyperboloid by the principal plane.*

Let the equation of the hyperboloid of one sheet be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \quad \dots (1)$$

Let the equations of *any* generator of the λ-system be

$$\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right),\ \frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right) \quad \dots (2)$$

[To find the equations of the projection of the line (2) on the xy-plane.]

The equation of *any* plane thro' the line (2) is

$$\frac{x}{a}+\frac{z}{c}-\lambda\left(1+\frac{y}{b}\right)+k\left[\frac{x}{a}-\frac{z}{c}-\frac{1}{\lambda}\left(1-\frac{y}{b}\right)\right]=0 \quad \dots (3)$$

If it is $\perp$ to the xy-plane, *i.e.*, $z=0$, then

$$\left(\frac{1}{a}+\frac{k}{a}\right)0+\left(-\frac{\lambda}{b}+\frac{k}{\lambda b}\right)0+\left(\frac{1}{c}-\frac{k}{c}\right)1=0$$

[$AA'+BB'+CC'=0$ (Art. 27, Cor. 1)]

$\therefore\ k=1.$

Substituting this value of k in (3),

$$\frac{x}{a}+\frac{z}{c}-\lambda\left(1+\frac{y}{b}\right)+\left[\frac{x}{a}-\frac{z}{c}-\frac{1}{\lambda}\left(1-\frac{y}{b}\right)\right]=0.$$

or $$\frac{2x}{a}-\lambda\left(1+\frac{y}{b}\right)-\frac{1}{\lambda}\left(1-\frac{y}{b}\right)=0$$

or $$\frac{2\lambda x}{a}-\lambda^2\left(1+\frac{y}{b}\right)-\left(1-\frac{y}{b}\right)=0$$

or $$\lambda^2\left(1+\frac{y}{b}\right)-2\lambda\frac{x}{a}+\left(1-\frac{y}{b}\right)=0.$$

$\therefore$ the equations of the projection of the line (2) on the xy-plane are

$$\lambda^2\left(1+\frac{y}{b}\right)-2\lambda\frac{x}{a}+\left(1-\frac{y}{b}\right)=0,\ z=0 \quad \dots (4)$$

The equations of the section of the hyperboloid (1) by the xy-plane are [putting $z=0$ in (1)]

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0 \quad \dots (5)$$

In order to prove that the projection (4) is a tangent to the section (5), let us *find the envelope of (4), λ being the parameter.*

The first equation of (4) is a quadratic in λ.

$\therefore$ the envelope is

$$\frac{4x^2}{a^2}=4\left(1+\frac{y}{b}\right)\left(1-\frac{y}{b}\right), \quad z=0$$

[$B^2 = 4AC$ (From Differential Calculus)]

[Cancel 4]

or $$\frac{x^2}{a^2}=1-\frac{y^2}{b^2}, \; z=0, \text{ or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad z=0,$$

which is the section (5).

$\therefore$ the projection (4) is a tangent to the section (5).

Similarly the projections of the generator (2) on the yz-, zx-planes are tangents to the sections of the hyperboloid by these planes.

Similarly the projection of *any* generator of the μ-system on a principal plane is a tangent to the section of the hyperboloid by the principal plane.

EXAMPLE

Prove that equal values of the parameters give two generators of opposite systems, of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, which project into the same tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \; z=0.$

SECTION II

RULED PARABOLOID

165. To prove that a hyperbolic paraboloid is a ruled surface.

Let the equation of the hyperbolic paraboloid be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z.$$

Factorising,

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{x}{a}-\frac{y}{b}\right)=2z$$

$$=\frac{z^*}{\lambda}.\, 2\lambda \quad ...(i)$$

or, interchanging the factors on the L.H.S., and changing λ to μ on the R.H.S.,

$$\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=\frac{z}{\mu}.\, 2\mu \quad ...(ii)$$

From (*i*) and (*ii*), the paraboloid is generated by the variable st. lines

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*To take the first factor ($\frac{z}{\lambda}$) on the R. H. S. of the same form as the first factor $\frac{x}{a}+\frac{y}{b}$ on the L. H. S.

$$(1)\ \frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda}\ ,\ \frac{x}{a}-\frac{y}{b}=2\lambda\ ;$$

and $$(2)\ \frac{x}{a}-\frac{y}{b}=\frac{z}{\mu}\ ,\ \frac{x}{a}+\frac{y}{b}=2\mu\ ;$$

where λ and μ are variable parameters.

As λ varies, taking in turn all real values, the line (1) moves so as to generate the paraboloid completely.

Similarly as μ varies, taking in turn all real values, the line (2) moves so as to generate the paraboloid completely.

$\therefore$ the hyperbolic paraboloid is a ruled surface.

[Def. (Art. 155)]

Cor. 1. *To prove that the two systems of generating lines are distinct.*

$\because$ it is impossible to give values to λ and μ so that the equations of the line (1) may become the same as those of the line (2)

$\therefore$ no member of the system of lines (1) coincides with any member of the system of lines (2),

i.e., the two systems of generating lines are distinct.

Note. λ-, and μ-systems of generators. Def.

The system of st. lines

$$\frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda}\ ,\ \frac{x}{a}-\frac{y}{b}=2\lambda\ ...(1)$$

where λ is a variable parameter, is called the **λ-system,** and the system of st. lines

$$\frac{x}{a}-\frac{y}{b}=\frac{z}{\mu}\ ,\ \frac{x}{a}+\frac{y}{b}=2\mu\ ...(2)$$

where μ is a variable parameter, is called the **μ-system.**

[**Rule to write down the equations of a generator of the μ-system, of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ from the equations of a generator of the λ-system :**

In the equations of a generator of the λ-system, change b to $-b$ and λ to μ.

For, changing b to $-b$ and λ to μ in (1), we get (2).]

Cor. 2. *The generating lines of the hyperbolic paraboloid*

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$$

are parallel to one of the fixed planes $\frac{x}{a}\pm\frac{y}{b}=0.$

The equations of *any* generator of the λ-system are

$$\frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda}\ ,\ \frac{x}{a}-\frac{y}{b}=2\lambda\ ...(1)$$

It lies in the plane $\frac{x}{a}-\frac{y}{b}=2\lambda$,

which is ∥ to the plane $\frac{x}{a}-\frac{y}{b}=0$.

$\therefore$ the generator (1) is ∥ to the fixed plane $\frac{x}{a}-\frac{y}{b}=0$.

Similarly the generating lines of the μ-system are ∥ to the fixed plane $\frac{x}{a}+\frac{y}{b}=0$.

EXAMPLES

1. A point, "t", on the parabola $y=0, x^2=2a^2z$, is $(2at, 0, 2t^2)$, and a point, "u", on the parabola $x=0, y^2=-2b^2z$, is $(0, 2bu, -2u^2)$. Find the locus of the lines joining the points for which,

(*i*) $t=u$, (*ii*) $t=-u$.

2. **Find the two systems of generators of the hyperbolic paraboloid $\frac{\mathbf{x}^2}{\mathbf{a}^2}-\frac{\mathbf{y}^2}{\mathbf{b}^2}=2\mathbf{z}$.**

Show that two generators pass through a given point on it, one of each system. [*P. U. H. 1957*]

[$\because$ the hyperbolic paraboloid is generated completely by either of the two systems of st. lines (Art. 165)

$\therefore$ thro' a given pt. P on it (hyperbolic paraboloid) there pass two generating lines, one of each system.]

166. To find the conditions that a given straight line should be a generator of a given paraboloid.

Let the equations of the line be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(1)

and the equation of the paraboloid be $ax^2+by^2=2z$ ...(2)

Any pt. on the line (1) is (x_1+lr, y_1+mr, z_1+nr).

If it lies on the paraboloid (2), then

$$a(x_1+lr)^2+b(y_1+mr)^2=2(z_1+nr)$$

or $$a(x_1+lr)^2+b(y_1+mr)^2-2(z_1+nr)=0$$

or $r^2(al^2+bm^2)+2r(alx_1+bmy_1-n)+(ax_1^2+by_1^2-2z_1)=0$ ...(3)

which is a quadratic in r.

If the line is a generator of the paraboloid, all pts. of the line lie on the paraboloid.

$\therefore$ the quadratic (3) is satisfied by *all* values of r,

$\therefore$ coeff. of $r^2=0$, coeff. of $r=0$, constant term $=0$,

[$\because$ $r^2(0)+r(0)+0=0$ for *all* values of r]

i.e.,
$$al^2+bm^2=0 \quad ...(4)$$
$$alx_1+bmy_1-n=0 \quad ...(5)$$
$$ax_1^2+by_1^2-2z_1=0 \quad ...(6)$$
(4), (5), (6) are the required conditions.

Cor. 1. To find the direction-ratios of the two generating lines of a paraboloid through any point (x_1, y_1, z_1) of the paraboloid.

From (4), $al^2=-bm^2$

or $\sqrt{a}.\, l=\pm\sqrt{-b}\, m$, or $\sqrt{a}.\, l\pm\sqrt{-b}\, m=0$

or $\sqrt{a}.\, l \pm \sqrt{-b}\, m+0.\, n=0 \quad ...(7)$

Solving (5) and (7) for l, m, n (by cross-multiplication),
$$\frac{l}{\pm\sqrt{-b}}=\frac{m}{-\sqrt{a}}=\frac{n}{\pm ax_1\sqrt{-b}-\sqrt{a}.by_1}.$$

Cor. 2. The only ruled paraboloid is the hyperbolic paraboloid.

For, from (7), dividing thro' out by $\sqrt{a}$,
$$l\pm\sqrt{\frac{-b}{a}}\, m=0$$

$\therefore$ the values of $l : m$ are real only if $-\frac{b}{a}$ is +ve

i.e., if $\frac{b}{a}$ is −ve, *i.e.,* if one of the two quantities a and b is +ve and the other −ve,

i.e., if, from (2), the paraboloid is a hyperbolic paraboloid (Art. 111, (*b*)).

EXAMPLES

1. Find the conditions that the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ should be a generator of the paraboloid $ax^2+by^2=2z$.

Find the direction-ratios of the two generating lines through (α, β, γ). [*P. U. H. 1956*]

2. Find the equations of the generating lines of the paraboloid $(x+y+z)(x+2y-z)=6z$ which pass through the point (1, 1, 1).

167. As in the case of a hyperboloid of one sheet the student can, and should prove the following results :

1. (*a*) *No two generators of the same system, of a hyperbolic paraboloid, intersect.*

(*b*) *Any generator of the λ-system, of a hyperbolic paraboloid, intersects any generator of the μ-system.*

[(*a*) Proceed as in Art. 159, (*a*). (*b*) Proceed as in Art. 159, (*b*).]

Cor. Co-ordinates of the point of intersection. The co-ordinates of the point of intersection of a generator of the λ-system and a generator of the μ-system, of the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z,$$

are given by $\frac{x}{a}=\mu+\lambda,\ \frac{y}{b}=\mu-\lambda,\ z=2\mu\lambda.$]

2. (*a*) *Section of a hyperbolic paraboloid by the tangent plane at a point.* **The section of a hyperbolic paraboloid by the tangent plane at a point is the two generating lines through the point.**

(*b*) **Any plane through a generating line.** *Any plane through a generating line of a hyperbolic paraboloid is a tangent plane.*

[(*a*) Proceed as in Art. 160, (*a*). (*b*) Proceed as in Art. 160, (*b*).]

3. *Locus of the points of intersection of perpendicular generators.* **The locus of the points of intersection of perpendicular generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ is the curve of intersection of the paraboloid and the plane $2z+a^2-b^2=0$.**

[Proceed as in Art. 161.]

4. **Co-ordinates of any point on a hyperbolic paraboloid in terms of two variables.** *The co-ordinates of any point on the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ are* **($ar\cos\theta$, $br\sin\theta$, $\frac{1}{2}r^2\cos 2\theta$).**

[Proceed as in Art. 162.]

Note. Point "r, θ". Def. *The point whose co-ordinates are* **($ar\cos\theta$, $br\sin\theta$, $\frac{1}{2}r^2\cos 2\theta$)** *is, for shortness, called the* **point "r,θ".**

5. **Projections of the generators on a principal plane.** *The projections of the generators of the hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ on the planes YOZ, ZOX are tangents to the principal sections*

$$y^2=-2b^2z,\ x=0\ ;\ x^2=2a^2z,\ y=0.$$

[Proceed as in Art. 164.]

EXAMPLES

1. Planes are drawn through the origin, O, and the generators through any point P of the paraboloid given by $x^2-y^2=az$. Prove that the angle between them is $\tan^{-1}\frac{2r}{a}$, where r is the length of OP.

[*Ag. U. 1947*]

2. **Angle between two generators.** Prove that the angle between the generating lines through the point (x_1, y_1, z_1) of the

hyperbolic paraboloid $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z$, is given by

$$\tan\theta=\frac{ab\sqrt{\dfrac{x_1^2}{a^4}+\dfrac{y_1^2}{b^4}+1}}{z_1+\dfrac{a^2-b^2}{2}}.$$

3. Find the equations of the generating lines through "r, θ" of the hyperbolic paraboloid $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z$.

[See Note, Art. 167, (4), and proceed as in Art. 166 and Cor. 1.]

4. Prove that equal values of the parameters give two generators of opposite systems, of the hyperbolic paraboloid $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z$, which project into the same tangent to the parabola $x^2=2a^2z$, $y=0$.

5. Find the locus of the perpendiculars from the vertex of the paraboloid $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=2z$ to the generators of one system.

[*P. U. H. 1956*]

[Let the equations of *any* generator of the λ-system be

$$\frac{x}{a}+\frac{y}{b}=\frac{z}{\lambda}\ ,\ \frac{x}{a}-\frac{y}{b}=2\lambda \quad ...(1)$$

The equations of *any* line thro' the vertex (0, 0, 0) are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(2)$$

Any pt. on this line is (lr, mr, nr).

(*i*) If it lies on the generator (1), then ? It will be found that

$$\frac{l}{a}+\frac{m}{b}=\frac{n}{\lambda} \quad ...(3)$$

(*ii*) If the line (2) is $\perp$ to the line (1) [direction-cosines proportional to $a, b, 2\lambda$], then ? ...(4)

(*iii*) Eliminate λ from (3) and (4), and then eliminate l, m, n from the result and the equations (2).

This gives the locus of the perpendiculars from the vertex of the paraboloid to the generators of the λ-system.

Changing b to $-b$ in the equation of the above locus, we get the locus of the perpendiculars from the vertex of the paraboloid to the generators of the μ-system. (See Note, Art. 165.)]

MISCELLANEOUS EXAMPLES ON CHAPTER XIII

1. (*a*) Name all the conicoids which are ruled surfaces.

[*P. U. H. 1951*]

(*b*) Prove that the line of intersection of two perpendicular planes which pass through two fixed non-intersecting lines generates a hyperboloid whose central circular sections are perpendicular to the lines and have their diameters equal to their shortest distance.

2. If P, P′ are the extremities of a diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, prove that the λ-generator through P and the μ-generator through P′ are parallel.

3. Show that the shortest distance between generators of the same system drawn at one end of each of the major and minor axes of the principal elliptic section of the hyperboloid

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is $2abc/\sqrt{a^2b^2+b^2c^2+c^2a^2}$. [*P. U. 1958*]

4. **Show that the shortest distances between generators of the same system drawn at the ends of diameters of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ lie on the surfaces whose equations are $\frac{cxy}{x^2+y^2}=\pm\frac{abz}{a^2-b^2}$.** [*Ag. U. 1955*]

[(*i*) Let the equations of the generator of one system drawn thro' one end $(a \cos \alpha, b \sin \alpha, 0)$ of a diameter be

$$\frac{x-a \cos \alpha}{a \sin \alpha}=\frac{y-b \sin \alpha}{-b \cos \alpha}=\frac{z}{c} \quad ...(1)$$

Then the equations of the generator of the same system drawn thro' the other end $(-a \cos \alpha, -b \sin \alpha, 0)$ of the diameter are [changing α to $\pi+\alpha$ in (1)],

$$\frac{x+a \cos \alpha}{-a \sin \alpha}=\frac{y+b \sin \alpha}{b \cos \alpha}=\frac{z}{c} \quad ...(2)$$

Solve the question.

(*ii*) *For the equations of generators of the opposite system, c is changed to* $-c$ (Ex. 4, Cor., Art. 158)

$\therefore$ change c to $-c$ in the answer obtained above.]

5. Show that the normals to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ at points of a generator meet the plane $z=0$ at points lying on a straight line, and for different generators of the same system this line touches a fixed conic. [*Ag. U. 1945*]

6. If A, A′ are the extremities of the major axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, and any gene-

rator meets two generators of the same system through A, A′ respectively in P, P′, prove that AP.A′P′ $=b^2+c^2$.

7. Show that, in general, two generators of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ can be drawn to cut a given generator at right angles, and that if they meet the plane $z=0$ in P and Q, PQ touches the ellipse $x^2/a^6+y^2/b^6=c^4/a^4b^4$. [*Ag. U. 1944*]

8. Show that for the hyperboloid $yz+zx+xy+1=0$, the equations of one system of generators are $x-\lambda y+\lambda+1=0$, $x+(\lambda+1)z+\lambda=0$, where λ is a variable parameter, and find the equations of generators of the other system.

[The equations of the line are

$$\left.\begin{aligned} x-\lambda y+\lambda+1 &= 0 \;\ldots (i), \\ x+(\lambda+1)z+\lambda &= 0 \;\ldots (ii) \end{aligned}\right\} \;\ldots (1)$$

(*Find the value of* λ *from* (*i*) *and of* $\dfrac{1}{\lambda}$ *from* (*ii*).)

From (*i*), $\lambda=\dfrac{x+1}{y-1}$; from (*ii*), $\dfrac{1}{\lambda}=\dfrac{-z-1}{z+x}$.

$$\therefore \quad \left(\frac{x+1}{y-1}\right)\left(\frac{-z-1}{z+x}\right)=\lambda.\frac{1}{\lambda} \;\ldots (2)$$

$$=1$$

It will be found that $yz+zx+xy+1=0$.

From (2), interchanging the factors on the L.H.S., and changing λ to μ on the R. H. S., [Compare Art. 156]

$$\left(\frac{-z-1}{z+x}\right)\left(\frac{x+1}{y-1}\right)=\mu.\frac{1}{\mu}$$

$$\therefore \qquad \frac{-z-1}{z+x}=\mu, \quad \frac{x+1}{y-1}=\frac{1}{\mu},$$

which are the required equations of generators of the other system.]

9. Prove that any point on the lines $(\mu+1)\,x=-\mu y=-(z+1)$ lies on the surface $yz+zx+xy+x+y=0$, and find the equations of the other system of lines which lies on the surface.

10. Show that tangent planes to $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, which are parallel to tangent planes to $\dfrac{b^2c^2x^2}{c^2-b^2}+\dfrac{c^2a^2y^2}{c^2-a^2}+\dfrac{a^2b^2z^2}{a^2+b^2}=0$, cut the surface in perpendicular generators. [*Ag. U. 1945*]

11. Show that the points on the quadric $ax^2+by^2+cz^2+d=0$ at which the generators are perpendicular lie on the cylinder

$$(c-a)x^2+(c-b)y^2+cd(a+b)\,/\,ab=0. \qquad [\textit{ P. U. H. 1961 }]$$

12. Show that the generators of the surface $x^2+y^2-z^2=1$ which intersect on XOY plane are at right angles. [*P. U. H. 1955*]

**13. Prove that the equations

$$\frac{x}{a}=\frac{\sin\theta\cos\phi}{\cos(\theta+\phi)},\ \frac{y}{b}=\frac{\cos\theta\sin\phi}{\cos(\theta+\phi)},\ \frac{z}{c}=\frac{\cos(\theta-\phi)}{\cos(\theta+\phi)}$$

determine a hyperboloid of one sheet, that θ is constant for points on a given generator of one system, and that ϕ is constant for points on a given generator of the other system.

[$$\frac{x}{a}+\frac{y}{b}=\frac{\sin(\theta+\phi)}{\cos(\theta+\phi)} \quad \dots (i)$$

$$\frac{x}{a}-\frac{y}{b}=\frac{\sin(\theta-\phi)}{\cos(\theta+\phi)} \quad \dots (ii)$$

$$\frac{z}{c}=\frac{\cos(\theta-\phi)}{\cos(\theta+\phi)} \quad \dots (iii)$$

Eliminating $(\theta-\phi)$ from (ii) and (iii) [by squaring (ii) and (iii), and adding],

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=\frac{1}{\cos^2(\theta+\phi)}=\sec^2(\theta+\phi) \quad \dots (iv)$$

Eliminating $(\theta+\phi)$ from (i) and (iv),

$$\left(\frac{x}{a}-\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1+\left(\frac{x}{a}+\frac{y}{b}\right)^2$$

or $\dfrac{z^2}{c^2}-\dfrac{4xy}{ab}=1$ or, transposing, $\dfrac{4xy}{ab}=\dfrac{z^2}{c^2}-1$

or, factorising, $\left(\dfrac{2x}{a}\right)\left(\dfrac{2y}{b}\right)=\left(\dfrac{z}{c}+1\right)\left(\dfrac{z}{c}-1\right)$.]

14. Prove that the generating lines through any point P on the section $z=c$ of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ meet the principal section by the plane $z=0$ at the ends of a pair of conjugate diameters. [*M. T.*]

15. If the generators through P, a point on the hyperboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$, whose centre is O, meet the plane $z=0$ in A and B, and the volume of the tetrahedron OAPB is constant and equal to $abc/6$, P lies on one of the planes $z=\pm c$. [*Ag. U. 1937*]

**16. Through a variable generator $x-y=\lambda$, $x+y=2z/\lambda$ of the paraboloid $x^2-y^2=2z$ a plane is drawn making a constant angle α with the plane $x=y$. Find the locus of the point at which it touches the paraboloid. [*P. U. H. 1957*]

17. R, S are the points of intersection of generators of opposite systems drawn at the extremities P, Q of conjugate semi-diameters of the principal elliptic section of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

Show that the volume of the tetrahedron PSQR is constant and equal to $\frac{1}{3}abc$. [*P. U. H. 1959*]

18. The generators through P on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

meet the plane $z=0$ in A and B. If A and B are the extremities of conjugate diameters of the principal elliptic section, prove that the median through P of the triangle PAB lies on the cone

$$\frac{2x^2}{a^2}+\frac{2y^2}{b^2}=\left(\frac{z}{c}\pm 1\right)^2.$$

19. The generators through R on the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

meet the plane $z=0$ in P and Q. If the median through R of the triangle RPQ is parallel to the fixed plane $lx+my+nz=0$, show that R lies on the surface $z(lx+my)+n(c^2+z^2)=0$.

[Let α, β be the eccentric angles of P, Q, so that their co-ordinates are $(a\cos\alpha, b\sin\alpha, 0)$, $(a\cos\beta, b\sin\beta, 0)$. Then the co-ordinates of R are

$$x=a\cos\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2}\ldots(i),\quad y=b\sin\frac{\beta+\alpha}{2}\sec\frac{\beta-\alpha}{2}\ldots(ii),$$

$$z=\pm c\tan\frac{\beta-\alpha}{2}$$ or, taking $+$ve sign on the R.H.S.,

$$z=c\tan\frac{\beta-\alpha}{2}\ldots(iii)$$

Find the co-ordinates of M, the mid-pt. of PQ, and then the direction-ratios of MR. If MR is $\parallel$ to the plane $lx+my+nz=0$, it will be found that

$$la\cos\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}+mb\sin\frac{\beta+\alpha}{2}\sin\frac{\beta-\alpha}{2}+nc=0\ldots(iv)$$

[To eliminate α, β from (*i*), (*ii*), (*iii*), (*iv*).]

Substitute the values of $\cos\frac{\beta+\alpha}{2}$ and $\sin\frac{\beta+\alpha}{2}$ from (*i*) and (*ii*) in (*iv*). It will be found that

$$(lx+my)\cos\frac{\beta-\alpha}{2}\sin\frac{\beta-\alpha}{2}+nc=0\ldots(v)$$

Substitute the values of $\cos\frac{\beta-\alpha}{2}$ and $\sin\frac{\beta-\alpha}{2}$ from (*iii*) in (*v*).]

20. Prove that the locus of the points of intersection of generators of the hyperbolic paraboloid $xy=az$ which are inclined at a constant angle α is the curve of intersection of the paraboloid and the hyperboloid $x^2+y^2-z^2\tan^2\alpha+a^2=0$.

21. Prove that the equations

$$x=ae^{\theta}\cosh\phi,\ y=be^{\theta}\sinh\phi,\ 2z=ce^{2\theta}$$

determine a hyperbolic paraboloid, and that $\theta+\phi$ is constant for points of a given generator of one system, and $\theta-\phi$ is constant for a given generator of the other.

ANSWERS

ANSWERS

CHAPTER I

Page 3.

1. Co-ordinates of N : $(x, y, 0)$; co-ordinates of A : $(x, 0, 0)$.

Page 5.

1. (a) $d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$.

[**Complete distance formula.

$$d=\pm\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2},$$

that sign being taken on the R.H.S. which gives a +ve result for d.]

(b) 6.

2. (i) $x-y+3z+3=0$, (ii) $x^2+y^2+z^2-6y+20-k^2=0$,

(iii) $2x-2y+6z+6+k^2=0$.

3. $2x-z=0$.

Pages 7—8.

1. (a) $x=\dfrac{\lambda x_2+x_1}{\lambda+1}$, $y=\dfrac{\lambda y_2+y_1}{\lambda+1}$, $z=\dfrac{\lambda z_2+z_1}{\lambda+1}$.

(b) $(\frac{3}{2}, -\frac{1}{2}, 1)$, $(0, -5, 7)$, $(-3, -14, 19)$.

2. 1 : 2.

CHAPTER II

Page 12.

1. $l^2+m^2+n^2=1$, where l, m, n are the direction-cosines.

2. 60° or 120°.

Page 14.

2. (a) See Art. 7 and Note 2, Art. 10.

(i) If the direction-ratios of a line are a, b, c, the direction-cosines are $\dfrac{a}{\sqrt{a^2+b^2+c^2}}$, $\dfrac{b}{\sqrt{a^2+b^2+c^2}}$, $\dfrac{c}{\sqrt{a^2+b^2+c^2}}$.

(ii) The sum of the squares of the direction-cosines of a line, $=1$, but the sum of the squares of the direction-ratios is not necessarily $=1$.

(b) $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.

[**Note. There are *four* lines which make equal angles with the axes. Their direction-cosines are $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$; $-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$; $\dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$; $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, -\dfrac{1}{\sqrt{3}}$. If it is assumed, *as is generally the case*, that the line makes equal *acute* angles with the +ve directions of the axes, its direction-cosines are $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}$.]

Page 15.

Ex. 13.

Page 16.

2. $\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}; \frac{3}{5\sqrt{2}}, -\frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}}; -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}$.

3. $-\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}$.

Pages 23—26.

1. (*b*) 60°.

2. $\theta = \cos^{-1}(ll' + mm' + nn')$, where l, m, n ; l', m', n' are the direction-cosines of the lines. (*i*) $\cos^{-1}\frac{13}{5\sqrt{7}}$. (*ii*) 90°.

4. $\cos^{-1}\left(-\frac{4\sqrt{2}}{15}\right)$.

5. $A=\cos^{-1}\frac{1}{\sqrt{3}}$, $B=90°$, $C=\cos^{-1}\sqrt{\frac{2}{3}}$.

7. $\frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{1}{\sqrt{14}}$. 9. $\cos^{-1}\frac{1}{3}$.

13. $\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}$.

[****Complete result.** $\pm\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}$.]

Page 27. Art. 14.

2. $-\frac{4}{3}$.

MISCELLANEOUS EXAMPLES ON CHAPTER II

Pages 27—28.

1. $\cos^{-1}\left(-\frac{3}{\sqrt{574}}\right)$. 8. $x_2-x_1, y_2-y_1, z_2-z_1$.

CHAPTER III

Page 32.

Ex. (*i*) A cylinder generated by a st. line which is || to the z-axis, and intersects the *circle* whose equation *in the xy-plane* is $x^2+y^2=a^2$. (It is called a **right circular cylinder.**)

(*ii*) A cylinder generated by a st. line which is || to the z-axis, and intersects the *parabola* whose equation *in the xy-plane* is $y^2=4ax$. (It is called a **parabolic cylinder.**)

Page 33. Art. 19.

1. (*i*) $y=b$; $z=c$; $x=a$. (*ii*) $x=a, y=b$.
2. (*i*) A plane || to the yz-plane, (*ii*) a line || to the z-axis.
3. (*i*) A line || to the z-axis ; (*ii*) a line || to the x-axis ; (*iii*) a line || to the z-axis, and in the zx-plane. (*iv*) z-axis.
4. (*i*) A circle in the xy-plane ; (*ii*) a circle in a plane || to the xy-plane ; (*iii*) a parabola in a plane || to the zx-plane.

5. (*i*) A pair of circles one in each of two planes $\parallel$ to the xy-plane;

(*ii*) two pairs of $\parallel$ lines, each pair being $\parallel$ to the z-axis.

6. $2y^2-2yz+3z^2+2y-2z-2=0$; $3z^2+2zx+2x^2-2z-2x-2=0$; $3x^2-4xy+3y^2-4x+4y-1=0$.

MISCELLANEOUS EXAMPLES ON CHAPTER III

Page 33.

1. (*i*) A system of planes $\parallel$ to the xy-plane.

(*ii*) A cylinder generated by a st. line which is $\parallel$ to the x-axis, and intersects the curve whose equation *in the yz-plane* is $f(y, z)=0$.

2. A cylinder generated by a st. line which is $\parallel$ to the y-axis, and intersects the *parabola* whose equation *in the zx-plane* is $z^2=4ax$. (It is called a **parabolic cylinder**.)

3. (*a*) In two dimensions* : a curve in the xy-plane ; in three dimensions† : a cylinder generated by a st. line which is $\parallel$ to the z-axis, and intersects the curve whose equation *in the xy-plane* is $f(x, y)=0$.

(*b*) In two dimensions : a rectangular hyperbola in the xy-plane ; in three dimensions : a cylinder generated by a st. line which is $\parallel$ to the z-axis, and intersects the rectangular hyperbola whose equation *in the xy-plane* is $x^2-y^2=a^2$. (It is called a **hyperbolic cylinder.**)

Pages 35—36.

2. $2:3,\ 4:5,\ -7:8$.

Page 38.

1. $\frac{3}{2}, 3, -\frac{3}{2}$; $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$. 2. $-\frac{2}{\sqrt{14}}, -\frac{3}{\sqrt{14}}, \frac{1}{\sqrt{14}}$.

Pages 39—40.

1. (*a*) $\begin{vmatrix} x, & y, & z, & 1 \\ x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \end{vmatrix} = 0,$

where (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) are the three given pts.

Pages 43—45.

1. (*i*) 60°, (*ii*) 90°. 2. $2x+2y-3z+3=0$.

4. $2x-3y+4z+11=0$.

7. $\cos^{-1}\frac{\sqrt{2}}{3}$, acute.

*i.e., in Analytical Plane Geometry.

†i.e., in Analytical Solid Geometry.

Note. Important. "*The angle between two planes*" means, for definiteness, "the *acute* angle between two planes."

Page 46.

Ex. On opposite sides.

Page 47.

1. $d = x' \cos\alpha + y' \cos\beta + z' \cos\gamma - p$.

[****Complete perpendicular distance formula.**

$$d = \pm\,(x' \cos\alpha + y' \cos\beta + z' \cos\gamma - p),$$

that sign being taken on the R.H.S. which gives a +ve result for d.]

Pages 49—50.

1. (*a*) $d = \dfrac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}$, where (x_1, y_1, z_1) is the given pt., and $Ax + By + Cz + D = 0$ the given plane.

[****Complete perpendicular distance formula.**

$$d = \pm\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}},$$

that sign being taken on the R.H.S. which gives a +ve result for d.]

(*b*) 2.

3. 2, 1 ; No.

4. $3x^2 + 8y^2 + 35z^2 - 36yz - 24zx + 12xy - 8x - 12y + 24z + 4 = 0$.

5. $x^2 + y^2 + z^2 - 4 = 0$.

Pages 51—52.

1. $\dfrac{ax + by + cz + d}{\sqrt{a^2 + b^2 + c^2}} = \pm\dfrac{a'x + b'y + c'z + d'}{\sqrt{a'^2 + b'^2 + c'^2}}$.

3. $23x - 13y + 32z + 45 = 0$.

Pages 54—55. Art. 32 (*b*).

2. 12π.

Page 55. Art. 33.

2. $\sqrt{\dfrac{609}{2}}$.

Pages 57—59.

1. (*a*) $V = \frac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$,

where (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) are the co-ordinates of the vertices of the tetrahedron.

[****Complete volume formula.** $V = \pm\frac{1}{6}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$,

that sign being taken on the R.H.S. which gives a +ve result for V.]

(b) $\frac{1}{3}$.

2. $2x+3y-2z+10=0$ or $2x+3y-2z-14=0$.

Page 61. Art. 35.

2. $abc+2a'b'c'-aa'^2-bb'^2-cc'^2=0$. 3. $\cos^{-1}\frac{16}{21}$.

CHAPTER V

Pages 65—66.

1. (b) $\frac{x-a}{0}=\frac{y-b}{0}=\frac{z-c}{1}$.

2. (a) $x-1=y-2=z-3$. (b) $\cos^{-1}\frac{1}{5}$.

Pages 67—68.

3. $\frac{x-\frac{3}{5}}{2}=y+\frac{6}{5}=z$. 4. $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{3}{\sqrt{14}}$.

5. $\cos^{-1}\frac{8}{\sqrt{406}}$. 7. $\frac{x-1}{-3}=\frac{y-2}{5}=\frac{z-3}{4}$.

Page 69.

3. $aa'+cc'+1=0$. 4. $a=1, b=2, c=-1, d=-2$.

Pages 74—75.

1. $d=\{(x'-a)^2+(y'-b)^2+(z'-c)^2$

$-[(x'-a)\cos\alpha+(y'-b)\cos\beta+(z'-c)\cos\gamma]^2\}^{\frac{1}{2}}$.

[**Complete perpendicular distance formula.

$d=\pm\{(x'-a)^2+(y'-b)^2+(z'-c)^2$

$-[(x'-a)\cos\alpha+(y'-b)\cos\beta+(z'-c)\cos\gamma]^2\}^{\frac{1}{2}}$,

that sign being taken on the R.H.S. which gives a +ve result for d.]

3. $\sqrt{\frac{14}{3}}$. 4. $\frac{3\sqrt{3}}{\sqrt{14}}$. 5. $\sqrt{13}, \sqrt{10}, \sqrt{5}$.

6. $a\sqrt{\frac{2}{3}}$, where a is an edge of the cube.

8. $\frac{x}{2}=\frac{y}{-1}=\frac{z}{-4}$; $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{4}{3}\right)$.

Pages 76—77.

2. 6. 3. 13. 4. $(-1, 0, 1)$. 5. 1.

6. $al+bm+cn=0$, and $a\alpha+b\beta+c\gamma+d=0$.

The condition $al+bm+cn=0$ means that the line is $\perp$ to the normal to the plane, and $a\alpha+b\beta+c\gamma+d=0$ means that its pt. (α, β, γ) lies on the plane.

Pages 79—82.

1. (a) (i) $al+bm+cn=0$, and $a\alpha+b\beta+c\gamma+d\neq0$.

(ii) $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$.

(b) $al+bm+cn=0$, and $a\alpha+b\beta+c\gamma+d=0$.

3. (b) $y+3z-2=0,\ x-z-1=0,\ 3x+y-5=0$.

4. (a) $\dfrac{x-1}{-2}=\dfrac{y-2}{1}=\dfrac{z-3}{0}$. 5. $(0, 1, -1)$.

6. $29x-27y-22z-85=0$. 8. $2x-3y+z-14=0$.

9. $x+5y-6z+19=0$. 10. $15x+y-7z-53=0$; $x+7y+9z=0$.

13. $\dfrac{x-1}{1}=\dfrac{-2}{5}=\dfrac{z-3}{3}$; $\dfrac{x-1}{3}=\dfrac{y-2}{-14}=\dfrac{z-3}{9}$;

$\left(\frac{3}{7}, -\frac{6}{7}, \frac{9}{7}\right)$; $\left(\frac{5}{2}, -5, \frac{15}{2}\right)$.

Pages 83—84.

2. $x-z+2=0$. 3. $u(a'l+b'm+c'n)-v(al+bm+cn)=0$.

5. $lx+my+nz=lf+mg+nh,\ \dfrac{ax+by+cz+d}{af+bg+ch+d}=\dfrac{a'x+b'y+c'z+d'}{a'f+b'g+c'h+d'}$.

Pages 85—86.

3. $\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ l, & m, & n \\ l', & m', & n' \end{vmatrix}=0.$

4. (i) $17y-25z=0$; $17x-5z-34=0$; $5x-y-10=0$.

(ii) $4y-3z+1=0$; $2x-z+1=0$; $3x-2y+1=0$.

Pages 87—90.

1. If the equations of the two lines are

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1},\quad \frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2},$$

the required condition is $\begin{vmatrix} x_2-x_1, & y_2-y_1, & z_2-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0.$

The equation of the plane is $\begin{vmatrix} x-x_1, & y-y_1, & z-z_1 \\ l_1, & m_1, & n_1 \\ l_2, & m_2, & n_2 \end{vmatrix}=0.$

4. $x-2y+z=0$.

Page 91.

2. $\begin{vmatrix} x, & y, & z \\ bc'-b'c, & ca'-c'a, & ab'-a'b \\ \beta\gamma'-\beta'\gamma, & \gamma\alpha'-\gamma'\alpha, & \alpha\beta'-\alpha'\beta \end{vmatrix}=0.$

Pages 93—97.

1. Length : $\dfrac{(\alpha'-\alpha)(mn'-m'n)+(\beta'-\beta)(nl'-n'l)+(\gamma'-\gamma)(lm'-l'm)}{\sqrt{\Sigma\,(mn'-m'n)^2}}$.

Equations :

$$\begin{vmatrix} x-\alpha, & y-\beta, & z-\gamma \\ l, & m, & n \\ mn'-m'n, & nl'-n'l, & lm'-l'm \end{vmatrix}=0,\quad \begin{vmatrix} x-\alpha', & y-\beta', & z-\gamma' \\ l', & m', & n' \\ mn'-m'n, & nl'-n'l, & lm'-l'm \end{vmatrix}=0.$$

3. $\frac{1}{3}$; $2x-2y-z=0,\ x+2y-2z+4=0.$

7. $2\sqrt{29}$; $\frac{x-3}{2}=\frac{y-5}{3}=\frac{z-7}{4}$; $(3, 5, 7), (-1, -1, -1).$

8. $(2, 8, -3), (0, 1, 2)$; $\sqrt{78}$. 10. $\frac{5m-10}{\sqrt{5m^2-16m+17}}$.

12. $\frac{13}{\sqrt{66}}$, $3x-y-z=0=x+2y+z-1.$

Pages 98—100.

3. If the equations of the two given lines are $y=mx,\ z=c$; $y=-mx,\ z=-c$; and the equation of the given plane is

$$ux+vy+wz=0^*,$$

the required locus is $m^2vx+uy+mwc=0,\ z=0,$ which is a st. line.

4. If the equations of the two given lines are $y=mx,\ z=c$; $y=-mx,\ z=-c$, the required locus is $mxy+(1+m^2)\,cz=0.$

Pages 101—102. Art. 48.

2. $x-1=0,\ 9x+15y-5z-19=0.$
3. $29x-7y-10z=0,\ 9x-2y-3z=0.$
4. $11x-2y-4z-10=0,\ x-4y+3z=0.$
5. $2x-2y-z-7=0,\ 4x-3y-z-10=0.$

Pages 102—104. Art. 49.

2. $mxz=cy.$ 5. $36x^2+16y^2-9z^2=144.$

7. If the equations of the two given lines are $y=mx,\ z=c$; $y=-mx,\ z=-c$, the required locus is $cy-mxz\pm m^2(yz-cmx)=0.$

8. If the equations of the two given lines are $y=mx,\ z=c$; $y=-mx,\ z=-c$, the required locus is $yz=cmx$ or $cy=mxz$.

9. If the equations of the two given lines are $y=mx,\ z=c$; $y=-mx,\ z=-c$, and the equation of the given plane is

$$ux+vy+wz=0,$$

the required locus is $u(cy-mxz)-vm(yz-cmx)-wm(z^2-c^2)=0.$

10. If the equations of the two given lines are $y=mx,\ z=c$; $y=-mx,\ z=-c$, the required locus is

$$-\frac{m^2}{1-m^2}x^2+\frac{y^2}{1-m^2}+z^2=c^2.$$

Pages 105—108.

3. $\frac{2}{3}$. 7. $\left(\frac{1}{3+\sqrt{6}},\ \frac{1}{3+\sqrt{6}},\ \frac{1}{3+\sqrt{6}}\right)$.

Pages 111—113.

6. The equation of the plane thro' the line of intersection of the

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*Why to take the equation of the plane as $ux+vy+wz=0$. In order to avoid confusion in writing c and C, we have taken the equation of the plane as $ux+vy+wz=0$, and not as $Ax+By+Cz=0$.

planes (1) and (2), and $\perp$ to the plane (3) is

$$(a_1x+b_1y+c_1z+d_1)\,(a_2a_3+b_2b_3+c_2c_3)$$
$$=(a_2x+b_2y+c_2z+d_2)(a_3a_1+b_3b_1+c_3c_1).$$

Similarly for the other planes.

7. $\dfrac{x}{0}=\dfrac{y-b}{-b}=\dfrac{z}{c}$;

$$\left(\frac{a^{-1}}{a^{-2}+b^{-2}+c^{-2}},\ \frac{b^{-1}}{a^{-2}+b^{-2}+c^{-2}},\ \frac{c^{-1}}{a^{-2}+b^{-2}+c^{-2}}\right).$$

Pages 114—116. Art. 54.

2. (*i*) The planes intersect at a pt.
   (*ii*) The planes have a common line of intersection.
   (*iii*) The planes form a prism.
3. (*i*) The planes form a prism.
   (*ii*) The planes have a common line of intersection.
   (*iii*) The planes intersect at a pt.
   (*iv*) The planes (3) and (1) are $\parallel$, and the plane (2) intersects them.
4. $\dfrac{1}{6\sqrt{3}}$.

MISCELLANEOUS EXAMPLES ON CHAPTER V

Pages 116—122.

6. $\left(-\dfrac{2}{29},-\dfrac{3}{29},\dfrac{4}{29}\right)$. 9. $x-4y+2z+4=0$.

11. $\dfrac{x-1}{1}=\dfrac{y}{2}=\dfrac{z}{-3}$. 13. $8x+y-26z+6=0$.

14. $9x-3y-z+14=0$. 16. $(0, 1, 2)$; $4x+y-2z+3=0$.

17. 60° ; $x-y+z=0$.

18. $(\alpha+l,\ \beta+m,\ \gamma+n)$; $x(m\gamma-n\beta)+y(n\alpha-l\gamma)+z\,(l\beta-m\alpha)=0$.

19. $\left(-\dfrac{4}{11},-\dfrac{3}{11},\dfrac{18}{11}\right)$. 20. $\dfrac{x-2}{-1}=\dfrac{y+3}{3}=\dfrac{z-1}{1}$; $(0, 3, 3)$.

21. $\dfrac{x}{1}=\dfrac{y}{2}=\dfrac{z}{-1}$, $\dfrac{x}{-1}=\dfrac{y}{1}=\dfrac{z}{-2}$.

25. $\left(-\dfrac{4}{29},-\dfrac{6}{29},\dfrac{8}{29}\right)$.

27. If the equations of the two given lines are $y=mx$, $z=c$; $y=-mx$, $z=-c$, and the ratio of the distances is $k:1$, the required locus is

$$m^2x^2+y^2+(1+m^2)(z^2+c^2)-2\left(\frac{1+k^2}{1-k^2}\right)\ [mxy+(1+m^2)\,cz\,]=0.$$

31. $c^2x^2z^2=(a^2z^2-c^2y^2)(2c-z)^2$. 32. $\dfrac{2x^2}{a^2}+\dfrac{2y^2}{b^2}-\dfrac{z^2}{c^2}=1$.

33. $x+2y-3z-4=0,\ x+y-2z-3=0.$ 35. $z^2-xy-c^2=0.$

36. $-m^2x^2+y^2+z^2=c^2.$

40. $2\sqrt{\frac{3}{7}},\ 3\sqrt{\frac{3}{14}},\ 3\sqrt{\frac{3}{14}}.$

CHAPTER VI

Page 124.

1. (3, 5, 7), (1, −1, 4). 2. In the obtuse angle.

Pages 127—128.

1. If $l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3$ are the direction-cosines of $O\xi$, $O\eta$, $O\zeta$, then

$$\left.\begin{aligned} l_1^2+m_1^2+n_1^2&=1,\\ l_2^2+m_2^2+n_2^2&=1,\\ l_3^2+m_3^2+n_3^2&=1.\end{aligned}\right\}\ldots(\mathrm{A})\qquad \left.\begin{aligned} l_2l_3+m_2m_3+n_2n_3&=0,\\ l_3l_1+m_3m_1+n_3n_1&=0,\\ l_1l_2+m_1m_2+n_1n_2&=0.\end{aligned}\right\}\ldots(\mathrm{B})$$

$$\left.\begin{aligned} l_1^2+l_2^2+l_3^2&=1,\\ m_1^2+m_2^2+m_3^2&=1,\\ n_1^2+n_2^2+n_3^2&=1.\end{aligned}\right\}\ldots(\mathrm{C})\qquad \left.\begin{aligned} m_1n_1+m_2n_2+m_3n_3&=0,\\ n_1l_1+n_2l_2+n_3l_3&=0,\\ l_1m_1+l_2m_2+l_3m_3&=0.\end{aligned}\right\}\ldots(\mathrm{D})$$

CHAPTER VII

Page 131.

1. If the centre of the sphere is the origin and radius a, the equation of the sphere is $x^2+y^2+z^2=a^2$.

3. $x^2+y^2+z^2+2x-2z-2=0.$

5. The curve of intersection of the sphere $x^2+y^2+z^2=a^2$ and the parabolic cylinder $y^2=4az$.

Pages 132—135. Art. 62.

3. (*i*) (−1, 2, −3), 4. (*ii*) $(\frac{1}{2}, -1, \frac{3}{2})$, 2. 5. 1 : 2, −1 : 2.

Pages 135—138. Art. 64.

2. $x^2+y^2+z^2-ax-by-cz=0.$ 4. $(-\frac{1}{2}, 1, -2).$

9. If the volume of the tetrahedron $=k^3$, the required locus is $4xyz=\pm 3k^3$.

Page 139. Art. 65.

3. $x^2+y^2+z^2-3x+5y+7=0.$

Pages 139—141. Art. 66.

2. $\left(-\frac{7}{3}, -\frac{5}{3}, -\frac{2}{3}\right)$, 3.

4. $\frac{32}{3}\pi,\ \frac{16}{3}\pi,\ \frac{32}{3}\pi;\ 16\pi.$

Pages 142—144. Art. 67.

2. $3(x^2+y^2+z^2)-2x-3y-4z-22=0.$

3. $\gamma(x^2+y^2+z^2-a^2)=z(\alpha^2+\beta^2+\gamma^2-a^2).$ 6. $x^2+y^2+z^2=9.$

Pages 144—145. Art. 68.

1. $(2, -6, 3), (6, -3, -2).$

2. The cylinder $x^2-2cz+c^2-k^2=0$, where c is the $\perp$ distance of the given pt. from the given line, and $2k$ the fixed length intercepted on the given line.

Page 146.

Ex. $x\alpha+y\beta+z\gamma=a^2.$

Pages 147—148.

1. $xx'+yy'+zz'+u(x+x')+v(y+y')+w(z+z')+d=0.$

3. $x\cos\theta\sin\phi+y\sin\theta\sin\phi+z\cos\phi=a.$

Pages 149—150.

1. $x^2+y^2+z^2+2x+4y+6z-11=0,$
$5(x^2+y^2+z^2)-4x-8y-12z-13=0.$

3. If (x_1, y_1, z_1) is the given pt., then
(*i*) the spheres are real if
$$x_1^2+y_1^2+z_1^2-2y_1z_1-2z_1x_1-2x_1y_1 \text{ is } -\text{ve}.$$
(*ii*) the spheres are coincident if
$$x_1^2+y_1^2+z_1^2-2y_1z_1-2z_1x_1-2x_1y_1=0.$$

4. The curve of intersection of the sphere
$$x^2+y^2+z^2-2cz+c^2-k^2=0$$
and the cylinder $x^2-2cz+c^2=0$, where c is the $\perp$ distance of the given pt. from the given line, and k the constant radius.

6. $\pm\sqrt{-a^{-2}+b^{-2}+c^{-2}}\,x\pm\sqrt{a^{-2}-b^{-2}+c^{-2}}\,y\pm\sqrt{a^{-2}+b^{-2}-c^{-2}}\,z=\sqrt{2};$

8.

Page 152.

1. $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0.$

Pages 154—155.

2. If the equations of the four given spheres are
$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0,$$
$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0,$$
$$x^2+y^2+z^2+2u_3x+2v_3y+2w_3z+d_3=0,$$
$$x^2+y^2+z^2+2u_4x+2v_4y+2w_4z+d_4=0,$$
then the equation of the required sphere is

$$\begin{vmatrix} x^2+y^2+z^2, & x, & y, & z, & 1 \\ -d_1, & u_1, & v_1, & w_1, & -1 \\ -d_2, & u_2, & v_2, & w_2, & -1 \\ -d_3, & u_3, & v_3, & w_3, & -1 \\ -d_4, & u_4, & v_4, & w_4, & -1 \end{vmatrix}=0.$$

3. $x^2+y^2+z^2+7x+10y-5z+12=0.$

MISCELLANEOUS EXAMPLES ON CHAPTER VII

Pages 162—165.

1. $\left(\frac{9}{5}, \frac{12}{5}, 4\right).$ 2. $\left(\frac{48}{7}, -\frac{16}{7}, \frac{24}{7}\right).$

5. $3(x^2+y^2+z^2)-2x-2y-2z-1=0.$ 6. $\left(-\frac{13}{98}, \frac{40}{49}, \frac{135}{98}\right).$

7. $$\left\| \begin{matrix} 2(u-u'), & 2(v-v'), & 2(w-w'), & d-d' \\ l, & m, & n, & -p \\ l', & m', & n', & -p' \end{matrix} \right\| =0 \;\ldots\text{(A)}$$

the notation indicating that each of the four determinants obtained by omitting the *fourth, third, second, first columns one by one* in (A) is zero (Notation, end of Art. 53).

8. $x^2+y^2+z^2-y-2z-14=0$; $4x-3y+6z-35=0,$

$$\frac{x-2}{4}=\frac{y+1}{-3}=\frac{z-4}{6}.$$

9. $4(x^2+y^2+z^2)+10x-25y-2z=0.$

10. $\left(\frac{7}{9}, \frac{7}{9}, \frac{7}{9}\right)$; $\frac{7}{9}$; $81(x^2+y^2+z^2)-126(x+y+z)+98=0.$

11. $x^2+y^2+z^2-2x-4y-5z+5=0$;
$5(x^2+y^2+z^2)-2x-4y-5z+1=0.$

12. $x^2+y^2+z^2-2x+28y+2z-2=0$;
$x^2+y^2+z^2-2x-4y+2z-2=0.$

13. $x^2+y^2+z^2-8a(x+y+z)+32a^2=0$;
$x^2+y^2+z^2-2a(x+y+z)+2a^2=0.$

14. $x^2+y^2+z^2\pm 2ax\pm 2ay\pm 2az+2a^2=0$; infinite number.

15. $2x-y+4z-5=0$; $4x-2y-z-16=0.$

16. $x+2y+2z-9=0$; $2x+y-2z-9=0.$

17. $x^2+y^2+z^2\pm 2ax\pm 2ay\pm 2az+a^2=0$; infinite number.

18. $(l^2+m^2+n^2)\,[(x-a)^2+(y-b)^2+(z-c)^2]$
$=[(b-g)n-(c-h)m]^2+[(c-h)l-(a-f)n]^2+[(a-f)m-(b-g)l]^2.$

19. 4 ; $x^2+y^2+z^2-10x\pm 8y\pm 8z+16=0$; $\left(\frac{16}{5}, 0, 0\right).$

20. $x\alpha+y\beta+z\gamma+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d=0.$

21. $x^2+y^2+z^2+\dfrac{b^2+c^2}{a}x+\dfrac{c^2+a^2}{b}y+\dfrac{a^2+b^2}{c}z+a^2+b^2+c^2=0.$

24. $(\frac{3}{2}, 1, \frac{7}{2})$. 26. $\cos^{-1}\frac{2}{3}$; $2(x^2+y^2+z^2)-14x-3y+8z=0.$

27. $9(x^2+y^2+z^2)-10x+20y-20z-31=0$; $\left(\dfrac{5}{9}, -\dfrac{10}{9}, \dfrac{10}{9}\right)$; $\frac{2}{3}\sqrt{14}$.

CHAPTER VIII

Pages 167—169.

2. (i) $p(ax^2+by^2)=2z(lx+my+nz).$

(ii) $p^2(x^2+y^2+z^2)+2pux(lx+my+nz)+d(lx+my+nz)^2=0.$

3. $b^2x^2=a^2(y^2+z^2).$ 4. $4z^2(ax^2+by^2+cz^2)=(\alpha x^2+\beta y^2)^2.$

Page 170. Art. 88.

1. If the pt. of concurrence of the five lines is taken as origin and the equations of the lines are

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1},\ \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2},\ \frac{x}{l_3}=\frac{y}{m_3}=\frac{z}{n_3},$$

$$\frac{x}{l_4}=\frac{y}{m_4}=\frac{z}{n_4},\ \frac{x}{l_5}=\frac{y}{m_5}=\frac{z}{n_5},$$

the equation of the cone is

$$\begin{vmatrix} x^2, & y^2, & z^2, & yz, & zx, & xy \\ l_1^2, & m_1^2, & n_1^2, & m_1n_1, & n_1l_1, & l_1m_1 \\ l_2^2, & m_2^2, & n_2^2, & m_2n_2, & n_2l_2, & l_2m_2 \\ l_3^2, & m_3^2, & n_3^2, & m_3n_3, & n_3l_3, & l_3m_3 \\ l_4^2, & m_4^2, & n_4^2, & m_4n_4, & n_4l_4, & l_4m_4 \\ l_5^2, & m_5^2, & n_5^2, & m_5n_5, & n_5l_5, & l_5m_5 \end{vmatrix}=0.$$

Pages 170—171. Art. 89.

1. $fyz+gzx+hxy=0.$

Page 172.

1. $\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}.$ 2. $-\dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}, \dfrac{1}{\sqrt{3}}.$

Pages 173—174.

1. $5(x^2+y^2+z^2)-8(yz+zx+xy)-4x+86y-58z+278=0.$

3. $4(x^2+y^2+z^2)+9(yz+zx+xy)=0.$

4. $x^2+y^2+z^2+6(yz+zx+xy)-16x-36y-4z-28=0.$

Pages 175—176.

1. (i) $a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=(z-\gamma)^2.$

(ii) $(\beta z-\gamma y)^2=4a(\alpha z-\gamma x)(z-\gamma).$

2. (a) $\dfrac{1}{a^2}(\alpha z-\gamma x)^2+\dfrac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2.$

Pages 177—178.

1. $(x^2+y^2+z^2+2ux+2vy+2wz+d)\times$
$$(x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d)$$
$$=[xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d]^2.$$

Page 179.

Ex. (1, 2, 3), (2, −3, 1).

Pages 184—185.

2. (*i*) $\cos^{-1}\frac{1}{6}$; (*ii*) $\cos^{-1}\frac{5}{\sqrt{39}}$. 4. $\frac{\pi}{3}$.

5. $\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$; $\frac{x}{-1}=\frac{y}{2}=\frac{z}{0}$.

Page 187.

1. (*a*) [See end of Note, Art. 100.]

Pages 190—191. Art. 102.

5. $x=y=-z$, $\frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$.

MISCELLANEOUS EXAMPLES ON CHAPTER VIII

Pages 191—197.

1. (*i*) $x^2+y^2=z^2$. (*ii*) $d^2(\alpha x^2+\beta y^2+\gamma z^2)=(ax+by+cz)^2$.
(*iii*) $x^2-z^2+3xy=0$.

8. $19x^2+13y^2+3z^2-24yz-12zx-8xy=0$. 9. $120°$.

16. $2x^2+y^2+5z^2-4zx=0$.

17. $x^2(aw^2-2gwu+cu^2)+2xy(cuv-fwu-gvw+hw^2)$
$$+y^2(bw^2-2fvw+cv^2)=0.$$

18. $8z^2+4yz-zx-5xy=0$.

21. $\frac{bn^2+cm^2}{fmn}=\frac{cl^2+an^2}{gnl}=\frac{am^2+bl^2}{hlm}$. 24. $\tan^{-1}\frac{1}{\sqrt{2}}$.

CHAPTER IX

Pages 198—199.

2. $10x^2+13y^2+5z^2+6yz-12zx+4xy+8x+10y-2z-123=0$.

3. $x^2+y^2=a^2$.

Page 202. Art. 107, (*b*).

Ex. $x^2+y^2+z^2-yz-zx-xy-4x+5y-z-2=0$.

MISCELLANEOUS EXAMPLES ON CHAPTER IX

Pages 202—205.

2. $5x^2+8y^2+5z^2-4yz-8zx-4xy+22x-16y-14z-10=0$.

3. $5x^2+8y^2+5z^2+4yz+8zx-4xy-144=0$.

4. $x^2+y^2+z^2+yz-zx+xy-9 = 0.$

5. $x^2+y^2+z^2-yz-zx-xy-1 = 0.$

7. $\frac{1}{a^2}(nx-z)^2+\frac{1}{b^2}(ny-mz)^2 = n^2.$

8. $3x^2+6y^2+3z^2+8yz-2zx+6x-24y-18z+24 = 0.$

9. $[(x-a)^2+(y-b)^2+(z-c)^2-r^2](l^2+m^2+n^2)$
$= [l(x-a)+m(y-b)+n(z-c)]^2.$

CHAPTER X

Page 215.

1. (*i*) If the equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and that of the given plane is $lx+my+nz = p$, the equation of the cone is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\left(\frac{lx+my+nz}{p}\right)^2$;

(*ii*) If the equation of the ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and that of the concentric sphere is $x^2+y^2+z^2=r^2$, the equation of the cone is $\left(\frac{1}{a^2}-\frac{1}{r^2}\right)x^2+\left(\frac{1}{b^2}-\frac{1}{r^2}\right)y^2+\left(\frac{1}{c^2}-\frac{1}{r^2}\right)z^2=0.$

Page 219.

Ex. (*a*) Hyperbolic paraboloid.

MISCELLANEOUS EXAMPLES ON CHAPTER X

Pages 222—223.

3. (*a*) Paraboloid of revolution.

CHAPTER XI

Page 227.

1. $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1.$

Pages 228—230.

1. $4x+6y+3z-5=0,\ 2x-12y+9z-5=0.$

4. (*b*) $lx+my+nz=\pm\sqrt{a^2l^2+b^2m^2+c^2n^2}.$

Page 231.

1. $x^2+y^2+z^2=a^2+b^2+c^2$, a sphere (called the *director sphere* of the ellipsoid).

Page 233.

2. The curve of intersection of the given ellipsoid and the ellipsoid $\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}=\frac{k^2}{a^4+b^4+c^4}.$

3. $2\dfrac{\left[\dfrac{x_1^2}{a^4}+\dfrac{y_1^2}{b^4}+\dfrac{z_1^2}{c^4}\right]^{\frac{3}{2}}}{\dfrac{x_1^2}{a^6}+\dfrac{y_1^2}{b^6}+\dfrac{z_1^2}{c^6}}$.

Page 239.

1. $\dfrac{xx'}{a^2}+\dfrac{yy'}{b^2}+\dfrac{zz'}{c^2}=1$.

Pages 241—242.

1. (See Note, Art. 122.) $ax\alpha+by\beta+cz\gamma=1$, $axl+bym+czn=0$.
2. $\dfrac{a(b-c)\alpha}{x-\alpha}+\dfrac{b(c-a)\beta}{y-\beta}+\dfrac{c(a-b)\gamma}{z-\gamma}=0$.
3. $ax_1x_2+by_1y_2+cz_1z_2=1$, $ax_1l_2+by_1m_2+cz_1n_2=0$,
 $ax_2l_1+by_2m_1+cz_2n_1=0$, $al_1l_2+bm_1m_2+cn_1n_2=0$.
4. $(ax_1x_2+by_1y_2+cz_1z_2-1)(al_1l_2+bm_1m_2+cn_1n_2)$
 $=(al_1x_2+bm_1y_2+cn_1z_2)(al_2x_1+bm_2y_1+cn_2z_1)$.
7. $y+5=0$, $5x+y-5z=0$.
8. $x-2y+3z+1=0$, $2x+3y+z=0$.

Pages 243—244.

1. $(ax^2+by^2+cz^2-1)(a\alpha^2+b\beta^2+c\gamma^2-1)=(ax\alpha+by\beta+cz\gamma-1)^2$;
 $a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c$.
2. (i) $z=\pm c$. (ii) $\dfrac{x^2+y^2}{a^2+b^2}+\dfrac{z^2}{c^2}=1$.
3. $\dfrac{y^2}{b^2-a^2}+\dfrac{z^2}{c^2}=1, x=0$; $\dfrac{x^2}{a^2-b^2}+\dfrac{z^2}{c^2}=1, y=0$.

Page 246.

1. [See Art. 106.]
 $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)\left(\dfrac{l^2}{a^2}+\dfrac{m^2}{b^2}+\dfrac{n^2}{c^2}\right)=\left(\dfrac{xl}{a^2}+\dfrac{ym}{b^2}+\dfrac{zn}{c^2}\right)^2$.
2. $(ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)=(axl+bym+czn)^2$.
3. $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}-1\right)$
 $=\left(\dfrac{xx_1}{a^2}+\dfrac{yy_1}{b^2}+\dfrac{zz_1}{c^2}-1\right)^2$;
 $\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1\right)\left(\dfrac{l^2}{a^2}+\dfrac{m^2}{b^2}+\dfrac{n^2}{c^2}\right)=\left(\dfrac{xl}{a^2}+\dfrac{ym}{b^2}+\dfrac{zn}{c^2}\right)^2$.

Pages 247—248.

1. $ax\alpha+by\beta+cz\gamma=a\alpha^2+b\beta^2+c\gamma^2$; $x+6y-10z+20=0$.
2. (a) $\dfrac{xf}{a^2}+\dfrac{yg}{b^2}+\dfrac{zh}{c^2}=\dfrac{f^2}{a^2}+\dfrac{g^2}{b^2}+\dfrac{h^2}{c^2}$.
5. $(ax^2+by^2+cz^2)^2=\dfrac{a^2x^2}{\alpha}+\dfrac{b^2y^2}{\beta}+\dfrac{c^2z^2}{\gamma}$;

6. If the equation of the conicoid is $ax^2+by^2+cz^2=1$, and k is the constant distance of the section from the centre, the required locus is $(ax^2+by^2+cz^2)^2=k^2(a^2x^2+b^2y^2+c^2z^2)$.

8. $xx_1+yy_1+zz_1=x_1^2+y_1^2+z_1^2$. 9. $(-2, 3, -1)$.

Pages 251—252.

1. $\dfrac{lx}{a^2}+\dfrac{my}{b^2}+\dfrac{nz}{c^2}=0$.

2. (*i*) $x+4y-3z-4=0$; $\dfrac{x+1}{1}=\dfrac{y-2}{4}=\dfrac{z-1}{-3}$.

(*ii*) $x-4y+6z=0$. (*iii*) $(-4, 2, \frac{8}{3})$.

3. $ax^2+by^2+cz^2=ax\alpha+by\beta+cz\gamma$.

Pages 255—257.

1. (*a*) [See Art. 130, Cor.]

6. The locus of the point of intersection of three mutually perpendicular tangent planes to the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$ is the director sphere $x^2+y^2+z^2=a^2+b^2+c^2$.

Pages 264—266.

1. $\dfrac{(b-c)x}{l}+\dfrac{(c-a)y}{m}+\dfrac{(a-b)\,z}{n}=0$.

Pages 269—270.

1. $ax\alpha+by\beta=z+\gamma$. 2. $(8, 9, 5)$.

5. $(al\alpha+bm\beta-n)^2=(al^2+bm^2)(a\alpha^2+b\beta^2-2\gamma)$;

$(ax^2+by^2-2z)(a\alpha^2+b\beta^2-2\gamma)=(ax\alpha+by\beta-z-\gamma)^2$.

6. $ab(x^2+y^2)-2(a+b)\,z-1=0$.

Pages 275—276. Art. 143.

3. $\dfrac{x-3}{9}=\dfrac{y-4}{10}=\dfrac{z-5}{15}$.

MISCELLANEOUS EXAMPLES ON CHAPTER XI

Pages 276—284.

4. $a^2x^2+b^2y^2+c^2z^2=c^4$.

33. $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=\dfrac{3}{2}$.

CHAPTER XII

Pages 286—288.

3. If $n=0$, the section is a parabola. If $n\neq0$, the section is an ellipse or hyperbola according as ab is +ve or −ve.

Pages 291—293.

1. If the equation of the ellipsoid is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$,

and that of the plane of the central section is $lx+my+nz=0$, then the equation giving the lengths (r) of the semi-axes of the section is

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2}\left[l^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+m^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+n^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right]$$
$$+\left(\frac{l^2}{b^2c^2}+\frac{m^2}{c^2a^2}+\frac{n^2}{a^2b^2}\right)=0.$$

2. $\dfrac{\sqrt{11}\pm\sqrt{13}}{6}$.

8. If the equation of the ellipsoid is $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, and that of the plane of the central section is $lx+my+nz=0$, then the equation giving the lengths (r) of the semi-axes of the section is

$$\frac{1}{r^4}(l^2+m^2+n^2)-\frac{1}{r^2}\left[l^2\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+m^2\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+n^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right]$$
$$+\left(\frac{l^2}{b^2c^2}+\frac{m^2}{c^2a^2}+\frac{n^2}{a^2b^2}\right)=0,$$

and the area$=\pi abc\dfrac{\sqrt{l^2+m^2+n^2}}{\sqrt{a^2l^2+b^2m^2+c^2n^2}}$.

Pages 298—301.

1. (*a*) If the equation of the central conicoid is $ax^2+by^2+cz^2=1$, that of the plane of the central section is $lx+my+nz=0$, and that of the plane of the parallel section is $lx+my+nz=p$; r_1, r_2 are the lengths of the semi-axes of the central section, r_1', r_2' the lengths of the corresponding semi-axes of the parallel section, then

$$r_1'=kr_1,\ r_2'=kr_2,$$

where $k^2=1-\dfrac{p^2}{p_0^2}$, and $p_0^2=\dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{n^2}{c}$.

(*b*) $\left(\frac{1}{5},\frac{1}{5},\frac{1}{5}\right)$; $\sqrt{\frac{11}{15}}$, $\frac{\sqrt{33}}{5}$. 4. $\dfrac{2\pi}{3\sqrt{3}}\sqrt{b^2c^2+c^2a^2+a^2b^2}$

5. $a^2b^2c^2\left(\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+\dfrac{z^2}{c^4}\right)\left(1-\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}\right)^2=k^4\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)$.

Pages 305—306.

1. $\dfrac{\sqrt{11}(3\pm\sqrt{3})}{4}$. 3. $a^2b^2\left(\dfrac{x^2}{a^4}+\dfrac{y^2}{b^4}+1\right)\left(\dfrac{x^2}{a^2}+\dfrac{y^3}{b^2}-2z\right)^2=k^4$.

Pages 309—310.

1. $x\pm z=0$.

Page 312.

2. $\dfrac{a^2x^2}{a^2-b^2}-\dfrac{c^2z^2}{b^2-c^2}=b^2$, $y=0$.

3. $\dfrac{x^2}{a^2-b^2} - \dfrac{z^2}{b^2-c^2} = 1-\dfrac{k^2}{b^2},\ y=0.$

Pages 313—315.

2. $4x-y-3z = 0,\ 2x+5y-z = 0.$

3. $x+y-z = \lambda,\ x-y+2z = \mu$, for all values of λ and μ.

6. $l=0,\ m^2(c-a)=n^2(a-b)$; or $m=0,\ n^2(a-b)=l^2(b-c)$; or $n=0,\ l^2(b-c) = m^2(c-a)$.

Page 317.

1. $y+2z=\lambda,\ y-2z=\mu$, for all values of λ and μ. 2. $\dfrac{4}{\sqrt{5}}$.

Pages 318—319. Art. 154.

7. $\left(0,\ \pm b\sqrt{a^2-b^2},\ \dfrac{a^2-b^2}{2}\right)$.

MISCELLANEOUS EXAMPLES ON CHAPTER XII

Pages 319—328.

10. (*i*) $\dfrac{\sqrt{3}}{2}$, 1. (*ii*) 1, 1.

CHAPTER XIII

Pages 333—336.

2. $\dfrac{x+2}{0} = \dfrac{y-1}{3} = \dfrac{z+\frac{4}{3}}{-4}$; $\dfrac{x+2}{3} = \dfrac{y-1}{6} = \dfrac{z+\frac{4}{3}}{10}$.

3. $\dfrac{x}{1} = \dfrac{y+1}{0} = \dfrac{z-3}{0}$; $\dfrac{x}{1} = \dfrac{y+1}{-1} = \dfrac{z-3}{-3}$.

Page 340.

1. The equation represents the two planes, one thro' the origin and one generator thro' (x_1, y_1, z_1), and the other thro' the origin and the other generator thro' (x_1, y_1, z_1).

Pages 346—347.

1. $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2,\ z = \pm c.$

Page 350.

1. (*i*) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z.$ (*ii*) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z.$

2. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{z}{\lambda},\ \dfrac{x}{a} - \dfrac{y}{b} = 2\lambda$;

$\dfrac{x}{a} - \dfrac{y}{b} = \dfrac{z}{\mu},\ \dfrac{x}{a} + \dfrac{y}{b} = 2\mu$;

where λ and μ are variable parameters.

Page 351.

1. $al^2+bm^2=0,\ al\alpha+bm\beta-n=0,\ a\alpha^2+b\beta^2-2\gamma=0$;

$$\frac{l}{\pm\sqrt{-b}}=\frac{m}{-\sqrt{a}}=\frac{n}{\pm a\alpha\sqrt{-b}-\sqrt{a}.b\beta}.$$

2. $\dfrac{x-1}{5}=\dfrac{y-1}{-4}=\dfrac{z-1}{-1},\ \dfrac{x-1}{3}=\dfrac{y-1}{-1}=\dfrac{z-1}{1}.$

Pages 352—353. Art. 167.

3. $\dfrac{x-ar\cos\theta}{\pm a}=\dfrac{y-br\sin\theta}{-b}=\dfrac{z-\frac{1}{2}r^2\cos 2\theta}{r(\pm\cos\theta+\sin\theta)}$, the ambiguous signs being taken with the upper signs thro' out or the lower signs thro' out.

5. $x^2+y^2+2z^2\pm\dfrac{a^2+b^2}{ab}xy=0.$

MISCELLANEOUS EXAMPLES ON CHAPTER XIII

Pages 353—358.

1. (*a*) Cone (and as a particular case, pair of planes), cylinder, hyperboloid of one sheet, and hyperbolic paraboloid.

8. $\mu x+(\mu+1)z+1=0,\ \mu x-y+\mu+1=0,$ where μ is a variable parameter.

9. $(\lambda+1)x=-y=-\lambda(z+1)$, where λ is a variable parameter.

16. The curve of intersection of the given paraboloid and the cylinder $(x+y)^2\tan^2\alpha=(x-y)^2+2$.